ANNALES
POLONICI MATHEMATICI
LXIII.2 (1996)

Some quadratic integral inequalities of Opial type

by Małgorzata Kuchta (Wrocław)

Abstract. We derive and investigate integral inequalities of Opial type: $\int_I s|h\dot{h}|\,dt \le \int_I r\dot{h}^2\,dt$, where $h \in H$, $I = (\alpha, \beta)$ is any interval on the real line, H is a class of absolutely continuous functions h satisfying $h(\alpha) = 0$ or $h(\beta) = 0$. Our method is a generalization of the method of [3]–[5]. Given the function r we determine the class of functions s for which quadratic integral inequalities of Opial type hold. Such classes have hitherto been described as the classes of solutions of a certain differential equation. In this paper a wider class of functions s is given which is the set of solutions of a certain differential inequality. This class is determined directly and some new inequalities are found.

Introduction. We derive and examine quadratic integral inequalities of Opial type, i.e. the inequalities of the form

$$\int_I s|h\dot{h}|\,dt \le \int_I r\dot{h}^2\,dt, \qquad h \in H, \tag{1}$$

where $I = (\alpha, \beta)$, $-\infty \le \alpha < \beta \le \infty$, r and s are fixed functions of t, H is a class of absolutely continuous functions and $\dot{h} \equiv dh/dt$. We extend the method used to examine inequalities of Sturm–Liouville type by Florkiewicz and Rybarski [5], Hardy type inequalities by Florkiewicz [3] and Opial type inequalities by Florkiewicz [4]. The method makes it possible, given a function r and an auxilliary function φ, to define a function s and an additional function v and next using r, s and v to define a class H of functions h for which (1) holds.

In this paper s is given as the solution of a certain differential inequality. In [5] and [3] s is calculated explicitly, in [4] it is described as the solution of a differential equation. In this way one obtains a larger class of functions s for which (1) holds. In the final section of the paper the class of functions s is determined directly and some new integral inequalities connected with the inequality (1) are found.

1991 *Mathematics Subject Classification*: Primary 26D10.
Key words and phrases: absolutely continuous function, integral inequality.

The integral inequalities of Opial type (1) have also been obtained and examined by different methods (see Beesack [1], Redheffer [9], Yang [11], Boyd [2]; further bibliography can be found in [6]).

1. Let $I=(\alpha,\beta)$, $-\infty\le\alpha<\beta\le\infty$. By $AC(I)$ we denote the class of absolutely continuous real functions on I. Let $r\in AC(I)$ and $\varphi\in AC(I)$ be such that $r,\varphi>0$ on I and $\dot\varphi\in AC(I)$. Let $s\in AC(I)$ satisfy the differential inequality

$$\tfrac12\dot s-(r\dot\varphi)^{\cdot}\varphi^{-1}\ge 0 \tag{2}$$

almost everywhere on I. Put further

$$v=-\tfrac12 s+r\dot\varphi\varphi^{-1}. \tag{3}$$

Let H denote the class of functions $h\in AC(I)$ such that $h\ge0$ on I and

$$\int_I r\dot h^2\,dt<\infty, \tag{4}$$

$$\int_I sh\dot h\,dt>-\infty, \tag{5}$$

$$\liminf_{t\to\alpha} vh^2<\infty,\qquad \limsup_{t\to\beta} vh^2>-\infty. \tag{6}$$

THEOREM 1. *For every $h\in H$ both limits in* (6) *are proper and finite*, *and*

$$\lim_{t\to\beta} vh^2-\lim_{t\to\alpha} vh^2\le\int_I (r\dot h^2-sh\dot h)\,dt. \tag{7}$$

If $h\not\equiv0$ and $\varphi\notin H$, then inequality (7) *is strict. If $\varphi\in H$, then inequality* (7) *becomes equality if and only if s satisfies the differential equation*

$$\tfrac12\dot s-(r\dot\varphi)^{\cdot}\varphi^{-1}=0 \tag{8}$$

a.e. on I and $h=c\varphi$ with $c=\mathrm{const}\ge0$.

