ANNALES
POLONICI MATHEMATICI
LXIII.1 (1996)

On balanced L^2-domains of holomorphy

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Abstract. We show that any bounded balanced domain of holomorphy is an $L^2_{\rm h}$-domain of holomorphy.

It is known [Pfl] that any bounded domain of holomorphy $G \subset \mathbb{C}^n$ with $G = \operatorname{int} \overline{G}$ is an $L^2_{\rm h}(G)$-domain of holomorphy, where $L^2_{\rm h}(G)$ denotes the space of all Lebesgue square integrable holomorphic functions on G. Observe that $L^2_{\rm h}(E) = L^2_{\rm h}(E \setminus \{0\})$, where $E := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. For Reinhardt domains there is a complete description of the $L^2_{\rm h}$-domains of holomorphy (cf. [Jar-Pfl]). In [Sic] J. Siciak formulated the following problem: Which balanced domains $G \subset \mathbb{C}^n$ are $L^2_{\rm h}(G)$-domains of holomorphy? Here "balanced" means that $\overline{E} \cdot G = G$.

The aim of this note is to give a partial solution of this problem (using a deep result due to T. Ohsawa and K. Takegoshi [Ohs-Tak]).

Theorem. *Any bounded balanced domain of holomorphy $G \subset \mathbb{C}^n$ is an $L^2_{\rm h}(G)$-domain of holomorphy. Consequently, the $L^2_{\rm h}(G)$-envelope of holomorphy of any bounded balanced domain $G \subset \mathbb{C}^n$ coincides with the standard envelope of holomorphy.*

Proof. We proceed by induction on n. Obviously, the result is true for $n = 1$. Assume that the assertion holds for $n - 1$.

Let $G \subset \mathbb{C}^n$ be a bounded balanced domain of holomorphy. Suppose that G is not an $L^2_{\rm h}(G)$-domain of holomorphy. Since G is balanced, we may assume that there are domains $\emptyset \neq V \subset U \cap G \subsetneq U$ such that:

(a) for any $f \in L^2_{\rm h}(G)$ there exists an $\widetilde{f} \in \mathcal{O}(U)$ with $\widetilde{f}|_V = f|_V$, where $\mathcal{O}(U)$ denotes the space of all holomorphic functions on U;

(b) there exists a continuous curve $\gamma : [0,1] \to U$ with $\gamma(0) \in V$, $\gamma(1) =: b \in \partial G$, $\gamma([0,1)) \subset G$, and $[0,1)b \subset G$;

(c) $b = (b_1, 0, \ldots, 0)$.

1991 *Mathematics Subject Classification*: 32D05, 32D10.
Key words and phrases: balanced domain of holomorphy, L^2-holomorphic function.

To see that (b) is always possible we can proceed as follows. Let U, V be as in (a) and let $\gamma : [0,1] \to U$ be an arbitrary curve with $\gamma(0) \in V$, $\gamma(1) =: b \in \partial G$, and $\gamma([0,1)) \subset G$. Observe that there is $\tau_0 \in (0,1]$ such that $[0,\tau_0)b \subset G$ and $\tau_0 b \in \partial G$. It suffices to show that we can replace U, V, γ by $\tau_0 U$, $\tau_0 V$, $\tau_0 \gamma$. Take an arbitrary function $f \in L^2_{\rm h}(G)$ and put $f_{\tau_0}(z) := f(\tau_0 z)$, $z \in G$. Then $f_{\tau_0} \in L^2_{\rm h}(G)$ and therefore there exists $\widetilde{f}_{\tau_0} \in \mathcal{O}(U)$ with $\widetilde{f}_{\tau_0}|_V = f_{\tau_0}|_V$. Define $\widetilde{f}(z) := \widetilde{f}_{\tau_0}(z/\tau_0)$, $z \in \tau_0 U$. Then $\widetilde{f} \in \mathcal{O}(\tau_0 U)$ with $\widetilde{f}|_{\tau_0 V} = f|_{\tau_0 V}$.

We choose $\tau \in (0,1)$ and a domain W with $\tau\gamma([0,1]) \subset W \subset G \cap U$, $\tau\gamma(0) \in V$, and $[\tau,1]b \subset U$. Take $(n-1)$-dimensional domains $\widetilde{U}$ and $\widetilde{V}$ and an $\varepsilon > 0$ such that $[\tau,1]b \subset \widetilde{U} \times (\varepsilon E) \subset U$ and $\tau b \in \widetilde{V} \times \{0\} \subset (\widetilde{U} \times \{0\}) \cap W$.

We define $\widetilde{G} := G \cap (\mathbb{C}^{n-1} \times \{0\})$ and consider $\widetilde{G}$ as a bounded balanced domain of holomorphy in $\mathbb{C}^{n-1}$.

Let $f \in L^2_{\rm h}(\widetilde{G})$. Using [Ohs-Tak] we find a function $F \in L^2_{\rm h}(G)$ with $F|_{\widetilde{G}} = f$. By our construction there exists $\widetilde{F} \in \mathcal{O}(U)$ such that $\widetilde{F}|_V = F|_V$. Then the identity theorem implies that $\widetilde{F}|_W = F|_W$; in particular, $\widetilde{F}(\cdot,0)|_{\widetilde{V}} = f|_{\widetilde{V}}$. Thus $f|_{\widetilde{V}}$ extends as a holomorphic function to $\widetilde{U}$ with $\widetilde{U} \not\subset \widetilde{G}$, which contradicts our assumption for the $(n-1)$-dimensional case. ∎

Remark. We do not know any characterization of unbounded balanced $L^2_{\rm h}$-domains of holomorphy. Even more, there is no description of unbounded balanced domains G with $L^2_{\rm h}(G) \neq \{0\}$. For Reinhardt domains see [Jar-Pfl].

References

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Reçu par la Rédaction le 1.6.1995