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Laplace ultradistributions on a half line and a strong quasi-analyticity principle

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Abstract. Several representations of the space of Laplace ultradistributions supported by a half line are given. A strong version of the quasi-analyticity principle of Phragmén–Lindelöf type is derived.

The theory of ultradistributions was founded by Buerling and Roumieu in the sixties as a generalization of the theory of Schwartz distributions. Since then it was extensively studied by many authors: Björk, Braun, Komatsu, Meise, Pilipović, Taylor , . . . , to mention but a few. The most systematic treatment was presented by Komatsu [2], [3]. He derived, in particular, the boundary value representation of the space $D^{(M_p)\prime}(\Omega)$ of ultradistributions on an open set $\Omega \subset \mathbb{R}^n$, structure theorems for $D^{(M_p)\prime}(\Omega)$ and described the image of the space $D_K^{(M_p)\prime}$ of ultradistributions with compact support in K under the Fourier–Laplace transformation. Following his approach Pilipović [9] recently introduced and investigated the space $S^{(M_p)\prime}(\mathbb{R})$ of tempered ultradistributions. On the other hand, in the study of the Laplace transformation it is convenient to consider the space $L'_{(\omega)}(\Gamma)$ of Laplace distributions of type $\omega \in \mathbb{R}$ supported by a half line Γ. Since in the logarithmic variables the Laplace transformation is the Mellin transformation we refer here to the book of Szmydt and Ziemian [11], where the latter transformation was systematically studied following the approach of Zemanian [12].

The aim of the present paper is to unify the theory of ultradistributions with that of Laplace distributions. We present it in the case of the space $L_{(\omega)}^{(M_p)\prime}(\Gamma)$ of Laplace ultradistributions of Buerling type. Our theory is based on the Seeley type extension theorems for ultradifferentiable functions re-

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cently proved by Langenbruch [4] and Meise and Taylor [7]. We describe the image of the space $L^{(M_p)\prime}_{(\omega)}(\Gamma)$ under the Laplace, Taylor and (modified) Cauchy transformations. In the latter case we follow the method of Morimoto [8]. As an application of our theory we give, in the final section, a version of the quasi-analyticity principle of Phragmén–Lindelöf type. It says that a function holomorphic and of exponential type in the half plane $\{\operatorname{Re} z > 0\}$ vanishes if it satisfies some growth conditions along vertical lines and decreases superexponentially along a ray in $\{\operatorname{Re} z > 0\}$.

0. Notation. Let $t > 0$. We denote by $\widetilde{B}(t)$ the universal covering of the punctured disc $B(t) \setminus \{0\}$ and by $\widetilde{\mathbb{C}}$ that of $\mathbb{C} \setminus \{0\}$. We treat $\widetilde{B}(t)$ and $\widetilde{\mathbb{C}}$ as Riemann manifolds. Recall that any point $x \in \widetilde{B}(t)$ can be written in the form $x = |x| \exp i \arg x$ with $|x| < t$.

We denote by $\mu : \mathbb{C} \to \widetilde{\mathbb{C}}$ the biholomorphism

$$\mu(z) = e^{-z} \quad \text{ for } z \in \mathbb{C},$$

i.e. $\mu(z) = x \in \widetilde{\mathbb{C}}$ with $|x| = e^{-\operatorname{Re} z}$, $\arg x = -\operatorname{Im} z$. Then the inverse mapping $\mu^{-1} : \widetilde{\mathbb{C}} \to \mathbb{C}$ is given by

$$\mu^{-1}(x) = -\ln x \quad \text{ for } x \in \widetilde{\mathbb{C}}.$$

Let $v \in \mathbb{R}$. We set

$$\Gamma_v = [v, \infty) \quad \text{and} \quad I_v = (0, e^{-v}].$$

Observe that $I_v = \mu(\Gamma_v)$. In the following we omit the subscript v as long as it is fixed. For $z \in \mathbb{C}$ we define the function $\exp_z : \mathbb{R} \to \mathbb{C}$ by

$$\exp_z y = e^{yz}, \quad y \in \mathbb{R}.$$

For $A \subset \mathbb{C}$ we set

$$A_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, A) < \varepsilon\}, \quad \varepsilon > 0.$$

We write D for the differential operator d/dx.