Proof. Let $h\in AC(I)$ and $h\ge0$ on I. By (3) and our assumptions we have $vh^2\in AC(I)$ and $h\varphi^{-1}\in AC(I)$. After simple calculations we get

$$(vh^2)^{\cdot}=-sh\dot h-\big(\tfrac12\dot s-(r\dot\varphi)^{\cdot}\varphi^{-1}\big)h^2+2r\dot\varphi\varphi^{-1}h\dot h-r\dot\varphi^2\varphi^{-2}h^2$$

and

$$r\varphi^2[(h\varphi^{-1})^{\cdot}]^2=r\dot h^2-2r\dot\varphi\varphi^{-1}h\dot h+r\dot\varphi^2\varphi^{-2}h^2$$

a.e. on I. Hence we obtain the identity

$$r\dot h^2-sh\dot h=(vh^2)^{\cdot}+\big(\tfrac12\dot s-(r\dot\varphi)^{\cdot}\varphi^{-1}\big)h^2+r\varphi^2[(h\varphi^{-1})^{\cdot}]^2 \tag{9}$$

a.e. on I.

Let now $h\in H$; we shall examine the summability of the functions that appear in (9). Condition (4) implies that $r\dot h^2$ is summable on I because

$r\dot{h}^2 \geq 0$ on I. Consider an arbitrary interval $[a,b] \subset I$. The function $sh\dot{h}$ is summable on $[a,b]$ because $sh \in AC(I)$ and therefore is bounded and $\dot{h}$ is summable on $[a,b]$. The function $(vh^2)^{\cdot}$ is summable on $[a,b]$ because $vh^2 \in AC(I)$. Similarly $\big(\frac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\big)h^2$ is summable on $[a,b]$ because $\dot{s}$ is summable on $[a,b]$ and $\frac{1}{2}h^2 \in AC(I)$, $(r\dot{\varphi})^{\cdot}$ is summable on $[a,b]$ and $\varphi^{-1}h^2 \in AC(I)$. It follows from (9) that $r\varphi^2[(h\varphi^{-1})^{\cdot}]^2$ is also summable on $[a,b]$ and

$$\int_a^b r\dot{h}^2\,dt - \int_a^b sh\dot{h}\,dt = \int_a^b \big(\tfrac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\big)h^2 dt + \int_a^b g\,dt + vh^2|_a^b, \tag{10}$$

where $g = r\varphi^2[(h\varphi^{-1})^{\cdot}]^2$. It follows from conditions (6) that there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that $\alpha < a_n < b_n < \beta$, $a_n \to \alpha$, $b_n \to \beta$ and

$$\lim_{n\to\infty} vh^2|_{a_n} = A < \infty, \qquad \lim_{n\to\infty}(-vh^2)|_{b_n} = B < \infty.$$

Thus there is a constant C such that

$$-vh^2|_{a_n}^{b_n} < C < \infty.$$

From (10) and the fact that $g \geq 0$ and $\big(\frac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\big)h^2 \geq 0$ it follows that

$$\int_{a_n}^{b_n} sh\dot{h}\,dt \leq \int_{a_n}^{b_n} r\dot{h}^2\,dt + C \leq \int_\alpha^\beta r\dot{h}^2\,dt + C.$$

Letting $n \to \infty$ we obtain

$$\int_I sh\dot{h}\,dt \leq \int_I r\dot{h}^2\,dt + C.$$

Thus in view of (4) and (5) we conclude that $sh\dot{h}$ is summable on I. In the analogous way using (10) we show that $\big(\frac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\big)h^2$ and g are summable on I. It follows that in (6) we can simply put lim instead of lim inf and lim sup and both limits are finite because all integrals in (10) have finite limits as $a \to \alpha$ or $b \to \beta$.

By (10) letting $a \to \alpha$ and $b \to \beta$ we obtain

$$\int_I r\dot{h}^2\,dt - \int_I sh\dot{h}\,dt = \int_I \big(\tfrac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\big)h^2\,dt + \int_I g\,dt + \lim_{t\to\beta} vh^2 - \lim_{t\to\alpha} vh^2, \tag{11}$$

whence (7) immediately follows because $g \geq 0$ and $\big(\frac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\big)h^2 \geq 0$ on I.

