Extremal plurisubharmonic functions

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Abstract. We study different notions of extremal plurisubharmonic functions.

1. Introduction. There are several different notions of extremal plurisubharmonic functions and the purpose of this note is to study some of them.

Definitions. Let Ω be an open and connected subset of $\mathbb{C}^n$, $n \geq 1$, and denote by $\mathrm{PSH}(\Omega)$ and $\mathrm{PH}(\Omega)$ the plurisubharmonic and pluriharmonic functions, respectively.

Extremal (A). Let K be a subset of Ω. The *relative extremal plurisubharmonic function* h_K^* is defined to be the upper regularization of

$$h_K(z) = \sup\{\varphi(z) \in \mathrm{PSH}(\Omega) : -1 \leq \varphi \leq 0, \ \varphi = -1 \text{ on } K\}$$

(cf. Siciak [8]).

Extremal (B$_1$). The class of functions $P(\Omega) = \{\varphi \in \mathrm{PSH}(\Omega) : -1 \leq \varphi \leq 0\}$ is a convex and compact space with locally convex topology inherited from $L^1(\Omega)$. An element $\varphi \in P(\Omega)$ is said to be *extremal* (B$_1$) if

$$\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2, \quad \varphi_1, \varphi_2 \in P(\Omega), \ 0 < \alpha < 1, \quad \text{implies } \varphi_1 = \varphi_2$$

(cf. Choquet [3, Vol. II, p. 95]).

Extremal (B$_2$). Define $\mathrm{PSH}^-(\Omega) = \{\varphi \in \mathrm{PSH}(\Omega) : \varphi \leq 0\}$. An element $\varphi \in \mathrm{PSH}^-(\Omega)$ is said to be *extremal* (B$_2$) if

$$\varphi = \varphi_1 + \varphi_2, \quad \varphi_1, \varphi_2 \in \mathrm{PSH}^-(\Omega),$$

implies that there exist non-negative constants λ_1 and λ_2 such that $\varphi_1 = \lambda_1\varphi$ and $\varphi_2 = \lambda_2\varphi$.

Extremal (C). The quotient space $\mathrm{PSH}(\Omega)/\mathrm{PH}(\Omega)$ is Hausdorff and a convex cone. An element $\varphi \in \mathrm{PSH}(\Omega)/\mathrm{PH}(\Omega)$ is called *extremal* (C) if

$$\varphi = \varphi_1 + \varphi_2, \qquad \varphi_1, \varphi_2 \in \mathrm{PSH}(\Omega)/\mathrm{PH}(\Omega),$$

implies that there exist non-negative constants λ_1 and λ_2 such that $\varphi_1 = \lambda_1 \varphi$ and $\varphi_2 = \lambda_2 \varphi$ (in $\mathrm{PSH}(\Omega)/\mathrm{PH}(\Omega)$) (cf. Lelong [5] and Demailly [4]).

2. The case $n = 1$. Let here $\Omega = D$ be the open unit disc in the complex plane and $\varphi \in \mathrm{SH}(D)$ with a harmonic majorant. Then, by the Riesz representation formula,

$$\varphi(z) = \int \, \log \left| \frac{z - \xi}{1 - z\bar{\xi}} \right| \Delta\varphi(\xi) + \int \, P(z, \xi) \, d\mu(\xi),$$

where P is the Poisson kernel. So that if $\varphi \leq 0$ and φ is extremal (B$_2$) then either $\Delta\varphi \equiv 0$ or $\mu = 0$. Therefore $\Delta\varphi = k\delta_{z_0}$ for some $z_0 \in \Omega$ so

$$\varphi(z) = k \log \left| \frac{z - z_0}{1 - z\bar{z}_0} \right|$$

or $\varphi(z) = kP(z, \xi_0)$ for some $\xi_0 \in \partial\Omega$, with k a constant.

Also, if $\varphi \in \mathrm{SH}(D)$ is extremal (C) then $\varphi(z) = k \log |z - z_0|$ modulo a harmonic function. Therefore, there are no functions bounded below and extremal (C).