Let $\{P_\tau\}_{\tau \in T}$ be a family of multivalued vector spaces. Then $\varinjlim_{\tau \in T} T_\tau$ (resp. $\varprojlim_{\tau \in T} P_\tau$) denotes the inductive limit (resp. projective limit) of P_τ, $\tau \in T$.

$\mathcal{O}(W)$ denotes the set of holomorphic functions on an open subset W of some Riemann manifold. The value of a functional S on a test function φ is denoted by $S[\varphi]$.

1. Laplace ultradistributions on a half line. Let $(M_p)_{p \in \mathbb{N}_0}$ be a sequence of positive numbers. Throughout the paper we assume that (M_p) satisfies the following conditions:

(M.1) (*Logarithmic convexity*)

$$M_p^2 \le M_{p-1}M_{p+1} \quad \text{for } p \in \mathbb{N};$$

(M.2) (*Stability under ultradifferential operators*) There are constants A, H such that

$$M_p \le AH^p \min_{0\le q\le p} M_q M_{p-q} \quad \text{for } p \in \mathbb{N}_0;$$

(M.3) (*Strong non-quasi-analyticity*) There is a constant A such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \le Ap\frac{M_p}{M_{p+1}} \quad \text{for } p \in \mathbb{N}.$$

Some results remain valid when (M.2), (M.3) are replaced by the following weaker conditions:

(M.2′) (*Stability under differential operators*) There are constants A, H such that

$$M_{p+1} \le AH^p M_p \quad \text{for } p \in \mathbb{N}_0;$$

(M.3′) (*Non-quasi-analyticity*)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

Define

$$m_p = M_p/M_{p-1} \quad \text{for } p \in \mathbb{N}.$$

Then (M.1) is equivalent to saying that the sequence m_p is non-decreasing, and by (M.3′) it follows that $m_p \to \infty$.

Note that the condition (M.3′) implies the following: for every $h > 0$ there exists $\delta > 0$ such that

$$M_p h^p \ge \delta \quad \text{for } p \in \mathbb{N}_0, \tag{1}$$

which is equivalent to the finiteness of the *associated function* M defined by

$$M(\varrho) = \sup_{p\in\mathbb{N}_0} \ln \frac{\varrho^p M_0}{M_p} \quad \text{for } \varrho > 0. \tag{2}$$

If $M_p/p!$ satisfies (1) the *growth function* M^* is defined by

$$M^*(\varrho) = \sup_{p\in\mathbb{N}_0} \ln \frac{\varrho^p p! M_0}{M_p} \quad \text{for } \varrho > 0. \tag{3}$$

Example 1. The *Gevrey sequence* of order $s > 1$ is defined by $M_p = (p!)^s$, $p \in \mathbb{N}_0$. It satisfies all conditions (M.1)–(M.3) and $M(\varrho) \sim \varrho^{1/s}$, $M^*(\varrho) \sim \varrho^{1/s-1}$ as $\varrho \to \infty$.

R e m a r k 1. It follows from Lemma 4.1 of [2] that if M_p satisfies (M.1) and (M.3′) then the associated function M is sublinear, i.e. $M(\varrho)/\varrho \to 0$ as $\varrho \to \infty$.

R e m a r k 2. If M_p satisfies (M.1) and (M.3′) then

$$\lim_{p\to\infty} (M_p/p!)^{1/p} = \infty. \tag{4}$$

P r o o f. Take any $l < \infty$. Then by (M.1) and (M.3′) there exists $p_l \in \mathbb{N}$ such that $M_p \ge lpM_{p-1}$ for $p \ge p_l$. Hence

$$M_p \ge C_l \cdot l^p \cdot p! \quad \text{for } p \ge p_l, \qquad \text{where } C_l = \frac{M_{p_l-1}}{M_0(p_l-1)!} l^{1-p_l},$$

and we easily get (4).

