ANNALES
POLONICI MATHEMATICI
LXIII.1 (1996)

On the versal discriminant of $J_{k,0}$ singularities

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Abstract. It is well known that the versal deformations of nonsimple singularities depend on moduli. The first step in deeper understanding of this phenomenon is to determine the versal discriminant, which roughly speaking is an obstacle to analytic triviality of an unfolding or deformation along the moduli. The versal discriminant of the Pham singularity ($J_{3,0}$ in Arnold's classification) was thoroughly investigated by J. Damon and A. Galligo [2], [3], [4]. The goal of this paper is to continue their work and to describe the versal discriminant of a general $J_{k,0}$ singularity.

1. Introduction. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with an isolated critical point at the origin. Let

$$F : U \to \mathbb{C}, \quad (0,0) \in U \subset \mathbb{C}^n \times \Lambda,$$

be an analytic deformation of f ($f(x) = F(x,0)_0$), which is miniversal for right equivalence. Obviously $F_\lambda(x) = F(x,\lambda)$ is versal for V-equivalence. Furthermore, if λ_1 is a free term, i.e. $F(x,\lambda) = F'(x,\lambda') + \lambda_1$, where $\lambda = (\lambda_1, \lambda') \in \Lambda = \mathbb{C} \times \Lambda'$, and the domain U splits as $U = \mathbb{C} \times U'$ with $U' \subset \mathbb{C}^n \times \Lambda'$ then the unfolding

$$\mathcal{F} : (U', (0,0)) \to (\Lambda, 0), \quad \mathcal{F}(x, \lambda') = (-F', \lambda'),$$

is right-left stable.

Let T be the moduli set, i.e. the subset of Λ consisting of λ such that $F_\lambda(x)$ has a critical point p of multiplicity $\mu = \mu(f)$ and $F_\lambda(p) = 0$.

Let $\pi : (\Lambda, 0) \to (T, 0)$ be an analytic projection (transversal to T). The *versal discriminant* V of the deformation F_λ (resp. of the unfolding $\mathcal{F}$) relative to the projection π is the subset of the fibre $\pi^{-1}(0)$ consisting of parameters λ such that the deformation $F_\lambda, \lambda \in \pi^{-1}(0)$, is not infinitesimally V-versal at λ (equivalently, the unfolding $\mathcal{F}$ restricted to the set $\mathcal{F}^{-1}(\pi^{-1}(0))$ is not right-left infinitesimally stable at λ).

1991 *Mathematics Subject Classification*: 32S30, 14B07.
Key words and phrases: versal deformation, versal unfolding, moduli, liftable vector fields, local algebra, topological triviality.

We remark that the versal discriminant has the following property. There exists an integer k such that if a nonsingular analytic subvariety $\Lambda_1 \subset \Lambda$, transversal to the moduli space T, is tangent to the fibre $\pi^{-1}(0)$ at V up to order k then the deformation $F_\lambda,\ \lambda \in \Lambda_1$, is V-equivalent to $F_\lambda,\ \lambda \in \pi^{-1}(0)$, in some neighbourhood of the origin. A similar fact is valid for unfoldings.

Now let the germ f be quasihomogeneous (weighted homogeneous). Then there exists a distinguished class of projections induced by quasihomogeneity. Indeed, let v be an associated quasihomogeneous weight. We consider the *quasihomogeneous* miniversal deformation:

