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Equations defining reducible Kummer surfaces in $\mathbb{P}^5$

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Abstract. Principally polarized abelian surfaces are the Jacobians of smooth genus 2 curves or of stable genus 2 curves of special type. In [S] we studied equations describing Kummer surfaces in the case of an irreducible principal polarization on the abelian surface. The aim of this note is to give a treatment of the second case. We describe intermediate Kummer surfaces coming from abelian surfaces carrying a product principal polarization. In Proposition 12 we give explicit equations of these surfaces in $\mathbb{P}^5$.

1. Introduction. This note is a continuation of [S1]. Here we study equations of Kummer surfaces induced by some partial linear system arising from a reducible principal polarization on an abelian surface. With a slight abuse of language we call the resulting surfaces reducible intermediate Kummer surfaces. These surfaces are projections of singular abelian surfaces which are complete intersections of 4 quadrics in $\mathbb{P}^6$ described first by Adler and van Moerbeke in [AvM1] and [AvM2]. The abelian surfaces were studied extensively from the algebro-geometric point of view by Barth in [B].

For preliminaries we refer to [M] and [S1]. As far as possible we stick to the notation of our previous paper. We recall it briefly in the next section.

The base field throughout the note is the field $\mathbb{C}$ of complex numbers.

2. The set-up. In [S1] we studied equations of Kummer surfaces coming from the Jacobians of smooth genus two curves. Let now A be the product of elliptic curves F_1 and F_2 and $\Theta = F_1 + F_2$ be a symmetric divisor on A with $\mathcal{L} = \mathcal{O}_A(\Theta)$. Thus $(A, \mathcal{L})$ is a principally polarized abelian surface. Let us denote the halfperiods on A as shown in Figure 1.

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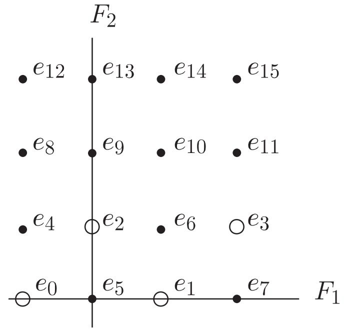

Fig. 1

Here e_0 is a neutral element of a torus action of A on itself and in this convention $e_3 = e_1 + e_2$. We denote by G the subgroup of the halfperiods on A consisting of $e_0, \ldots, e_3$. This subgroup can be lifted in a natural way to the total space of $\mathcal{O}_A(4\Theta)$. Thus G acts on $H^0(\mathcal{O}_A(4\Theta))$. The liftings of e_1 and e_2 can be chosen to be again involutions, which we denote by σ and τ respectively.

Let $Bl : \widetilde{A} \to A$ be the blowing up of A at $e_0, \ldots, e_3$ and $Bl_s : A_s \to A$ the blowing up at all 16 halfperiods. In both cases we denote the exceptional divisor over e_i by E_i. Let $\widetilde{\iota} = Bl^*(\iota)$ and $\iota_s = Bl_s^*(\iota)$, where $\iota : A \ni a \to -a \in A$ is the inverse element mapping on A. The quotients $\widetilde{K} = \widetilde{A}/\widetilde{\iota}$ and $K_s = A_s/\iota_s$ are called the *intermediate* and the *smooth Kummer surface* of A respectively. The quotient mappings are denoted by $\widetilde{\pi}$ in the first case and by π_s in the second.

In what follows we deal mostly with the surface $\widetilde{K}$ which is singular. If it appears to be disturbing one can always think of divisors and line bundles on $\widetilde{K}$ as push-downs from the smooth model K_s. This should exclude any possible confusion.

For a symmetric divisor D on an abelian surface A we denote by $H^0(D)^{\mathrm{ev}}$ and $H^0(D)^{\mathrm{odd}}$ the eigenspaces of 1 and -1 respectively of the mapping $H^0(D) \ni s \to \iota_L s \iota \in H^0(D)$. Here ι_L is the lifting of ι to an involution on the total space of $L = \mathcal{O}_A(D)$. The elements of $H^0(D)^{\mathrm{ev}}$ are called *even sections*, and elements of $H^0(D)^{\mathrm{odd}}$ *odd sections* of the line bundle L.