Definition. Let $\Gamma = [v, \infty)$ with $v \in \mathbb{R}$. The space $D^{(M_p)\prime}(\Gamma)$ of *ultradistributions on Γ of class (M_p)* is defined as the dual space of

$$D^{(M_p)}(\Gamma) = \varinjlim_{K\subset\Gamma} \varprojlim_{h>0} D^{(M_p)}_{K,h}(\Gamma),$$

where for any compact set $K \subset \Gamma$ and $h > 0$,

$$D^{(M_p)}_{K,h}(\Gamma) = \Big\{\varphi \in C^\infty(\Gamma) : \operatorname{supp}\varphi \subset K \text{ and } \|\varphi\|^{(M_p)}_{K,h} = \sup_{y\in K} \sup_{\alpha\in\mathbb{N}_0} \frac{|D^\alpha \varphi(y)|}{h^\alpha M_\alpha} < \infty\Big\}.$$

By $\varphi \in C^\infty(\Gamma)$ we mean a restriction to Γ of some function $\widetilde{\varphi} \in C^\infty(\mathbb{R})$.

Definition. Let $\omega \in \mathbb{R} \cup \{\infty\}$. We define the space $L^{(M_p)\prime}_{(\omega)}(\Gamma)$ of *Laplace ultradistributions* as the dual space of

$$L^{(M_p)}_{(\omega)}(\Gamma) = \varinjlim_{a<\omega} L^{(M_p)}_a(\Gamma),$$

where for any $a \in \mathbb{R}$,

$$L^{(M_p)}_a(\Gamma) = \varprojlim_{h>0} L^{(M_p)}_{a,h}(\Gamma),$$

$$L^{(M_p)}_{a,h}(\Gamma) = \Big\{\varphi \in C^\infty(\Gamma) : \|\varphi\|^{(M_p)}_{a,h} = \sup_{y\in\Gamma} \sup_{\alpha\in\mathbb{N}_0} \frac{|e^{-ay} D^\alpha \varphi(y)|}{h^\alpha M_\alpha} < \infty\Big\}.$$

Lemma 1. *Assume that (M_p) satisfies* (M.1) *and* (M.3′). *Then $D^{(M_p)}(\Gamma)$ is a dense subspace of $L^{(M_p)}_{(\omega)}(\Gamma)$. Thus, $L^{(M_p)\prime}_{(\omega)}(\Gamma)$ is a subspace of the space of ultradistributions $D^{(M_p)\prime}(\Gamma)$.*

P r o o f. Making a translation if necessary we can assume that $\Gamma = \overline{\mathbb{R}}_+$. Let $\varphi \in L^{(M_p)}_{(\omega)}(\Gamma)$. Then there exist $a < b < \omega$ such that $\varphi \in L^{(M_p)}_a(\Gamma) \subset L^{(M_p)}_b(\Gamma)$. By the Denjoy–Carleman–Mandelbrojt theorem (cf. [2], [6]) there exists a function $\psi \in D^{(M_p)}(\Gamma)$ such that $0 \le \psi(y) \le 1$ for $y \in \Gamma$, $\psi(y) = 1$ for $0 \le y \le 1$ and $\psi(y) = 0$ for $y \ge 2$. Put $\varphi_\nu(y) = \psi(y/\nu)\varphi(y)$ for $y \in \Gamma$, $\nu \in \mathbb{N}$. Then $\varphi_\nu \in D^{(M_p)}(\Gamma)$ and we shall show that $\varphi_\nu \to \varphi$ in $L_b^{(M_p)}(\Gamma)$ as $\nu \to \infty$. To this end take any $h > 0$. Noting that (M.1) implies $M_q M_{p-q} \le M_0 M_p$ for $0 \le q \le p$, by the Leibniz formula we get

$$\|\varphi_\nu - \varphi\|_{b,h}^{(M_p)} = \sup_{y\in\Gamma}\sup_{\alpha\in\mathbb{N}_0} \frac{|e^{-by} D^\alpha(\varphi(y)(\psi(y/\nu) - 1))|}{h^\alpha M_\alpha}$$