$$F_\lambda = f + \sum_{i=1}^{\mu} \lambda_i e_i,$$

where $e_1, \ldots, e_\mu$ is a quasihomogeneous base of the local algebra $\mathcal{O}_n/I_f$ and I_f is the ideal spanned by the partial derivatives $\partial f/\partial x_i$, $i = 1, \ldots, n$ (compare [1], §8). We say that the parameter λ_i is *underdiagonal*, *overdiagonal*, or *diagonal* if the weight of e_i is less than, greater than or equal to the weight of f respectively. In this case the moduli set T is a linear subspace of the base $\Lambda = \mathbb{C}^\mu$ spanned by overdiagonal and diagonal λ's. Moreover, there is a canonical projection π onto T—"forgetting" the underdiagonal λ's. For that projection the restriction F_λ, $\lambda \in \pi^{-1}(0)$, is a part of the deformation consisting of underdiagonal terms, the so-called underdiagonal deformation (also called the deformation of negative weight). Since quasihomogeneous germs are germs of polynomials, F_λ is defined globally and we may put for example $U = \mathbb{C}^n \times \mathbb{C}^\mu$. However, since the versal discriminant depends on the choice of the domain where the deformation is defined, we restrict ourselves to domains U such that:

- U splits: $U = \{(x, \lambda_1, \lambda') : \lambda_1 \in \mathbb{C},\ (x, \lambda') \in U'\}$,
- U is a neighbourhood of $\mathbb{C}^n \times \pi^{-1}(0)$,
- the factor algebra of the analytic functions on U modulo the ideal generated by the partial derivatives $\partial F_\lambda/\partial x_i$, $i = 1, \ldots, n$, is a free module over the ring of analytic functions on $\Pi(U)$ (the projection of U on Λ) generated by the polynomials $e_1, \ldots, e_\mu$ introduced above, i.e.

$$\mathcal{O}(U)/I_{F_\lambda} = \mathcal{O}(\Pi(U)) \otimes_{\mathbb{C}} \mathcal{O}_n/I_f.$$

Our goal is to describe the versal discriminant, relative to the canonical projection onto the moduli set, of quasihomogeneous miniversal deformations of $J_{k,0}$ singularities, restricted to the domain U as above.

The author would like to mention here that he has been recently informed that A. Du Plessis and C. T. C. Wall are dealing with similar problems in their forthcoming book.

Acknowledgements. I would like to express my heart-felt gratitude to Professor James Damon, whose kind hospitality during my stay at the University of North Carolina enabled me to work on this subject.

2. The main results. In this paper we consider the quasihomogeneous analytic functions

$$f(x,y) = (y+\alpha x^k)^3 + \beta(y+\alpha x^k)x^{2k} + \gamma x^{3k}, \qquad k = 2,3,\ldots,$$

where $4\beta^3 + 27\gamma^2 \neq 0$. They have an isolated singular point at the origin. In Arnold's classification (see [1], §15) such singularities are called $J_{k,0}$. We remark that they are classified by the j-invariant:

$$j = \frac{4\beta^3}{4\beta^3 + 27\gamma^2},$$

the same which classifies the elliptic curves

$$z^2 = (y+\alpha x)^3 + \beta(y+\alpha x)x^2 + \gamma x^3$$

(compare [5], §IV.4), therefore they can be described by normal forms which depend on one modulus only. The most commonly used are the following:

$$\begin{aligned} &y^3 + 3\alpha y^2 x^k + x^{3k}, && 4\alpha^3 + 1 \neq 0, && \text{for all } j,\\ &y^3 + \beta y x^{2k} + x^{3k}, && 4\beta^3 + 27 \neq 0, && \text{for } j \neq 1,\\ &y^3 + y x^{2k} + \gamma x^{3k}, && 4 + 27\gamma^2 \neq 0, && \text{for } j \neq 0. \end{aligned}$$

Our aim is to describe the versal discriminant of these singularities relative to the projection π onto overdiagonals and diagonals:

$$\pi(\lambda_{\text{underdiagonal}}, \lambda_{\text{diagonal}}, \lambda_{\text{overdiagonal}}) = (\lambda_{\text{diagonal}}, \lambda_{\text{overdiagonal}}).$$

This shows that the versal discriminants do not depend on the choice of the particular quasihomogeneous deformation.