3. Relations

LEMMA 1. *If* $0, -1 \not\equiv \varphi \in P(\Omega)$ *is extremal* (B$_1$), *then* $\inf_{z \in \Omega} = -1$ *and* $\sup_{z \in \Omega} \varphi(z) = 0$.

Proof. If $\beta = \inf_{z \in \Omega} \varphi(z) > -1$, then $\varphi(z)/\gamma \in P(\Omega)$, where $\gamma = \max(1/2, -\beta) < 1$. Choose k so that $1/\gamma + k = 2$. Since $1 < 1/\gamma \leq 2$ it follows that $0 \leq k \leq 1$ so $k\varphi(z) \in P(\Omega)$ and

$$\varphi(z) = \frac{1}{2} \left(\frac{\varphi(z)}{\gamma} + k\varphi(z) \right)$$

so φ is not extremal (B$_1$). If $\sup_{z \in \Omega} \varphi(z) = \delta < 0$ then

$$\varphi(z) = \varphi(z) + \delta - \delta = \left(\frac{\varphi(z) + \delta}{1 - \delta} \right)(1 - \delta) + \delta \cdot (-1)$$

so φ is not extremal.

PROPOSITION 1. *If* K *is a relatively compact subset of* Ω *then* h_K^* *is extremal* (B$_1$).

Proof. If K is pluripolar, then $h_K^* \equiv 0$; we can assume that K is not pluripolar. If $h_K^* = \alpha u_1 + (1 - \alpha)u_2$, where $u_1, u_2 \in P(\Omega)$, then $u_1 = u_2 = -1$ on $\{z \in \Omega : h_K^* = -1\}$ so it follows from the definition of h_K (since

$h_K = h_K^*$ outside a pluripolar set) that $u_1 \leq h_K^*$ and $u_2 \leq h_K^*$. Therefore $u_1 = u_2 = h_K^*$.

COROLLARY. *There exist discontinuous functions extremal* (B$_1$).

PROPOSITION 2. *If* $\varphi \in \text{PSH}^-(\Omega)$, $\lim_{z \to \xi} \varphi(z) = 0$ *for all* $\xi \in \partial\Omega$ *and if* φ *is extremal* (C), *then* φ *is extremal* (B$_2$). *If* $\varphi \in P(\Omega)$ *with* $\varphi(\xi_0) = -1$ *for some* $\xi_0 \in \Omega$ *and if* φ *is extremal* (B$_2$) *then* φ *is extremal* (B$_1$).

P r o o f. Suppose $\varphi \in \text{PSH}^-(\Omega)$ and that φ is extremal (C). If $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \text{PSH}^-(\Omega)$, then

$$\varphi_1 + h_1 = \lambda_1 \varphi, \quad \lambda_1 \geq 0, \, h_1 \in \text{PH},$$
$$\varphi_2 + h_2 = \lambda_2 \varphi, \quad \lambda_2 \geq 0, \, h_2 \in \text{PH}.$$

Since $\varphi = \varphi_1 + \varphi_2$ and $\varphi_1, \varphi_2 \leq 0$, also $\varphi_1(z), \varphi_2(z) \to 0$ as $z \to \xi$, $\xi \in \partial\Omega$. Hence $h_1(z), h_2(z) \to 0$ as $z \to \xi$, $\xi \in \partial\Omega$, since $h_1 = \lambda_1 \varphi - \varphi_1$ and $h_2 = \lambda_2 \varphi - \varphi_2$. Therefore, being harmonic, h_1 and h_2 vanish identically on Ω.

Let now $\varphi \in P(\Omega)$ be extremal (B$_2$) and assume $\varphi(\xi_0) = -1$ for some $\xi_0 \in \Omega$. If

$$\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2, \quad 0 < \alpha < 1, \, \varphi_1, \varphi_2 \in P(\Omega),$$

then

$$\alpha\varphi_1 = \lambda_1 \varphi, \quad \lambda_1 \geq 0,$$
$$(1-\alpha)\varphi_2 = \lambda_2 \varphi, \quad \lambda_2 \geq 0.$$

At ξ_0, we have $-1 = \varphi(\xi_0) = \varphi_1(\xi_0) = \varphi_2(\xi_0)$ so $\alpha = \lambda_1$ and $1 - \alpha = \lambda_2$, which proves that φ is extremal (B$_1$).