For a divisor D on a surface X, a point $x \in X$ and a natural number n we denote by $|D - nx|$ those divisors in the linear system $|D|$ which pass through x with multiplicity at least n. Equivalently one can think of sections in $\mathcal{O}_X(D)$ vanishing at x to order at least n or of sections in the sheaf $\mathcal{I}_x^{\otimes n}.\mathcal{O}_X(D)$, where $\mathcal{I}_x$ is the ideal sheaf of x.

3. The linear systems on A and $\widetilde{K}$. We are interested in the equations of the image X of $\widetilde{K}$ in $\mathbb{P}^5$ under the morphism $\varphi : \widetilde{A} \to \mathbb{P}^5$ defined by the linear system $L = |4Bl^*\Theta - 2(E_0 + E_1 + E_2 + E_3)|^{\mathrm{ev}}$. This morphism factors over $\psi : \widetilde{K} \to \mathbb{P}^5$. Moreover, both mappings are G-equivariant. We begin the study of the linear system L with the following

PROPOSITION 1. *For* $L = |4Bl^*\Theta - 2(E_0 + E_1 + E_2 + E_3)|^{\mathrm{ev}}$ *we have* $h^0(L) = 6$.

Proof. $H^0(4\Theta)$ can be written as a direct sum $H^0(4\Theta)^{\mathrm{ev}} \oplus H^0(4\Theta)^{\mathrm{odd}}$. According to [LB, formula 4.7.5] we have $h^0(4\Theta)^{\mathrm{ev}} = 10$ and $h^0(4\Theta)^{\mathrm{odd}} = 6$. The linear system $|4\Theta - (e_0 + \ldots + e_3)|$ has dimension 12 since the four imposed conditions are clearly independent. Moreover, we also have

$$|4\Theta - (e_0 + \ldots + e_3)| = |4\Theta - (e_0 + \ldots + e_3)|^{\mathrm{ev}} \oplus |4\Theta - (e_0 + \ldots + e_3)|^{\mathrm{odd}}.$$

Since 4Θ is totally symmetric the odd sections vanish at each halfperiod to order at least one. Hence $H^0(4\Theta)^{\mathrm{odd}} = |4\Theta - (e_0 + \ldots + e_3)|^{\mathrm{odd}}$ and it follows that $\dim |4\Theta - (e_0 + \ldots + e_3)|^{\mathrm{ev}} = 12 - 6 = 6$. This proves the assertion since again by the total symmetry $|4\Theta - (e_0 + \ldots + e_3)|^{\mathrm{ev}} = |4\Theta - 2(e_0 + \ldots + e_3)|^{\mathrm{ev}}$ and the system in question is the pull-back under the blowing-up of the last system. ∎

The following lemma turns out to be useful in the explicit computation of the action of G on L.

LEMMA 2. *Let* $\sigma^{\pm}$, $\tau^{\pm}$ *be the eigenspaces of* ± 1 *for* σ, τ *respectively. Then* $\dim \sigma^+ = \dim \tau^+ = 4$ *and* $\dim \sigma^- = \dim \tau^- = 2$.

Proof. Since the procedure for σ, τ is the same we consider σ only.

Let w_1, w_2 be complex numbers with $\operatorname{Im} w_i > 0$ and $F_i = \mathbb{C}/(\mathbb{Z}w_i \oplus \mathbb{Z})$ for $i = 1, 2$. Then $A = \mathbb{C}^2/\Lambda$, where

$$\Lambda = \begin{pmatrix} w_1 \\ 0 \end{pmatrix}\mathbb{Z} \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{Z} \oplus \begin{pmatrix} 0 \\ w_2 \end{pmatrix}\mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{Z}$$

is a principally polarized abelian surface and $A = F_1 \times F_2$. We denote the period matrix of A by Π.

Consider the matrices

$$\Pi_\varepsilon = \begin{pmatrix} w_1 & 1 & \varepsilon & 0 \\ \varepsilon & 0 & w_2 & 1 \end{pmatrix}.$$

For $\varepsilon \in D = \{z \in \mathbb{C} : |z| < (\operatorname{Im} w_1 \operatorname{Im} w_2)^{1/2}\}$ the matrix Π_ε defines a principally polarized abelian surface A_ε (see [LB, 4.2]). For $\varepsilon \neq 0$ the surface A_ε is not a product of elliptic curves (compare [LB, 10.6.1]). Hence it must be the Jacobian surface of some smooth curve C_ε of genus 2. We denote by Θ_ε the image of C_ε in A_ε under the Abel–Jacobi mapping. Thus we have a family $\mathcal{A} = \bigcup A_\varepsilon$ of principally polarized abelian surfaces $(A_\varepsilon, \Theta_\varepsilon)$ over the disc D in the complex plane. Let $\pi : \mathcal{A} \to D$ be the obvious mapping $A_\varepsilon \ni x \to \varepsilon \in D$.