THEOREM 1. *The versal discriminant consists of parameters λ such that, after a substitution $y = y - a(x)$, F_λ is one of the following polynomials*:

$$\begin{aligned} &\text{A:} && y^3 + ed(x)^2 y + d(x)^3,\\ &\text{B:} && y^3 + b(x)y,\\ &\text{C:} && y^3 + c(x), \end{aligned}$$

where $27+4e^3 \neq 0$, $b(x), c(x), d(x)$ are polynomials of degree respectively $2k$, $3k$, k with no more than $k-1$ different roots, and $a(x)$ is any polynomial of degree k or less.

Remark. The cases A, B, C of the above theorem occur for $J_{k,0}$ quasihomogeneous singularities with j-invariant equal respectively to $27/(27+4e^3)$, 1, 0.

In [6] we investigate the above polynomials in more detail and determine the singular points of the associated curves.

3. Liftable vector fields. Let $F(x,\lambda), (x,\lambda) \in U \subset \mathbb{C}^n \times \Lambda$, be a right miniversal deformation (of $f(x) = F(x,0)_0$). We assume that λ_1 is the free term, i.e.

$$F(x,\lambda) = F'(x,\lambda') + \lambda_1, \qquad \lambda = (\lambda_1, \lambda') \in \mathbb{C} \times \Lambda'.$$

Furthermore, we assume that $U = \mathbb{C} \times U'$, where $U' \subset \mathbb{C}^n \times \Lambda'$. Then the associated unfolding is given by the formula

$$\mathcal{F}(x,\lambda') = (-F'(x,\lambda'), \lambda'), \qquad (x,\lambda') \in U'.$$

We choose the minus sign to simplify formulas.

Let Π be the canonical projection $\mathbb{C}^n \times \Lambda \to \Lambda$. We say that an analytic vector field η defined on $\Pi(U) \subset \Lambda$ is respectively *V-liftable* or *A-liftable* if there exists respectively an analytic vector field ξ_v on U such that

$$\xi_v(g \circ \Pi) = \eta(g) \circ \Pi, \qquad \xi_v(F) = HF,$$

or an analytic vector field ξ_a on U' such that

$$\xi_a(g \circ \mathcal{F}) = \eta(g) \circ \mathcal{F},$$

where g is any analytic function on $\Pi(U)$ and H is an analytic function on U.

In coordinates these conditions may be stated as follows: The vector field

$$\eta = \sum_{i=1}^{\mu} \eta_i(\lambda) \frac{\partial}{\partial \lambda_i}$$

is V-liftable if there exists a vector field

$$\xi_v = \sum_{i=1}^{n} a_i(x,\lambda) \frac{\partial}{\partial x_i} + \sum_{i=1}^{\mu} \eta_i(\lambda) \frac{\partial}{\partial \lambda_i},$$

such that

$$\sum_{i=1}^{n} a_i(x,\lambda) \frac{\partial F}{\partial x_i} + \sum_{i=1}^{\mu} \eta_i(\lambda) \frac{\partial F}{\partial \lambda_i} = H(x,\lambda) F(x,\lambda).$$

Similarly, η is A-liftable if there exists a vector field

$$\xi_a = \sum_{i=1}^{n} b_i(x,\lambda') \frac{\partial}{\partial x_i} + \sum_{i=2}^{\mu} \eta_i(-F',\lambda') \frac{\partial}{\partial \lambda_i},$$

such that

$$\sum_{i=1}^{n} b_i(x,\lambda') \frac{\partial F'}{\partial x_i} + \sum_{i=2}^{\mu} \eta_i(-F',\lambda') \frac{\partial F'}{\partial \lambda_i} + \eta_1(-F',\lambda') = 0.$$

LEMMA 1. *The vector field η is V-liftable if and only if it is A-liftable.*

Proof. $\Leftarrow$. We substitute $F' = F - \lambda_1$ and expand η_i's:

$$\eta_i(\lambda_1 - F, \lambda') = \eta_i(\lambda_1, \lambda') + H_i F.$$

Note that $\partial F/\partial\lambda_1 = 1$. Then we put $a_i(x, \lambda_1, \lambda') = b_i(x, \lambda')$.