By (11), (2) and the condition $g \geq 0$ on I, inequality (7) becomes equality for a non-vanishing function $h \in H$ if and only if $\int_I \big(\frac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\big)h^2\,dt = 0$ and $\int_I g\,dt = 0$. Now $\int_I g\,dt = 0$ if and only if $g = r\varphi^2[(h\varphi^{-1})^{\cdot}]^2 = 0$ a.e. on I. Hence $(h\varphi^{-1})^{\cdot} = 0$ a.e. on I and $h = c\varphi$, where c is a positive constant, since by assumption $h\varphi^{-1} \in AC(I)$. Thus $\varphi \in H$. Further $\int_I \left(\frac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\right)h^2\,dt = 0$ if and only if s is a solution of the differential equation (8) a.e. on I, since $h = c\varphi > 0$ on I.

2. In further considerations we use the following lemmas.

LEMMA 1. *Let $h \in AC(I)$ and $\int_I r\dot{h}^2\,dt < \infty$. If $\int_\alpha^t r^{-1}\,d\tau < \infty$ (resp. $\int_t^\beta r^{-1}d\tau < \infty$) for some $t \in I$, then the limit*

$$h(\alpha) = \lim_{t\to\alpha} h \qquad (\textit{resp. } h(\beta) = \lim_{t\to\beta} h)$$

exists and is finite.

For the proof see [3].

Let us denote by H_0 (resp. H^0) the class of functions $h \in AC(I)$ satisfying the integral condition

$$\int_I r\dot{h}^2\,dt < \infty$$

and the limit condition

$$\lim_{t\to\alpha} h = 0 \qquad (\text{resp. } \lim_{t\to\beta} h = 0).$$

LEMMA 2. *If $h \in H_0$ (resp. $h \in H^0$), $\int_\alpha^t r^{-1}\,d\tau < \infty$ (resp. $\int_t^\beta r^{-1}d\tau < \infty$) for some $t \in I$ and $w\int_\alpha^t r^{-1}d\tau = O(1)$ as $t \to \alpha$ (resp. $w\int_t^\beta r^{-1}d\tau = O(1)$ as $t \to \beta$), where w is an arbitrary measurable function on I, then*

$$\lim_{t\to\alpha} wh^2 = 0 \qquad (\textit{resp. } \lim_{t\to\beta} wh^2 = 0).$$

Proof. We prove the lemma for the case $h \in H_0$ (the case $h \in H^0$ is similar). Using the Schwarz inequality we obtain

$$0 \le |h(t) - h(a)|^2 \le \int_a^t r^{-1}\,d\tau \int_a^t r\dot{h}^2\,d\tau,$$

where $\alpha < a < t < \beta$. Letting $a \to \alpha$ shows that

$$0 \le |wh^2| \le \left|w \int_\alpha^t r^{-1}\,d\tau\right| \int_\alpha^t r\dot{h}^2\,d\tau.$$

Letting $t \to \alpha$ we obtain the assertion. ∎

The derivative of the function $v \in AC(I)$ defined by (3) is

$$\dot{v} = -r\dot{\varphi}^2\varphi^{-2} - \left(\tfrac{1}{2}\dot{s} - (r\dot{\varphi})^{\cdot}\varphi^{-1}\right) \qquad \text{a.e. on } I.$$

From (2) it follows that

$$\dot{v} \le -r\dot{\varphi}^2\varphi^{-2} \tag{12}$$

on I, so $\dot v \le 0$ on I. Hence v is decreasing on I, the limit values $v(\alpha) = \lim_{t\to\alpha} v$ and $v(\beta) = \lim_{t\to\beta} v$ exist and $v(\alpha) \ge v(\beta)$.

LEMMA 3. *Let $s \in AC(I)$ satisfy the differential inequality* (2) *a.e. on I.*

(i) *Let $s \ge 0$ on I. If $v(\alpha) > 0$, then $\int_\alpha^t r^{-1} d\tau < \infty$ for some $t \in I$ and $v\int_\alpha^t r^{-1} d\tau = O(1)$ as $t \to \alpha$. If $v(\beta) > 0$, then $\int_t^\beta r^{-1} d\tau < \infty$ for some $t \in I$.*

(ii) *Let $s \le 0$ on I. If $v(\beta) < 0$, then $\int_t^\beta r^{-1} d\tau < \infty$ for some $t \in I$ and $v\int_t^\beta r^{-1} d\tau = O(1)$ as $t \to \beta$. If $v(\alpha) < 0$, then $\int_\alpha^t r^{-1} d\tau < \infty$ for some $t \in I$.*

Proof. We prove Lemma 3 only in one case: $s \ge 0$ on I and $v(\alpha) > 0$. The remaining cases can be proved similarly.