There is a section $s_0 : D \to \mathcal{A}$ such that $s_0(\varepsilon) =$ a neutral element e_0^ε of A_ε. This section can be translated to the sections s_1, s_2 in such a way that for $i = 1, 2$ we have $s_i(\varepsilon) = e_i^\varepsilon$, where e_i^ε are two even halfperiods on A_ε and $e_i^0 = e_i$. Thus for each ε we also have the involutions σ_ε, τ_ε operating on $L_\varepsilon = H^0(\mathcal{I}_\varepsilon.\mathcal{O}_{A_\varepsilon}(4\Theta_\varepsilon))^{\mathrm{ev}}$, where $\mathcal{I}_\varepsilon$ denotes the ideal sheaf of $e_0^\varepsilon, e_1^\varepsilon, e_2^\varepsilon, e_3^\varepsilon = e_1^\varepsilon + e_2^\varepsilon$. By Proposition 1 and [S1, Proposition 6] the vector spaces L_ε have dimension 6 for each $\varepsilon \in D$. These vector spaces patched together yield a vector bundle $\mathcal{L}$ on $\mathcal{A}$. It can be easily seen that the mapping $\widetilde{\sigma}(x) := \sigma_{\pi(x)}(x)$ for x in the total space of $\mathcal{L}$ is a vector bundle automorphism. Moreover, $\widetilde{\sigma}$ is an involution. Let $\mathcal{E}_\lambda$ denote $\ker(\widetilde{\sigma} - \lambda\,\mathrm{id}_{\mathcal{L}})$ for $\lambda = \pm 1$. Then according to Grauert's semicontinuity theorem [BPV, Theorem 1.8.5.ii], $\dim \mathcal{E}_{\pm 1}(\varepsilon) = \dim \sigma_\varepsilon^{\pm 1}$ are upper semicontinuous functions of ε, hence these dimensions cannot drop. But they cannot jump up either because 1, -1 are the only eigenvalues of $\widetilde{\sigma}$ and $\dim \sigma_\varepsilon^{+1} + \dim \sigma_\varepsilon^{-1} = H^0(L_\varepsilon) = H^0(\pi_* \mathcal{L})$ according to the base change theorem [BPV, Theorem 1.8.5.iv]. The assertion follows now from [S1, Prop. 6]. ∎

LEMMA 3. *Let* $M = |4\Theta - 2(e_0 + \ldots + e_{15})|$. *Then* $h^0(M) = 1$.

Proof. The divisor $D = \Theta + t_{e_3}^* \Theta + t_{e_9}^* \Theta + t_{e_{14}}^* \Theta$ is clearly in M, hence $h^0(M) \geq 1$. Let $x \in A \setminus \operatorname{supp} D$. If $h^0(M) \geq 2$ then there is an effective divisor $D' \in M$ such that $x \in \operatorname{supp} D'$. It follows that $D \neq D'$. On the other hand, $D.D' = 4\Theta.4\Theta = 32$ and $D \cap D'$ contains all 16 halfperiods with multiplicity at least 4. Hence the two divisors must have common components. A somehow tedious computation on the components of D, D' shows that $D = D'$. Hence $h^0(M) = 1$. ∎

Let s_0 be a generator of $H^0(\Theta)$. Since the line bundles $\mathcal{O}_A(2\Theta)$ and $\mathcal{O}_A(2t_{e_\bullet}^* \Theta)$ are isomorphic for any halfperiod $e_\bullet$ the section s_0^2 can be translated to a section $s_\bullet^2$ doubly vanishing on $\Theta_\bullet = t_{e_\bullet}^*(\Theta)$. These translates are canonically defined as soon as the theta structure is fixed. Furthermore, let $w_4 = Bl^* s$ for some section s with the divisor of s in the linear system M. Let us note that w_4 is thus fixed up to a constant in view of the previous lemma.

Now we are in a position to write a basis for the linear system L explicitly.