$\Rightarrow$. We substitute $\lambda_1 = -F'$ (i.e. $F = 0$).

We shall denote the module of liftable vector fields by $\mathcal{M}$ and by $\mathcal{M}(\lambda)$ the linear space of their values at the point λ.

We recall the following:

A point $\lambda \in \pi^{-1}(0)$ does not belong to the versal discriminant relative to the projection π if and only if the module $\mathcal{M}$ of liftable vector fields is transversal to the fibre $\pi^{-1}(0)$ at λ:

$$\mathcal{M}(\lambda) \oplus T_\lambda(\pi^{-1}(0)) = T_\lambda \Lambda.$$

The property mentioned in the introduction may be proved by integration of liftable vector fields near the origin (compare with [3], §2; see also the local description of V-versality in [1], §8, and of right-left stability in [1], §6).

4. Multiplications in the local algebra. In this section we shall consider the local algebra of the germ

$$f(x, y) = y^3 + \beta y x^{2k} + \gamma x^{3k}, \qquad k = 2, 3, \ldots,$$

where $4\beta^3 + 27\gamma^2 \neq 0$. The above germ is quasihomogeneous, the associated weight is

$$v(x) = 1, \qquad v(y) = k.$$

Let $e_1, \ldots, e_\mu$, $\mu = 6k - 2$, be the basis of the local algebra $\Omega = \mathcal{O}_2/I_f$:

$$\mathcal{O}_2 = \mathcal{O}_2\{\partial f/\partial x, \partial f/\partial y\} \oplus \mathbb{C}\{e_1, \ldots, e_\mu\},$$

consisting of quasihomogeneous polynomials ordered by their weights

$$v(e_i) \le v(e_{i+1}).$$

For example, if $\gamma \neq 0$ then we may choose

$$\begin{aligned} e_i &= x^{i-1} && \text{for } i = 1, \ldots, k, \\ e_{k+2i-1} &= x^{k+i-1} && \text{for } i = 1, \ldots, 2k-1, \\ e_{k+2i} &= y x^{i-1} && \text{for } i = 1, \ldots, 2k-1, \\ e_{5k+i-2} &= y x^{2k+i-2} && \text{for } i = 1, \ldots, k. \end{aligned}$$

We remark that the number of base elements of a given weight does not depend on the choice of the base (see [1], §12.2). Hence we have

$$\begin{aligned} v(e_i) &= i-1 && \text{for } i=1,\ldots,k,\\ v(e_{k+2i-1}) &= v(e_{k+2i}) = k+i-1 && \text{for } i=1,\ldots,2k-1,\\ v(e_{5k+i-2}) &= 3k+i-2 && \text{for } i=1,\ldots,k. \end{aligned}$$

Multiplication by x in the local algebra

$$\mathcal{O}_2/I_f \approx \mathbb{C}\{e_1,\ldots,e_\mu\}$$

is "shift-like". The product $x\cdot e_i$, $i<\mu$, is a linear combination of base elements of weight one greater than the weight of e_i. The kernel of this multiplication is two-dimensional; it is spanned by e_μ and a combination of e_{5k-3} and e_{5k-2}.