$$\le \sup_{y\in\Gamma}\sup_{\alpha\in\mathbb{N}_0} \frac{e^{-ay}|D^\alpha\varphi(y)|}{h^\alpha M_\alpha} e^{(a-b)y}|\psi(y/\nu) - 1|$$

$$+ \sup_{y\in\Gamma}\sup_{\alpha\in\mathbb{N}} \sum_{0\le\beta<\alpha} \binom{\alpha}{\beta} \frac{e^{-ay}|D^\beta\varphi(y)|}{h^\beta M_\beta} \cdot \frac{e^{(a-b)y}|D^{\alpha-\beta}(\psi(y/\nu) - 1)|M_0}{h^{\alpha-\beta}M_{\alpha-\beta}}.$$

Since $\psi(y/\nu) = 1$ for $0 \le y \le \nu$ the first summand tends to zero as $\nu \to \infty$. Put $K = [1, 2]$. Then for $\beta < \alpha$ and any $h_1 > 0$ we have

$$|D^{\alpha-\beta}(\psi(y/\nu) - 1)| = |\nu^{-(\alpha-\beta)}\psi^{(\alpha-\beta)}(y/\nu)| \le \nu^{-1}\|\psi\|_{K,h_1}^{(M_p)} h_1^{\alpha-\beta} M_{\alpha-\beta}.$$

We also have for any $h_2 > 0$ and $\beta \ge 0$, $e^{-ay}|D^\beta\varphi(y)| \le \|\varphi\|_{a,h_2}^{(M_p)} h_2^\beta M_\beta$. So the second summand is bounded by

$$\frac{M_0 e^{(a-b)\nu}}{\nu} \sup_{\alpha\in\mathbb{N}} \sum_{\beta<\alpha} \binom{\alpha}{\beta} \|\varphi\|_{a,h_2}^{(M_p)} \left(\frac{h_2}{h}\right)^\beta \|\psi\|_{K,h_1}^{(M_p)} \left(\frac{h_1}{h}\right)^{\alpha-\beta}$$

$$\le \frac{M_0}{\nu} \|\varphi\|_{a,h_2}^{(M_p)} \|\psi\|_{K,h_1}^{(M_p)} \quad \text{if } h_2 + h_1 \le h$$

and thus tends to zero as $\nu \to \infty$, proving the lemma.

Example 2. Let (M_p) satisfy (1). Then the function

$$\Gamma \ni y \to \exp_z y = e^{yz}$$

belongs to $L_{(\omega)}^{(M_p)}(\Gamma)$ if and only if $\operatorname{Re} z < \omega$. Furthermore, in this case for any $a < \omega$ and $h > 0$ we have

$$\|\exp_z\|_{a,h}^{(M_p)} = M_0^{-1} \exp\{(\operatorname{Re} z - a)v + M(|z|/h)\}.$$

Let (M_p) satisfy (M.1) and (1), and let $z \in \mathbb{C}$. Then the operation of multiplication

$$\exp_z : L_{(\omega)}^{(M_p)}(\Gamma) \to L_{(\omega+\operatorname{Re} z)}^{(M_p)}(\Gamma)$$

is continuous. Thus the formula

$$\exp_z S[\varphi] = S[\exp_z \varphi] \quad \text{for } \varphi \in L_{(\omega+\operatorname{Re} z)}^{(M_p)}(\Gamma),\ S \in L_{(\omega)}^{(M_p)\prime}(\Gamma)$$

defines a continuous operation

$$\exp_z : L_{(\omega)}^{(M_p)\prime}(\Gamma) \to L_{(\omega-\operatorname{Re} z)}^{(M_p)\prime}(\Gamma).$$