PROPOSITION 3. *Suppose* $0 \geq \varphi \in \text{PSH}(\Omega \times \Omega)$ *and that* φ *is separately extremal* (B$_2$). *Then* φ *is extremal* (B$_2$) *on* $\Omega \times \Omega$.

P r o o f. Suppose $\varphi = \varphi_1 + \varphi_2$, where $0 \geq \varphi_1, \varphi_2 \in \text{PSH}(\Omega \times \Omega)$. Then

$$\varphi_1(x, y) = \lambda_1(y)\varphi(x, y) = \beta_1(x)\varphi(x, y),$$
$$\varphi_2(x, y) = \lambda_2(y)\varphi(x, y) = \beta_2(x)\varphi(x, y),$$

where $\lambda_1, \lambda_2, \beta_1, \beta_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1 = \beta_1 + \beta_2$ since φ is separately extremal (B$_2$). Therefore $\lambda_1 \equiv \beta_1 \equiv \text{const}$ and $\lambda_2 \equiv \beta_2 \equiv \text{const}$, which proves that φ is extremal (B$_2$).

LEMMA 2. *Suppose* $0 \geq u \in \text{PSH}(G)$, *where* G *is a ball in* $\mathbb{C}^n$ *with center at* 0. *Then*

$$\lim_{r \searrow 0} \sup_{\substack{\xi \in \mathbb{C} \\ |\xi| = r}} \frac{u(\xi\nu)}{-\log r} = \text{const}$$

for all $\nu \in G$ *outside a pluripolar set.*

Proof. We may assume that $G = B$, the unit ball. Suppose $0 \geq u \in$ PSH(B). For $0 < r < 1$ and $\nu \in B$ we define

$$\Psi_r(\nu) = \sup_{\substack{\xi \in \mathbb{C} \\ |\xi| = r}} \frac{u(\xi\nu)}{-\log r} \leq 0.$$

Then $\Psi_r(\nu) \in$ PSH(B) and is decreasing in r. For let $0 < r_1 < r_2 < 1$. If $z \in \mathbb{C}$ with $|z| = r_1$, then $u(z\nu) \leq \Psi_{r_1}(\nu) \cdot (-\log|z|)$, which also holds for $|z| = 1$. Therefore $u(z\nu) \leq \Psi_{r_1}(\nu) \cdot (-\log|z|)$ for $r_1 \leq |z| \leq 1$. In particular,

$$\frac{u(z\nu)}{-\log|z|} \leq \Psi_{r_1}(\nu) \quad \text{for } |z| = r_2$$

and so $\Psi_{r_2}(\nu) \leq \Psi_{r_1}(\nu)$.

Hence $\Psi^*(\nu) = (\lim_{r \searrow 0} \Psi_r(\nu))^* \in$ PSH($\mathbb{C}^n$) and since Ψ^* is negative it is constant. Since $\Psi = \Psi^*$ outside a pluripolar set (cf. [1]), the lemma follows.

Remark 1. The constant $-\Psi^*$ can be shown to be equal to the Lelong number of u at zero. We do not need to use that in this paper.

4. Examples

EXAMPLE 1. Denote by D the unit disc in $\mathbb{C}$. Then

$$h^*_{\{|z_1|<e^{-1}\} \times \{|z_2|<e^{-1}\}}(z) = \max(\log|z_1|, \log|z_2|, -1)$$

so $\max(\log|z_1|, \log|z_2|, -1)$, $z = (z_1, z_2) \in D \times D$, is extremal (A) and thus by Proposition 1 also extremal (B_1).

EXAMPLE 2. Denote by B the unit ball in $\mathbb{C}^n$. Then

$$h^*_{\{z \in \mathbb{C}^n : |z| < e^{-1}\}}(z) = \max(\log|z|, -1)$$

is extremal (A) and therefore extremal (B_1).