Now let F be a quasihomogeneous deformation of f:

$$F(x,y,\lambda) = f(x,y) + \sum_{i=1}^{\mu} \lambda_i e_i.$$

Let π be the projection onto overdiagonals and diagonals:

$$\pi(\lambda_1,\ldots,\lambda_\mu) = (\lambda_{5k},\ldots,\lambda_\mu).$$

The condition from the previous section may be restated in the following way:

Proposition 1. *A point $\lambda \in \pi^{-1}(0)$ does not belong to the versal discriminant relative to the projection π if and only if the polynomials $F_\lambda, xF_\lambda, \ldots, x^m F_\lambda, \ldots, yF_\lambda, xyF_\lambda, \ldots, x^m yF_\lambda, \ldots$ and the underdiagonal base elements generate the whole algebra $\Omega_\lambda = \mathbb{C}[x,y]/I_{F_\lambda}$.*

Proof. The tangent space to the fibre of the projection is spanned by the underdiagonal vectors $\partial/\partial\lambda_1,\ldots,\partial/\partial\lambda_{5k-1}$; hence λ does not belong to the versal discriminant if

$$\mathcal{M}(\lambda) \oplus \mathbb{C}(\partial/\partial\lambda_1,\ldots,\partial/\partial\lambda_{5k-1}) = \mathbb{C}(\partial/\partial\lambda_1,\ldots,\partial/\partial\lambda_\mu).$$

Since the mapping $\eta \to \eta(F_\lambda)$ is a $\mathbf{C}$-linear isomorphism of the tangent space $T_\lambda\Lambda$ onto the algebra Ω_λ, the above condition is equivalent to

$$\mathbb{C}\Big(\sum_{i=1}^{\mu} \eta_i(\lambda)e_i : \eta \in \mathcal{M}_\lambda\Big) \oplus \mathbb{C}(e_1,\ldots,e_{5k-1}) = \Omega_\lambda.$$

We remark that we have chosen the domain U on which $F(x,\lambda)$ is defined in such a way that the analytic factor algebra $\mathcal{O}(U)/I_{F_\lambda}$ is a free $\mathcal{O}(\Pi(U))$-module generated by $e_i = \partial F/\partial\lambda_i$; hence, if the vector field

$$\eta = \sum_{i=1}^{\mu} \eta_i(\lambda)\frac{\partial}{\partial\lambda_i}$$

is liftable in λ, then in the factor algebra Ω_λ,

$$\sum_{i=1}^{\mu} \eta_i(\lambda) e_i = H(x,y) F_\lambda(x,y),$$

where H is a polynomial in x and y. Since $f_y = 3y^2 + \beta x^{2k}$, we may assume that $H(x,y) = H_0(x) + H_1(x)y$. Therefore the condition from the previous section is equivalent to that from the proposition.

Proposition 1 reduces our problem to the investigation of the orbits of F and yF in the algebra $\Omega_\lambda = \mathbb{C}[x,y]/I_F$ under iterated multiplication by x. We remark that the polynomials e_i form a basis of the algebra Ω_λ. Moreover, multiplication in this algebra is a deformation of multiplication in the local algebra Ω and is also shift-like: $x \cdot e_i$, $i < \mu$, is a linear combination of base elements of weight not greater than $v(e_i) + 1$.

5. The proof of Theorem 1. First we consider the following particular case:

$$F_\lambda = y^3 + b(x)y + c(x),$$

where b and c are polynomials of degree not greater than $2k$ and $3k$ respectively:

$$b(x) = \sum_{i=0}^{2k} b_i(\lambda) x^i, \qquad c(x) = \sum_{i=0}^{3k} c_i(\lambda) x^i,$$

which satisfy the nondegeneracy condition

$$27 c_{3k}^2 + 4 b_{2k}^3 \neq 0.$$

We start with the following observation.

PROPOSITION 2. *If λ belongs to the versal discriminant then*

$$2b(x)c'(x) - 3b'(x)c(x) \equiv 0.$$

Proof. For $F = y^3 + b(x)y + c(x)$ we have

$$F_y = \frac{\partial F}{\partial y} = 3y^2 + b(x), \qquad F_x = \frac{\partial F}{\partial x} = b'(x)y + c'(x).$$

We put

$$\widetilde{F} = 3F - yF_y = 2b(x)y + 3c(x).$$

We investigate the orbit of $\widetilde{F}$ in Ω_λ under multiplication by x. Since $k\widetilde{F} - xF_x$ is underdiagonal and multiplication by x is shift-like, it follows that if j is the lowest power for which $x^j\widetilde{F}$ is not underdiagonal mod F_x, then it is diagonal. Thus, successive $x^{j+r}\widetilde{F}$ give the overdiagonal terms. Therefore, if λ is not in the versal discriminant, then all $x^j\widetilde{F}$ are underdiagonal mod F_x.