There exists a neighbourhood U of α such that $v > 0$ on U. By (3) it follows that $0 < v \le r\dot\varphi\varphi^{-1}$ on U. Hence, by (12), $\dot v \le -r^{-1}v^2$ a.e. on U. Thus $r^{-1} \le -\dot v v^{-2}$ a.e. on U and we obtain

$$\int_a^t r^{-1} d\tau \le -\int_a^t \dot v v^{-2} d\tau = v^{-1} - v^{-1}(a) < v^{-1}$$

for any $a, t \in U$ such that $\alpha < a < t < \beta$. Letting $a \to \alpha$ shows that $\int_\alpha^t r^{-1} d\tau < \infty$ and $0 < v\int_\alpha^t r^{-1} d\tau \le 1$. Thus $v\int_\alpha^t r^{-1} d\tau = O(1)$ as $t \to \alpha$. ■

THEOREM 2. *Let $s \in AC(I)$ satisfy the differential inequality* (2) *a.e. on I.*

(i) *If $s \ge 0$ on I and $v(\beta) \ge 0$, then*

$$\int_I s|h\dot h|\,dt + \lim_{t\to\beta} vh^2 \le \int_I r\dot h^2\,dt \tag{13}$$

for every $h \in H_0$. If $v(\beta) > 0$, then the limit value $h(\beta)$ exists and is finite, and (13) *takes the form*

$$\int_I s|h\dot h|\,dt + v(\beta)h^2(\beta) \le \int_I r\dot h^2\,dt. \tag{14}$$

If $h \not\equiv 0$, then (13) *becomes equality if and only if s satisfies the differential equation* (8) *a.e. on I, $h = c\varphi$ with $c = \text{const} \ne 0$, $\varphi \in H_0$ and $\dot\varphi \ge 0$ on I.*

(ii) *If $s \le 0$ on I and $v(\alpha) \le 0$, then*

$$\int_I |sh\dot h|\,dt - \lim_{t\to\alpha} vh^2 \le \int_I r\dot h^2\,dt \tag{15}$$

for every $h \in H^0$. If $v(\alpha) < 0$, then the limit value $h(\alpha)$ exists and is finite, *and* (15) *takes the form*

$$\int_I |sh\dot{h}|\,dt - v(\alpha)h^2(\alpha) \le \int_I r\dot{h}^2\,dt. \tag{16}$$

If $h \not\equiv 0$, *then* (15) *becomes equality if and only if* s *satisfies* (8) *a.e. on* I, $h = c\varphi$ *with* $c = \mathrm{const} \ne 0$, $\varphi \in H^0$ *and* $\dot{\varphi} \le 0$ *on* I.

Proof. (i) If $v(\beta) \ge 0$, then $v \ge 0$ on I. Then $\limsup_{t\to\beta} vh^2 \ge 0$ for every $h \in AC(I)$ whence the second condition of (6) is valid. If $v(\alpha) = 0$, then $v \equiv 0$ on I. Then $s \equiv 0$ on I and (13) trivially holds. If $v(\alpha) > 0$, then by Lemma 3(i) we have $\int_\alpha^t r^{-1}d\tau < \infty$ and $v\int_\alpha^t r^{-1}d\tau = O(1)$ as $t \to \alpha$. Thus by Lemma 2 for every $h \in H_0$ we have $\lim_{t\to\alpha} vh^2 = 0$, whence the first condition of (6) is valid. Let $h_+ \in H_0$ be such that $h_+ \ge 0$ and $\dot{h}_+ \ge 0$ on I. Then $\int_I sh_+\dot{h}_+\,dt \ge 0$ and so (5) is satisfied. To sum up, $h_+ \in H$ and from Theorem 1 we obtain

$$\int_I sh_+\dot{h}_+\,dt + \lim_{t\to\beta} vh_+^2 \le \int_I r(\dot{h}_+)^2\,dt. \tag{17}$$