EXAMPLE 3. If $\alpha > 0$ then $\varphi(z) = |z|^\alpha - 1$ is not extremal (B_1) on D. For let

$$\varphi_1(z) = \frac{4}{3}\left(|z|^\alpha - \frac{|z|^{2\alpha}}{4} - \frac{3}{4}\right), \quad \varphi_2(z) = \frac{2}{3}\left(|z|^\alpha + \frac{|z|^{2\alpha}}{2} - \frac{3}{2}\right).$$

Then $\varphi = (\varphi_1 + \varphi_2)/2$, and since $\Delta\varphi_1 = \alpha^2(|z|^{\alpha-2} - |z|^{2\alpha-2}) \geq 0$ for $0 \leq |z| < 1$, φ_1 is subharmonic on D and therefore $\varphi_1, \varphi_2 \in P(D)$.

EXAMPLE 4. $\psi(z_1, z_2) = \max(\log|z_1|, \log|z_2|)$ is not extremal (C). Since

$$\frac{1}{2\pi} \int_0^{2\pi} \log|\omega - e^{i\theta}| \, d\theta = \max(\log|\omega|, 0), \quad \omega \in \mathbb{C},$$

it follows that

$$\psi(z_1, z_2) = \max(\log|z_1|, \log|z_2|) = \frac{1}{2\pi} \int\limits_0^{2\pi} \log|z_1 + z_2 e^{i\theta}|\, d\theta.$$

Thus

$$\psi(z_1, z_2) = \frac{1}{2\pi} \int\limits_0^{\pi} \log|z_1 + z_2 e^{i\theta}|\, d\theta + \frac{1}{2\pi} \int\limits_\pi^{2\pi} \log|z_1 + z_2 e^{i\theta}|\, d\theta$$

so ψ is not extremal (C).

However, ψ is extremal (B_2) in $D \times D$. For suppose $\psi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \mathrm{PSH}^-(D \times D)$. If $z = (z_1, z_2)$, then $\psi(\lambda z)$, $\lambda \in \mathbb{C}$, is extremal as a function of λ, so

$$\varphi_1(\lambda z) = \alpha_2 \psi(\lambda z) = \alpha_2(\log|\lambda| + \psi(z)), \quad \alpha_2 \geq 0,$$
$$\varphi_2(\lambda z) = \beta_2 \psi(\lambda z) = \beta_2(\log|\lambda| + \psi(z)), \quad \beta_2 \geq 0.$$

By Lemma 2, $\alpha_2 = \mathrm{const} = \alpha$ a.e., $\beta_2 = \mathrm{const} = \beta$ a.e. Therefore $\varphi_1(z) = \alpha\varphi(z)$ and $\varphi_2(z) = \beta\varphi(z)$, which means that ψ is extremal (B_2).

R e m a r k 2. Example 4 answers a question by El Mir [6].

EXAMPLE 5 (Poletsky [7]). The function $\log|z|$ is extremal (B_2) in the unit ball of $\mathbb{C}^n$. This can be shown exactly as in Example 4, using Lemma 2.

5. The pluricomplex Green function. The functions $\max(\log|z_1|, \log|z_2|)$ and $\log|z|$ are pluricomplex Green functions with pole at zero for the domains $D \times D$ and B respectively. Example 4 shows that the pluricomplex Green function is not extremal (C) in the bidisc. Using a suitable holomorphic transformation, Examples 4 and 5 show that the pluricomplex Green function for the bidisc and the ball is extremal (B_2). This is a special example of the following theorem.

THEOREM. *Let Ω be a domain in $\mathbb{C}^n$ and let $G_\Omega(x, y)$ be the pluricomplex Green function for $\Omega \times \Omega$. Then $G_\Omega(x, y_0)$ is extremal (B_2) for every $y_0 \in \Omega$.*

P r o o f. We can assume that $y_0 = 0$. Suppose $G_\Omega = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \mathrm{PSH}^-(\Omega)$. Using Lemma 2, we find three constants

$$\alpha = \left(\lim_{r \searrow 0} \sup_{\substack{\xi \\ |\xi|=r}} \frac{G_\Omega(\xi\nu)}{-\log|\xi|} \right)^*,$$

$$\alpha_1 = \left(\lim_{r \searrow 0} \sup_{\substack{\xi \\ |\xi|=r}} \frac{\varphi_1(\xi\nu)}{-\log|\xi|} \right)^* \quad \text{and} \quad \alpha_2 = \left(\lim_{r \searrow 0} \sup_{\substack{\xi \\ |\xi|=r}} \frac{\varphi_2(\xi\nu)}{-\log|\xi|} \right)^*.$$

Since $G_\Omega(x, 0) - \log|x|$ is bounded near $x = 0$, we have $\alpha = -1$, $0 \geq \alpha_1, \alpha_2 \geq -1$ and $\alpha_1 + \alpha_2 \geq -1$.