The space of underdiagonal polynomials being finite-dimensional, there exist two nonzero polynomials $w_1(x)$ and $w_2(x)$ such that

$$w_1(x)\widetilde{F} + w_2(x)F_x = 0.$$

Thus we have a system of two equations:

$$2bw_1 + b'w_2 = 0, \qquad 3cw_1 + c'w_2 = 0.$$

The polynomial $R = 2bc' - 3b'c$ is the determinant of this system. Hence nontrivial solutions may exist only when $R \equiv 0$.

Now we investigate in more detail the condition $R \equiv 0$.

LEMMA 2. *The condition $R = 2bc' - 3b'c \equiv 0$ is satisfied only in three cases*:

$$\begin{aligned} &\text{A:} \quad y^3 + ed(x)^2y + d(x)^3, \\ &\text{B:} \quad y^3 + b(x)y, \\ &\text{C:} \quad y^3 + c(x), \end{aligned}$$

where $d(x), b(x)$ and $c(x)$ are polynomials of degree k, $2k$ and $3k$ respectively, and e is a constant. Moreover, the j-invariant of the leading part equals $27/(27+4e^3)$, 1 and 0 respectively.

Proof. Obviously $R \equiv 0$ if $c \equiv 0$ (case B) or $b \equiv 0$ (case C). Otherwise we have

$$2\frac{c'}{c} = 3\frac{b'}{b}.$$

Therefore for some constant C,

$$Cc(x)^2 = b(x)^3.$$

Hence b is a square and c is a cube. We put

$$c(x) = d(x)^3, \qquad b(x) = ed(x)^2.$$

Next we investigate the orbit of $y\widetilde{F}$ under multiplication by x and get sufficient conditions.

LEMMA 3. *The point λ belongs to the versal discriminant if and only if F_λ is of type* A, B *or* C *and respectively the polynomial $d(x)$, $b(x)$ or $c(x)$ has at most $k-1$ different roots.*

Proof. Case A. In this case we have

$$F = y^3 + ed(x)^2y + d(x)^3,$$
$$F_y = 3y^2 + ed(x)^2, \qquad F_x = (2ey + 3d(x))d(x)d'(x),$$
$$\widetilde{F} = (2ey + 3d(x))d(x)^2.$$

Let $d_1(x)$ be the greatest common divisor of d and d'. We put $d' = d_0 d_1$ and $d = d_2 d_1$. Then

$$d_0 \widetilde{F} = (2ey + 3d(x))d(x)^2 d_0(x) = (2ey + 3d(x))d(x)d'(x)d_2(x) = d_2 F_x.$$

Hence if the degree of $d_0(x)$ is smaller than $k-1$ (i.e. d has multiple roots) then the polynomials $\widetilde{F}, x\widetilde{F}, \ldots$ are underdiagonal modF_x and the polynomials $y\widetilde{F}, xy\widetilde{F}, \ldots$ and the underdiagonal base elements do not generate the whole algebra Ω_λ. Indeed, the dimension of the subspace of Ω_λ spanned by $y\widetilde{F}, xy\widetilde{F}, \ldots$ is smaller than the number of diagonal and overdiagonal base elements.