Let now $h \in H_0$ be arbitrary and let $h_+ = \int_\alpha^t |\dot{h}|\,d\tau$. Then $h_+ \ge 0$ on I and $\dot{h}_+ \ge 0$ on I and $h_+(\alpha) = 0$. Thus $h_+ \in H_0$ and h_+ satisfies inequality (17). Notice that

$$|h| = \Big|\int_\alpha^t \dot{h}\,d\tau\Big| \le \int_\alpha^t |\dot{h}|\,d\tau = h_+ \tag{18}$$

and equality holds if and only if $\dot{h}$ does not change sign on I. Hence

$$\int_I s|h\dot{h}|\,dt \le \int_I sh_+\dot{h}_+\,dt$$

and since $v \ge 0$ on I we obtain

$$\int_I s|h\dot{h}|\,dt + \lim_{t\to\beta} vh^2 \le \int_I sh_+\dot{h}_+\,dt + \lim_{t\to\beta} vh_+^2. \tag{19}$$

Since

$$\int_I r(\dot{h}_+)^2\,dt = \int_I r\dot{h}^2\,dt, \tag{20}$$

inequality (13) follows from (17) and (19).

If $v(\beta) > 0$, then by Lemma 3(i) we have $\int_t^\beta r^{-1}d\tau < \infty$ and Lemma 1 yields the existence of a finite limit value $h(\beta)$. Hence inequality (13) takes the form (14).

Let now $h \in H_0$, $h \not\equiv 0$ and suppose that equality holds in (13). Then by (17), (19) and (20) there are equalities in (17) and (20) for $h_+ = \int_\alpha^t |\dot{h}|\,d\tau$. The assumptions $s \ge 0$ and $v \ge 0$ on I and equality in (19) yield equality in (18). Hence $\dot{h}$ does not change sign on I. We know that $h_+ \in H$ and by Theorem 1 inequality (17) becomes equality if and only if $h_+ = c\varphi$, where $c = \text{const} > 0$, $\varphi \in H$ and s satisfies (8) a.e. on I. Hence $h = c\varphi$, where $c = \text{const} \neq 0$ and $\varphi \in H_0$ with $\dot{\varphi} \geq 0$ on I.

Conversely, if $h = c\varphi$, where $c = \text{const} \neq 0$ and $\varphi \in H_0$ with $\dot{\varphi} \geq 0$ on I, then $h_+ = |h| \in H$. Let s satisfy (8). Then by Theorem 1, inequality (13) becomes equality for h.

(ii) This case can be proved analogously to (i) if we consider the function $h_- = \int_t^\beta |\dot{h}|\, d\tau \in H$, where $h \in H^0$. ∎

Example 1. Let $I = (0, \beta)$, $0 < \beta < \infty$. Put $r = e^{-2t}$ and $\varphi = e^t$ on I. Then $s = e^{-2t}$ satisfies (8) on I and it follows from Theorem 2(i) that every $h \in H_0$ satisfies the inequality

$$\int_0^\beta e^{-2t}|h\dot{h}|\, dt + \frac{1}{2} e^{-2\beta} h^2(\beta) \leq \int_0^\beta e^{-2t} \dot{h}^2\, dt,$$

called *Hlavka's inequality*. Note that $\varphi \notin H_0$ whence the inequality is strict for $h \not\equiv 0$. This inequality was considered by Redheffer [10].

3. Now we prove the existence of functions s satisfying the hypothesis of Theorem 2 and give their explicit form.

Every function $s \in AC(I)$ satisfying the differential inequality (2) has the form

$$s = 2\Big(\int_{t_0}^t (r\dot{\varphi})^{\cdot} \varphi^{-1} d\tau + \psi \Big),$$

where $t_0 \in I$ is an arbitrary point and $\psi \in AC(I)$ with $\dot{\psi} \geq 0$ on I. Since $v(\beta) \geq 0$ we have $v \geq 0$ on I. Thus s satisfies the hypothesis of Theorem 2(i) if and only if there exists ψ satisfying

$$-\int_{t_0}^t (r\dot{\varphi})^{\cdot} \varphi^{-1}\, d\tau \leq \psi \leq r\dot{\varphi}\varphi^{-1} - \int_{t_0}^t (r\dot{\varphi})^{\cdot} \varphi^{-1} d\tau. \tag{21}$$