PROPOSITION 4. *The sections* $w_1 = s_0^2 s_1^2$, $w_2 = s_2^2 s_3^2$, $w_3 = s_1^2 s_2^2 + s_0^2 s_3^2$, w_4, $w_5 = s_0^2 s_2^2$ *and* $w_6 = s_1^2 s_3^2$ *form a basis of* $H^0(L)$ *in which* σ *and* τ *are represented by the matrices*

$$\sigma = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 1 & 0 & & \\ & & 0 & -1 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & -1 & & & & \\ -1 & 0 & & & & \\ & & -1 & 0 & & \\ & & 0 & 1 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{bmatrix}.$$

P r o o f. It is convenient to view the zero sets of the above sections as in Figure 2.

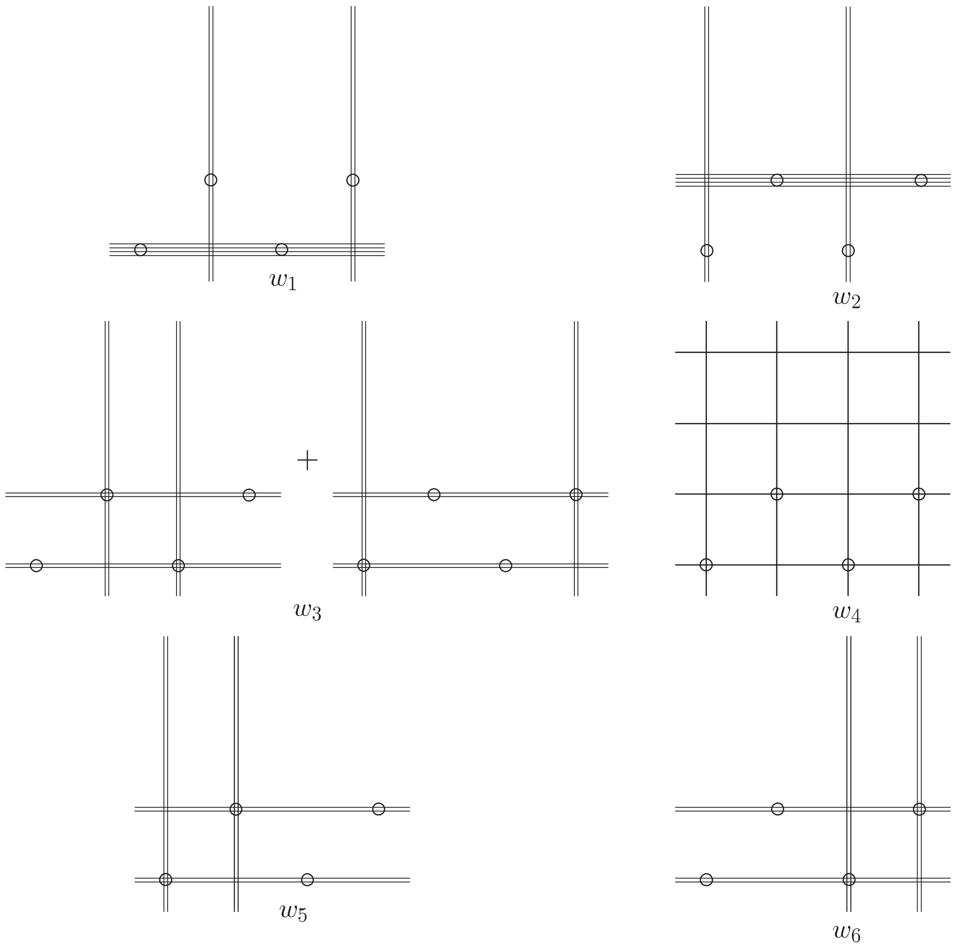

Fig. 2

It is clear that the sections $w_1, \dots, w_6$ are in L. We prove the linear independence and the representation of σ, τ simultaneously.