Otherwise we put

$$\begin{aligned} \widehat{F} &= r_1(x)\widetilde{F} + r_2 F_x = (2ey + 3d(x))d(x)d_1(x), \\ G &= 3y\widehat{F} - 2ed(x)d_1(x)F_y = (9y - 2e^2 d(x))d(x)^2 d_1(x), \end{aligned}$$

where r_1 and r_2 are polynomials such that

$$d_1(x) = r_1(x)d(x) + r_2(x)d'(x).$$

By the nondegeneracy condition the polynomials $2ey+3d(x)$ and $9y-2e^2d(x)$ are linearly independent. Furthermore, the maximal weight of a monomial in G is $3k$. Hence the polynomials $G, xG, \ldots$ and the underdiagonal base elements generate the whole algebra Ω_λ.

Therefore, in this case, λ belongs to the versal discriminant if and only if the polynomial $d(x)$ has at least one multiple root, or equivalently at most $k-1$ different roots.

Case B. In this case we have

$$F = y^3 + b(x)y, \quad F_y = 3y^2 + b(x), \quad F_x = b'(x)y, \quad \widetilde{F} = 2b(x)y.$$

Let $b_1(x)$ be the greatest common divisor of b and b'. We put $b' = b_0 b_1$ and $b = b_2 b_1$. Then

$$b_0 \widetilde{F} = 2b(x)b_0(x)y = 2b_2(x)b'(x) = 2b_2 F_x.$$

Hence if the degree of $b_0(x)$ is smaller than $k-1$ (i.e. b has less than k different roots) then the polynomials $y\widetilde{F}, xy\widetilde{F}, \ldots$ and the underdiagonal base elements do not generate the whole algebra Ω_λ.

Otherwise we put

$$\begin{aligned} \widehat{F} &= r_1(x)\widetilde{F} + 2r_2 F_x = 2b_1(x)y, \\ G &= 3y\widehat{F} - 2b_1(x)F_y = -2b(x)b_1(x), \end{aligned}$$

where

$$b_1(x) = r_1(x)b(x) + r_2(x)b'(x).$$

The maximal weight of a monomial in G is $2k + \deg b_1$. Hence if $\deg b_1$ is equal to k or less then the polynomials $G, xG, \ldots$ and the underdiagonal base elements generate the whole algebra Ω_λ.

Therefore, in this case, λ belongs to the versal discriminant if and only if the polynomial $b(x)$ has at most $k-1$ different roots.

Case C. In this case we have

$$F = y^3 + c(x), \quad F_y = 3y^2, \quad F_x = c'(x), \quad \widetilde{F} = 3c(x).$$

Let $c_1(x)$ be the greatest common divisor of c and c'. We put $c' = c_0 c_1$ and $c = c_2 c_1$. Then

$$c_0 \widetilde{F} = 3c(x)^2 c_0(x) y = 3c_2(x) c'(x) = 3c_2 F_x.$$

Hence if the degree of $c_0(x)$ is smaller than $k-1$ (i.e. c has less than k different roots) then the polynomials $y\widetilde{F}, xy\widetilde{F}, \ldots$ and the underdiagonal base elements do not generate the whole algebra Ω_λ.

Otherwise we put

$$\widehat{F} = r_1(x)\widetilde{F} + 3r_2(x)F_x = 3c_1(x), \qquad G = y\widehat{F} = 3c_1(x)y,$$

where

$$c_1(x) = r_1(x)c(x) + r_2(x)c'(x).$$

The maximal weight of a monomial in G is $k + \deg c_1$. Hence if $\deg c_1$ is equal to $2k$ or less then the polynomials $G, xG, \ldots$ and the underdiagonal base elements generate the whole algebra Ω_λ.

Therefore, in this case, λ belongs to the versal discriminant if and only if the polynomial $c(x)$ has at most $k-1$ different roots.

To prove the theorem it is enough to notice that Proposition 2 and Lemmas 2 and 3 remain valid if we replace y by $y - a(x)$, where $a(x)$ is any polynomial of degree k or less. As a matter of fact such transformation induces an isomorphism of unfoldings and deformations which preserves the versal discriminant.

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Reçu par la Rédaction le 3.3.1995