Then $\dot{\varphi} \geq 0$ on I. Note that the function $r\dot{\varphi}\varphi^{-1} - \int_{t_0}^t (r\dot{\varphi})^{\cdot} \varphi^{-1} d\tau$ is nonincreasing in I, thus the following condition is necessary for the existence of ψ satisfying (21):

$$\sup_{t \in I} \Big(-\int_{t_0}^t (r\dot{\varphi})^{\cdot} \varphi^{-1} d\tau \Big) \leq \lim_{t \to \beta} \Big(r\dot{\varphi}\varphi^{-1} - \int_{t_0}^t (r\dot{\varphi})^{\cdot} \varphi^{-1} d\tau \Big). \tag{22}$$

Therefore assume that r and φ satisfy the condition

$$\Big| \int_t^\beta (r\dot{\varphi})^{\cdot} \varphi^{-1} d\tau \Big| < \infty \qquad \text{for some } t \in I.$$

Then $\lim_{t\to\beta} r\dot\varphi\varphi^{-1} < \infty$ and condition (22) takes the form

$$\sup_{t\in I}\int_t^\beta (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau \le \lim_{t\to\beta} r\dot\varphi\varphi^{-1}. \tag{23}$$

If (23) holds then

$$\psi = \lim_{t\to\beta} r\dot\varphi\varphi^{-1} - \int_{t_0}^\beta (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau = \text{const}$$

is the maximal function satisfying (21). Then

$$s = 2\Big(\lim_{t\to\beta} r\dot\varphi\varphi^{-1} - \int_t^\beta (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau\Big) \ge 0 \quad \text{and} \quad v(\beta) = 0.$$

If $r\dot\varphi$ is nonincreasing on I, then $-\int_{t_0}^t (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau$ is nondecreasing on I and condition (22) is trivially satisfied. Then $\psi = -\int_{t_0}^t (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau$ is the minimal function satisfying (21). Thus $s \equiv 0$ and $v = r\dot\varphi\varphi^{-1} \ge 0$ on I.

If $r\dot\varphi$ is nondecreasing on I, then $-\int_{t_0}^t (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau$ is nonincreasing on I. Then condition (22) takes the form

$$\int_I (r\dot\varphi)^{\cdot}\varphi^{-1}\,dt \le \lim_{t\to\beta} r\dot\varphi\varphi^{-1}. \tag{24}$$

If (24) holds then $\psi = \int_\alpha^{t_0} (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau = \text{const}$ is the minimal function satisfying (21) and $s = 2\int_\alpha^t (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau \ge 0$.

From the above considerations and from Theorem 2(i) one immediately infers the following theorem.

THEOREM 3. *Let $h \in H_0$ and r and φ satisfy $|\int_t^\beta (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau| < \infty$ for some $t \in I$ and $\dot\varphi \ge 0$ on I.*

(i) *If $\sup_{t\in I}\int_t^\beta (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau \le \lim_{t\to\beta} r\dot\varphi\varphi^{-1}$ on I, then*

$$\int_I s|h\dot h|\,dt \le \int_I r\dot h^2\,dt \tag{25}$$

where $s = 2(\lim_{t\to\beta} r\dot\varphi\varphi^{-1} - \int_t^\beta (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau) \ge 0$.

(ii) *If $r\dot\varphi$ is nonincreasing on I, then*

$$v(\beta)h^2(\beta) \le \int_I r\dot h^2\,dt, \tag{26}$$

where $v(\beta) = \lim_{t\to\beta} r\dot\varphi\varphi^{-1} \ge 0$.

(iii) *If* $r\dot\varphi$ *is nondecreasing on* I *and* $\int_I (r\dot\varphi)^{\cdot}\varphi^{-1}\,dt \le \lim_{t\to\beta} r\dot\varphi\varphi^{-1}$ *then*

$$\int_I s|h\dot h|\,dt + v(\beta)h^2(\beta) \le \int_I r\dot h^2\,dt, \tag{27}$$

where $s = 2\int_\alpha^t (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau \ge 0$ *and* $v(\beta) = \lim_{t\to\beta} r\dot\varphi\varphi^{-1} - \int_I (r\dot\varphi)^{\cdot}\varphi^{-1}\,dt \ge 0$.