Let (M_p) satisfy (M.2). An *ultradifferential operator* $P(D)$ *of class* (M_p) is defined by

$$P(D)=\sum_{\alpha=0}^{\infty}a_\alpha D^\alpha,$$

where $a_\alpha\in\mathbb{C}$ satisfy the following condition: there are constants $K<\infty$ and $C<\infty$ such that

$$|a_\alpha|\le C\frac{K^\alpha}{M_\alpha}\quad\text{ for }\alpha\in\mathbb{N}_0. \tag{5}$$

The entire function $\mathbb{C}\ni z\to P(z)$ is called a *symbol of class* (M_p). As in the proof of Theorem 2.12 of [2] one can show that an ultradifferential operator of class (M_p) defines linear continuous mappings

$$P(D):L^{(M_p)}_{(\omega)}(\Gamma)\to L^{(M_p)}_{(\omega)}(\Gamma),\qquad P(D):L^{(M_p)\prime}_{(\omega)}(\Gamma)\to L^{(M_p)\prime}_{(\omega)}(\Gamma),$$

where for $S\in L^{(M_p)\prime}_{(\omega)}(\Gamma)$ and $\varphi\in L^{(M_p)}_{(\omega)}(\Gamma)$,

$$P(D)S[\varphi]=S[P^*(D)\varphi]\quad\text{ with }\quad P^*(D)=\sum_{\alpha=0}^{\infty}(-1)^\alpha a_\alpha D^\alpha.$$

For $a\in\mathbb{R}$ and $\omega\in\mathbb{R}\cup\{\infty\}$ we define

$$Y_a=\operatorname{span}\{\exp_c\}_{c\le a},\qquad Y_{(\omega)}=\bigcup_{a<\omega}Y_a. \tag{6}$$

PROPOSITION 1. *Let $b<a$. Then $L^{(M_p)}_b(\Gamma)$ is contained in the closure of Y_a in $L^{(M_p)}_a(\Gamma)$. Thus $Y_{(\omega)}$ is dense in $L^{(M_p)}_{(\omega)}(\Gamma)$.*

Proof. Since the multiplication by $\exp_{-a}$ is a continuous isomorphism of $L^{(M_p)}_c(\Gamma)$ onto $L^{(M_p)}_{c-a}(\Gamma)$ and Y_c onto Y_{c-a}, where $c\in\mathbb{R}$, it is sufficient to assume that $a=0$. Let $\varphi\in L^{(M_p)}_b(\Gamma)$. It is enough to show that for every $\varepsilon>0$ and $h>0$ there exists $\psi\in Y_0$ such that $\|\varphi-\psi\|^{(M_p)}_{0,h}<\varepsilon$. To this end fix $\varepsilon>0$ and $h>0$. By the proof of Lemma 1 there exists $\widetilde{\psi}\in D^{(M_p)}(\Gamma)$ such that $\|\varphi-\widetilde{\psi}\|^{(M_p)}_{0,h}<\varepsilon/2$. Put $\eta(x)=\psi\circ\mu^{-1}(x)$ for $x\in I$. Then η has compact support in $I=\mu(\Gamma)$ and by the Roumieu theorem ([10], Théorème 13), $\eta\in D^{(M_p)}(I)$. By the Weierstrass type theorem ([2], Theorem 7.3) for any $\delta>0$ and $h_1>0$ there exists a polynomial $p=\sum_{\nu=0}^N c_\nu x^\nu$ such that

$$\|\eta-p\|^{(M_p)}_{\bar I,h_1}<\delta. \tag{7}$$

Put $\psi(y)=p\circ\mu(y)=\sum_{\nu=0}^N c_\nu e^{-\nu y}$ for $y\in\Gamma$. Then $\psi\in Y_0$ and we shall show that for a suitable choice of δ and h_1, $\|\widetilde{\psi}-\psi\|^{(M_p)}_{0,h}<\varepsilon/2$. To this end put $f = \eta - p$. Following the proof of Théorème 13 of [10] one can show that the derivatives $D^\alpha f \circ \mu$ are estimated by

$$\|f\|^{(M_p)}_{\bar{I},h_1} \cdot \sum_{\beta=1}^{\alpha} e^{-v\beta} \frac{M_\beta}{\beta!} h_1^\beta \frac{(\alpha-1)!\alpha!}{(\alpha-\beta)!(\beta-1)!}. \tag{8}$$