Assume that

$$\sum_{i=1}^{6} \lambda_i w_i \equiv 0.$$

Restricting this identity to the elliptic curves F_1 and $t_{e_2}^*(F_1)$ we get at once $\lambda_1 = \lambda_2 = 0$. Let $V = \operatorname{span}\{w_3, \dots, w_6\}$. We are done if we show $\dim V = 4$. Since w_3, w_5, w_6 are obviously linearly independent the contrary assumption is $\dim V = 3$. Before we show that this is not possible we have to compute the action of σ and τ. The method of [S1, Prop. 6] applies directly to all sections but w_4 and we have:

	w_1	w_2	w_3	w_5	w_6
σ	w_1	w_2	w_3	w_6	w_5
τ	$-w_2$	$-w_1$	$-w_3$	w_5	w_6

Now suppose $\dim V = 3$. Then $V = V^1 \oplus V^{-1}$, where $V^{\pm 1}$ is the eigenspace of ± 1 for τ. What is more, $V^1 = \operatorname{span}\{w_5, w_6\}$, $V^{-1} = \operatorname{span}\{w_3\}$. Since w_4 is invariant under τ up to sign we must have $w_4 \in V^1$ or $w_4 \in V^{-1}$. In the first case we would have $w_4 = \alpha w_5 + \beta w_6$. This is not possible: just compute its restriction to F_2 and $t^*_{e_1}(F_2)$. The second case is absurd, hence $\dim V = 4$.

We must still compute the action of σ and τ on w_4. As already mentioned, the only problem is to decide whether w_4 is a $+1$ or -1 eigenvector. According to Lemma 2 we must have $\sigma(w_4) = -w_4$ and $\tau(w_4) = w_4$. ∎

The line bundle $\mathcal{O}_A(4Bl^*\Theta - 2(E_0 + \ldots + E_3))$ defines a rank 2 vector bundle $\mathcal{M} = \widetilde{\pi}_* L$ on the Kummer surface $\widetilde{K}$ which splits into a direct sum of two line bundles $\mathcal{M} = \mathcal{M}^+ \bigoplus \mathcal{M}^-$. There is a canonical isomorphism between $H^0(L)$ and $H^0(\mathcal{M}^+)$. Therefore we can denote the coordinates in the second space again by $w_1, \ldots, w_6$.

The next theorem describes the morphism defined by $\mathcal{M}^+$.

PROPOSITION 5. *The line bundle $\mathcal{M}^+$ defines a birational morphism $\psi : \widetilde{K} \to X \subset \mathbb{P}^5$ which is an isomorphism away of the contracted curves $\widetilde{\pi}_* F_1$ and $\widetilde{\pi}_*(t^*_{e_2} F_1)$.*

P r o o f. The proof is based on Saint-Donat's theorem [S-D] and is similar to that of [S1, Prop. 8]. The only difference is the contraction of the two curves. To conclude, it is enough to observe that $\widetilde{\pi}_* F_1.\mathcal{M}^+ = \widetilde{\pi}_*(t^*_{e_2} F_1).\mathcal{M}^+ = 0$. The projective coordinates of the image points can be easily computed. In the basis $w_1, \ldots, w_6$ they are

$$p_{01} = \psi(\widetilde{\pi}_* F_1) = (0 : 1 : 0 : 0 : 0 : 0),$$
$$p_{23} = \psi(\widetilde{\pi}_*(t^*_{e_2} F_1)) = (1 : 0 : 0 : 0 : 0 : 0). \ ∎$$

4. Geometric properties. As in the case of an irreducible principal polarization of A the surface X contains 4 conics and 4 lines. They are now arranged in what we can call a degenerate 4_3 configuration.

LEMMA 6. *The curves $C_i = \varphi(E_i)$ are smooth conics in $\mathbb{P}^5$ for $i = 0, \ldots, 3$.*

P r o o f. For $i = 0, \ldots, 3$ we have

$$\deg(C_i) = \deg(\varphi(E_i)) = \mathcal{M}^+.D_i = 2$$

and C_i is irreducible. It cannot be a double line according again to Saint-Donat's theorem. Hence it is a smooth conic. ■

LEMMA 7. *The elliptic curves $t^*_{e_i}(F_2)$ go $2:1$ under φ onto lines L_{3-i} for $i = 0, \ldots, 3$. There are exactly 3 singular points of X on each of these lines.*

Proof. It is enough to prove the lemma for a chosen curve, say F_2, since all the others are images of $\varphi(F_2)$ under the group G. Let $C = \widetilde{\pi}_* Bl^* F_2$. Then we have

$$\begin{aligned}\deg L_0 &= \mathcal{M}^+.C = \tfrac{1}{2}(4Bl^*\Theta - 2(E_0 + \ldots + E_3)).\widetilde{\pi}^*C \\ &= 2Bl^*(F_1 + F_2).(Bl^*F_2 - E_2) - 1 = 1.\end{aligned}$$

The two double points on L_0 are $\varphi(e_9)$ and $\varphi(e_{13})$. The remaining singular point $\varphi(Bl^*(t^*_{e_2}(F_1)) - E_3)$ is of type A_3. ■

COROLLARY 8. *$\widetilde{X}$ has 8 double points of type A_1 and 2 singularities of type A_3.*

In the sequel the geometric interpretation in Figure 3 of the degenerate 4_3 configuration of lines and conics will be useful. The dotted lines are the conics, • denotes the A_1 singularities and p_{01}, p_{23} are the A_3 singularities.