If $h \not\equiv 0$ *then inequalities* (25)–(27) *become equalities if and only if* $h = c\varphi$, *where* $c = \text{const} \ne 0$ *and* $\varphi \in H_0$ *and in the case of* (26) *the additional condition* $r\dot\varphi = \text{const}$ *holds.*

Similar considerations apply to $s \in AC(I)$ satisfying the hypothesis of Theorem 2(ii). Thus we obtain the following theorem.

THEOREM 4. *Let* $h \in H^0$ *and* r *and* φ *satisfy* $|\int_\alpha^t (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau| < \infty$ *for some* $t \in I$ *and* $\dot\varphi \le 0$ *on* I.

(i) *If* $\sup_{t\in I}\int_\alpha^t (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau \le -\lim_{t\to\alpha} r\dot\varphi\varphi^{-1}$ *on* I, *then*

$$\int_I s|h\dot h|\,dt \le \int_I r\dot h^2\,dt, \tag{28}$$

where $s = -2(\lim_{t\to\alpha} r\dot\varphi\varphi^{-1} + \int_\alpha^t (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau) \ge 0$.

(ii) *If* $r\dot\varphi$ *is nonincreasing on* I, *then*

$$-v(\alpha)h^2(\alpha) \le \int_I r\dot h^2\,dt, \tag{29}$$

where $v(\alpha) = \lim_{t\to\alpha} r\dot\varphi\varphi^{-1} \le 0$.

(iii) *If* $r\dot\varphi$ *is nondecreasing on* I *and* $\int_I (r\dot\varphi)^{\cdot}\varphi^{-1}\,dt \le -\lim_{t\to\alpha} r\dot\varphi\varphi^{-1}$ *on* I *then*

$$\int_I s|h\dot h|\,dt - v(\alpha)h^2(\alpha) \le \int_I r\dot h^2\,dt, \tag{30}$$

where $s = 2\int_t^\beta (r\dot\varphi)^{\cdot}\varphi^{-1}d\tau \ge 0$ *and* $v(\alpha) = \lim_{t\to\alpha} r\dot\varphi\varphi^{-1} + \int_I (r\dot\varphi)^{\cdot}\varphi^{-1}\,dt \le 0$.

If $h \not\equiv 0$ *then inequalities* (28)–(30) *become equalities if and only if* $h = c\varphi$, *where* $c = \text{const} \ne 0$ *and* $\varphi \in H^0$ *and in the case of* (29) *the additional condition* $r\dot\varphi = \text{const}$ *holds.*

Inequalities of the form (25) and (28), which do not contain explicitly the limit conditions, are said to be of *Opial type* (cf. [6]).

EXAMPLE 2. Let $I = (0,\beta)$, $0 < \beta \le \infty$. Let r be an arbitrary function absolutely continuous on I such that $r > 0$ and $\int_0^\beta r^{-1}\,dt < \infty$.

Put $\varphi = \int_0^t r^{-1} d\tau$ on I. Then $s = 2(\int_0^\beta r^{-1}\,dt)^{-1}$ satisfies (8) and by Theorem 3(i), (25) holds for every $h \in H_0$.

Put $\varphi = \int_t^\beta r^{-1} d\tau$ on I. Then $s = 2(\int_0^\beta r^{-1}\,dt)^{-1}$ satisfies (8) and by Theorem 4(i), (28) holds for every $h \in H^0$. Hence we get the following:

If $h \in AC((0,\beta))$, $0 < \beta \le \infty$, *satisfies the integral condition* $\int_0^\beta r\dot{h}^2\,dt < \infty$ *and the limit condition* $h(0) = 0$ *or* $h(\beta) = 0$, *then*

$$\int_0^\beta |h\dot{h}|\,dt \le \frac{1}{2}\int_0^\beta r^{-1}\,dt \int_0^\beta r\dot{h}^2\,dt, \tag{31}$$

with equality if and only if $h = c\int_0^t r^{-1}d\tau$ *or* $h = c\int_t^\beta r^{-1}d\tau$ *according as* $h(0) = 0$ *or* $h(\beta) = 0$, *where* $c = \text{const}$.