Let V be a ball in Ω centered at zero. We consider only the points $\nu \in V$ where the regularization $*$ is not necessary. This is enough, since the complement of this set is pluripolar. Without loss of generality, we can also assume that $V = B$, the unit ball.

Note that

$$\frac{\varphi_1(\xi\nu)}{-\log|\xi|} \leq \alpha_1, \quad \frac{\varphi_2(\xi\nu)}{-\log|\xi|} \leq \alpha_2, \quad |\xi| \leq 1.$$

Hence $\varphi_1(\xi\nu) \leq -\alpha_1 \log|\xi|$ and $\varphi_2(\xi\nu) \leq -\alpha_2 \log|\xi|$ so $\varepsilon_1(\xi) = \varphi_1(\xi\nu) + \alpha_1 \log|\xi|$ and $\varepsilon_2(\xi) = \varphi_2(\xi\nu) + \alpha_2 \log|\xi|$ both extend to subharmonic functions on the unit disc. Furthermore, since

$$\sup_{\substack{r\searrow 0 \\ \xi\in\mathbb{C} \\ |\xi|=r}} \frac{\varepsilon_1(\xi)}{-\log|r|} = \sup_{\substack{r\searrow 0 \\ \xi\in\mathbb{C} \\ |\xi|=r}} \frac{\varepsilon_2(\xi)}{-\log|r|} = 0,$$

we have $\Delta\varepsilon_1\{0\} = \Delta\varepsilon_2\{0\} = 0$. Also, since

$$G_\Omega(\xi\nu,0) = \varphi_1(\xi\nu) + \varphi_2(\xi\nu) = \varepsilon_1(\xi) + \varepsilon_2(\xi) - \alpha_1 \log|\xi| - \alpha_2 \log|\xi|$$

$$= \log|\xi| + \varepsilon_1(\xi) + \varepsilon_2(\xi) - (1 + \alpha_1 + \alpha_2) \log|\xi|$$

and since $\xi \mapsto G_\Omega(\xi\nu,0) - \log|\xi|$ is subharmonic on the unit disc, it follows that $-(1 + \alpha_1 + \alpha_2) \geq 0$. Thus $\alpha_1 + \alpha_2 \leq -1$ and as we already know that $\alpha_1 + \alpha_2 \geq -1$, we conclude that $\alpha_1 + \alpha_2 = -1$.

Now

$$\frac{\varphi_1(\xi\nu)}{-\alpha_1} \leq \log|\xi|, \quad \frac{\varphi_2(\xi\nu)}{-\alpha_2} \leq \log|\xi|, \quad |\xi| \leq 1,$$

so

$$\frac{\varphi_1(z)}{-\alpha_1} \leq G_\Omega(z,0), \quad \frac{\varphi_2(z)}{-\alpha_2} \leq G_\Omega(z,0), \quad z \in B,$$

by the definition of G_Ω. Since $\varphi_1 + \varphi_2 \equiv G_\Omega$, it follows that $\varphi_1/(-\alpha_1) = \varphi_2/(-\alpha_2) = G_\Omega$.

R e m a r k 3. From the proof of Proposition 3, it follows that if Ω is such that G_Ω is symmetric (for example, if Ω is convex), then G_Ω is "extremal (B_2)" among the negative separately plurisubharmonic functions.

R e m a r k 4. In the unit disc, every bounded subharmonic function is an (infinite) sum of subharmonic and continuous functions. We do not know if this is true in the bidisc (cf. [2]), but this problem is one of the motivations for us to study extremal plurisubharmonic functions.

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