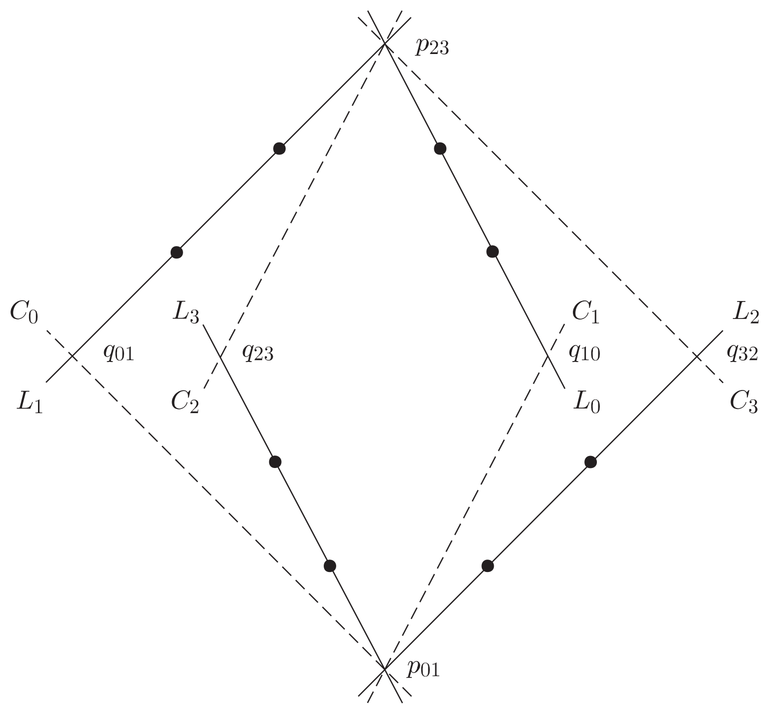

Fig. 3

R e m a r k 9. We already know the coordinates of the points p_{01}, p_{23}. The other four points form an orbit under the group operation. In the sequel we need the projective coordinates of one of them:

$$q_{23} = (0 : 0 : \alpha : 0 : 0 : \beta),$$

where $(\alpha : \beta) \in \mathbb{P}^1$ depends a priori on the elliptic curves F_1, F_2. In fact, we will show later that $(\alpha : \beta)$ only depends on the curve F_1, namely α^2/β^2 turns out to be its cross-ratio.

5. The equations. Let us use the notation $u_3 = s_1^2 s_2^2$, $u_4 = s_0^2 s_3^2$ introduced in [S1] and begin with the following

LEMMA 10. *There are complex numbers μ_1 and μ_2 such that $v := u_3 - u_4 = \mu_1(w_5 - w_6) + \mu_2 w_4$.*

P r o o f. It is enough to notice that v is σ-antiinvariant and w_5–w_6, w_4 are a basis for the -1-eigenspace of σ according to Lemma 2 and Proposition 4. In fact, we will show later that $\mu_2 = 0$. ∎

There are two obvious quadrics containing the image surface X:

- $Q_1 = \{w_1 w_2 - w_5 w_6 = 0\}$,
- $Q_2 = \{w_1 w_2 - u_3 u_4 = 0\} = \{4w_1 w_2 - w_3^2 + (\mu_1(w_5 - w_6) + \mu_2 w_4)^2 = 0\}$.

To find the next equation let us consider the divisors of the two sections shown in Figure 4.