The inequality (31) was considered by Beesack [1].

In the particular case where $\beta < \infty$ and $r = 1$ on $(0,\beta)$ we obtain the well-known Opial inequality

$$\int_0^\beta |h\dot{h}|\,dt \le \frac{\beta}{2}\int_0^\beta \dot{h}^2\,dt, \tag{32}$$

which holds for every absolutely continuous function h such that $\int_0^\beta \dot{h}^2\,dt < \infty$ and $h(0) = 0$ or $h(\beta) = 0$. Equality occurs in (32) only for the function $h = ct$ or $h = c(\beta - t)$, where $c = \text{const}$ (see Opial [8], Olech [7]).

Example 3. Take $I = (0,\beta)$, $0 < \beta < \infty$, $r = at^{p+1} + bt^{q+1}$, $\varphi = t^k$, where a, b, k, p, q are constants satisfying $a > 0$, $k > 0$, $p < q$ and $b \ge -a\beta^{p-q}$. Then $r > 0$, $\varphi > 0$ and $\dot{\varphi} > 0$ on I.

If

(i) $p + k < 0$ and $b(q + k) < 0$ or $q + k < 0$, $b < 0$ or $q + k > 0$, $0 < b \le -a(p + q)/(q + k)$, or

(ii) $p + k > 0$ and $b \le -(aq/p)\beta^{p-q}$, or

(iii) $p + k = 0$, $q > 0$ and $0 < b \le -(aq/p)\beta^{p-q}$,

then by Theorem 3(i) we get the following:

If $h \in H_0$ *then*

$$2k\int_0^\beta (At^p + Bt^q + C)|h\dot{h}|\,dt \le \int_0^\beta (at^{p+1} + bt^{q+1})\dot{h}^2\,dt, \tag{33}$$

where $A = \frac{a}{p}(p + k)$, $B = \frac{b}{q}(q + k)$, $C = -k\big(\frac{a}{p}\beta^p + \frac{b}{q}\beta^q\big)$.

Inequality (33) becomes equality if and only if $h = ct^k$, where $c = \text{const}$.

If $\beta = 1$ and $b = -aq/p$ we get the inequality considered by Redheffer [9].

If $p+k<0$ and either $q+k<0$, or $q+k>0$ and $b<-a(p+k)/(q+k)\beta^{p-q}$, then by Theorem 3(ii) we get the following:

If $h \in H_0$ *then*

$$k(a\beta^p + b\beta^q)h^2(\beta) \le \int\limits_0^\beta (at^{p+1} + bt^{q+1})\dot{h}^2\,dt. \tag{34}$$

If $h \not\equiv 0$, then inequality (34) is strict.

References

[1] P. R. Beesack, *On an integral inequality of Z. Opial*, Trans. Amer. Math. Soc. 104 (1962), 470–475.

[2] D. W. Boyd, *Best constants in inequalities related to Opial's inequality*, J. Math. Anal. Appl. 25 (1969), 378–387.

[3] B. Florkiewicz, *Some integral inequalities of Hardy type*, Colloq. Math. 43 (1980), 321–330.

[4] —, *On some integral inequalities of Opial type*, to appear.

[5] B. Florkiewicz and A. Rybarski, *Some integral inequalities of Sturm–Liouville type*, Colloq. Math. 36 (1976), 127–141.

[6] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer, Dordrecht, 1991, 114–142.

[7] C. Olech, *A simple proof of a certain result of Z. Opial*, Ann. Polon. Math. 8 (1960), 61–63.

[8] Z. Opial, *Sur une inégalité*, ibid. 8 (1960), 29–32.

[9] R. Redheffer, *Inequalities with three functions*, J. Math. Anal. Appl. 16 (1966), 219–242.

[10] —, *Integral inequalities with boundary terms*, in: Inequalities II, Proc. II Symposium on Inequalities, Colorado (USA), August 14-22, 1967, O. Shisha (ed.), New York and London 1970, 261–291.

[11] G. S. Yang, *On a certain result of Z. Opial*, Proc. Japan Acad. 42 (1966), 78–83.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY OF TECHNOLOGY
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW, POLAND

Reçu par la Rédaction le 16.6.1994
Révisé le 15.12.1995 et 16.10.1995