$y_1 = s_0^2 s_{14}^2 = s_1^2 s_{12}^2$ $\qquad\qquad$ $y_2 = s_2^2 s_{11}^2 = s_3^2 s_9^2$

Fig. 4

These sections are in L. Furthermore, they are σ-invariant and τ exchanges them without changing the sign. Hence there are complex numbers $\lambda_1, \ldots, \lambda_6$ such that

$$y_1 = \sum \lambda_i w_i.$$

Restricting this equality to F_1 we get immediately $\lambda_2 = 0$ and from σ-invariance we have $\lambda_4 = 0$, $\lambda_5 = \lambda_6$. Let $\lambda_1 = a$, $\lambda_3 = b$, $\lambda_5 = c$. Then

$$\begin{aligned} y_1 &= aw_1 + bw_3 + c(w_5 + w_6), \\ y_2 &= -aw_2 - bw_3 + c(w_5 + w_6). \end{aligned}$$

Since $\operatorname{div}(w_4^2) = \operatorname{div}(y_1^2) + \operatorname{div}(y_2^2)$ we are in a position to write down our third equation:

$$Q_3 = \{w_4^2 + (aw_1 + bw_3 + c(w_5 + w_6))(aw_2 + bw_3 - c(w_5 + w_6)) = 0\}.$$

Remark 11. Notice that in the above equation the parameters a, b, c can be considered only up to a multiplicative constant, hence as a point $(a : b : c) \in \mathbb{P}^2$. The reason is that w_4 as a section is fixed only up to a constant.

In the rest of this section we state relations between the parameters α, β, μ_1, μ_2, a, b, c appearing in our equations. As expected there will be only three (homogeneous) left, depending on the moduli of elliptic curves defining A.

To get the relations between the parameters we have to use the information coded in the singular locus of X.

Let $x = sp_{01} + tq_{23}$, $(s : t) \in \mathbb{P}^1$, be a point on L_3 and let

$$W = \begin{pmatrix} \frac{dQ_1}{d(w_1,\ldots,w_6)}(x) \\ \frac{dQ_2}{d(w_1,\ldots,w_6)}(x) \\ \frac{dQ_3}{d(w_1,\ldots,w_6)}(x) \end{pmatrix}.$$

We can compute explicitly W to be

$$\begin{pmatrix} s & 0 & 0 & 0 & -\beta t & 0 \\ 2s & 0 & -\alpha t & -t\beta\mu_1\mu_2 & -\mu_1^2\beta t & \mu_1^2\beta t \\ a(as+bt\alpha-ct\beta) & abt\alpha + act\beta & abs + 2b^2t\alpha & 0 & acs - 2c^2t\beta & acs - 2c^2t\beta \end{pmatrix}.$$

This matrix carries much information about the Kummer surface X. Let W_{ijk} denote the minor of W consisting of the ith, jth and kth column of W. At the singular points on the line L_3 all these determinants must vanish. Thus we get a system of degree 3 equations in s and t. The crucial observation is that there are three *distinct* singularities on this line. Hence the obtained equations must be either trivial or have exactly 3 different zeroes. Evaluating this information we get the following conditions:

- $a, b, c, \alpha, \beta, \mu_1 \neq 0$, $b^2 \neq c^2$, $4(b^2 - c^2) \neq a^2$,
- $\mu_2 = 0$,
- $\alpha b + \beta c = 0$, $\mu_1^2\beta b + \alpha c = 0$, $\mu_1^2 b^2 - c^2 = 0$.

Calculations leading to the above conditions are tedious and therefore omitted here. In what follows we set $\mu_1^2 = c^2/b^2$ and $\beta = -b$, $\alpha = c$.

In the next section and in [S2] we need the equation of singularities on L_3, which we get from $\det W_{156} = 0$:

$$2ab^2c^2t^3 - bc(4c^2 - 4b^2 - a^2)st^2 + 2a(b^2 - c^2)s^2t = 0. \tag{1}$$

We conclude this section with the following

Proposition 12. *The reducible intermediate Kummer surface X in $\mathbb{P}^5$ is a complete intersection of a net of quadrics spanned by*

$$Q_1 = \left[\begin{array}{cc|cc|cc} 0 & 1 & & & & \\ 1 & 0 & & & & \\ \hline & & & & & \\ & & & & & \\ \hline & & & & 0 & -1 \\ & & & & -1 & 0 \end{array}\right], \quad Q_2 = \left[\begin{array}{cc|cc|cc} 0 & 2b^2 & & & & \\ 2b^2 & 0 & & & & \\ \hline & & -b^2 & 0 & & \\ & & 0 & 0 & & \\ \hline & & & & c^2 & -c^2 \\ & & & & -c^2 & c^2 \end{array}\right],$$

$$Q_3 = \left[\begin{array}{cc|cc|cc} 0 & a^2 & ab & 0 & -ac & -ac \\ a^2 & 0 & ab & 0 & ac & ac \\ \hline ab & ab & 2b^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \hline -ac & ac & 0 & 0 & -2c^2 & -2c^2 \\ -ac & ac & 0 & 0 & -2c^2 & -2c^2 \end{array}\right],$$

where $(a:b:c) \in \mathbb{P}^2$ depends on elliptic curves defining the abelian surface A and satisfies $b^2 \neq c^2$, $4(b^2 - c^2) \neq a^2$, $a, b, c \neq 0$.

6. Parameters vs cross-ratios. In this section we show how the parameters a, b, c depend on the moduli of the elliptic curves F_1, F_2 defining A. We begin with the following

Lemma 13. *$C := \varphi(t^*_{e_8}F_1)$ is a conic.*

Proof. $\deg C = C.\mathcal{M}^+ = \frac{1}{2}(4Bl^*(F_1+F_2) - 2(E_0 + \ldots + E_3)).t^*_{e_8}F_1 = 2$. ∎

Our aim is to parametrize the conic C. Let E denote the plane spanned by the conic. Then we have

Lemma 14. *The equations of E are*

$$w_4 = 0,$$
$$w_2 = 4\frac{b^2 - c^2}{a^2}w_1 + 2\frac{c}{a}w_5 + 2\frac{c}{a}w_6,$$
$$w_3 = 4\frac{c^2 - b^2}{ab}w_1 - \frac{c}{b}w_5 - \frac{c}{b}w_6.$$

Proof. From the equation (1) and the action of G we get easily

$$\begin{aligned}\varphi(e_8) &= (-abc : 0 : 2c(b^2-c^2) : 0 : 0 : 2b(b^2-c^2)),\\ \varphi(e_9) &= (0 : 2bc : -ac : 0 : 0 : ab),\\ \varphi(e_{10}) &= (-abc : 0 : 2c(b^2-c^2) : 0 : 2b(b^2-c^2) : 0),\\ \varphi(e_{11}) &= (0 : 2bc : -ac : 0 : ab : 0).\end{aligned}$$

Now to verify our assertion one has to solve a system of linear equations. ∎

Using the above lemma we can parametrize the plane E in the following way:

$$w_1 = a^2bx, \quad w_2 = 4b(b^2-c^2)x + 2abcy + 2abcz,$$
$$w_3 = 4a(c^2-b^2)x - a^2cy - a^2cz, \quad w_4 = 0, \quad w_5 = a^2by, \quad w_6 = a^2bz.$$

In the coordinates $(x : y : z)$ the conic C is given by the equation

$$4(b^2-c^2)x^2 + 2acxy + 2acxz - a^2yz = 0$$

and the four points determining the cross-ratio for F_1 are

$$(0:0:1), \quad (0:1:0), \quad (-ac : 0 : 2(b^2-c^2)), \quad (-ac : 2(b^2-c^2) : 0).$$

One verifies easily that the following is a parametrization of the conic C

$$x = ast, \quad y = 2bs^2 - 2cst, \quad z = 2cst + 2bt^2,$$

and the four points in the $(s : t)$-coordinates are $(0 : 1)$, $(1 : 0)$, $(c : -b)$, $(b : -c)$.

Now we are in a position to state the following

Proposition 15. *The cross-ratios r_1, r_2 of the elliptic curves F_1, F_2 are given by $r_1 = c^2/b^2$ and $r_2 = 4(b^2-c^2)/a^2$.*

Proof. For F_1 there is nothing to do because of the above considerations. For F_2 we first observe that the mapping $\varphi|F_2 : F_2 \to L_3$ is a $2:1$ covering branched over 4 points. To know F_2 it is enough to compute the cross-ratio of the branch points. Two of them are $t_1 = q_{23} = (0 : 1)$ and $t_2 = p_{01} = (1 : 0)$ written in the $(s : t)$ coordinates and the two others are the A_1 singularities $t_3 = \varphi(e_9)$, $t_4 = \varphi(e_{13})$. Their coordinates can be easily computed from equation (1). Thus the cross-ratio is

$$r_2 = \frac{bc(4c^2-4b^2-a^2) + \Delta^{1/2}}{bc(4c^2-4b^2-a^2) - \Delta^{1/2}},$$

where $\Delta = (bc(4(c^2-b^2)+a^2))^2$. This proves the assertion for r_2. ∎

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