For $\gamma \in \mathbb{N}_0$ put

$$H_\gamma = \sup_{\beta \geq \gamma} (\beta!/M_\beta)^{1/\beta}. \tag{9}$$

Then by Remark 2, $H_\gamma \to 0$ as $\gamma \to \infty$. Hence we can find $\gamma \in \mathbb{N}_0$ and $h_1 > 0$ such that

$$(\sqrt{e^{-v}h_1} + \sqrt{H_\gamma})^2 \cdot H < h, \tag{10}$$

where H is the constant in (M.2). Since by (M.2), $M_{p+q} \leq AH^{p+q} M_p M_q$ for $p, q \in \mathbb{N}_0$ and by (M.1), $M_\beta M_{\alpha-\beta} \leq M_1 M_\alpha$ for $0 \leq \beta \leq \alpha$, we get for $\alpha \in \mathbb{N}_0$, $0 \leq \beta \leq \alpha$,

$$M_\beta M_{\alpha-\beta+\gamma} \leq M_\beta A H^{\alpha-\beta+\gamma} M_{\alpha-\beta} M_\gamma \leq C_\gamma H^\alpha M_\alpha, \quad \text{where } C_\gamma = AM_1 M_\gamma \max(H^\gamma, 1). \tag{11}$$

Observe also that

$$\alpha < \beta(\alpha - \beta + \gamma) \quad \text{for } 1 \leq \beta \leq \alpha,\ \alpha \in \mathbb{N},$$

and

$$\sum_{\beta=1}^{\alpha} \binom{\alpha-1}{\beta-1}^2 x^{\beta-1} y^{\alpha-\beta} \leq (\sqrt{x} + \sqrt{y})^{2\alpha-2} \quad \text{for } \alpha \in \mathbb{N},\ x \geq 0,\ y \geq 0.$$

Hence using (8), (9) and (11) we derive for $\alpha \in \mathbb{N}$, $y \in \Gamma$,

$$\begin{aligned}
&|D^\alpha f \circ \mu(y)| \\
&\leq \|f\|^{(M_p)}_{\bar{I},h_1} \cdot \sum_{\beta=1}^{\alpha} e^{-v\beta} h_1^\beta H_\gamma^{\alpha-\beta+\gamma} \frac{(\alpha-1)!\alpha!}{(\alpha-\beta)!(\alpha-\beta+\gamma)!\beta!(\beta-1)!} M_\beta M_{\alpha-\beta+\gamma} \\
&\leq \|f\|^{(M_p)}_{\bar{I},h_1} \cdot C_\gamma H_\gamma^\gamma e^{-v} h_1 \sum_{\beta=1}^{\alpha} (e^{-v}h_1)^{\beta-1} H_\gamma^{\alpha-\beta} \left(\frac{(\alpha-1)!}{(\alpha-\beta)!(\beta-1)!} \right)^2 H^\alpha M_\alpha \\
&\leq \|f\|^{(M_p)}_{\bar{I},h_1} \cdot C_\gamma H_\gamma^\gamma e^{-v} h_1 (\sqrt{e^{-v}h_1} + \sqrt{H_\gamma})^{-2} ((\sqrt{e^{-v}h_1} + \sqrt{H_\gamma})^2 H)^\alpha M_\alpha \\
&\leq \widetilde{C}_\gamma M_\alpha L^\alpha \|f\|^{(M_p)}_{\bar{I},h_1}, \quad \text{where } L = (\sqrt{e^{-v}h_1} + \sqrt{H_\gamma})^2 H.
\end{aligned}$$

Finally, choosing $\delta < \varepsilon/(2\widetilde{C}_\gamma)$ in (7) we get by (10),

$$\|\widetilde{\psi}-\psi\|_{0,h}^{(M_p)}=\sup_{y\in\Gamma}\sup_{\alpha\in\mathbb{N}_0}\frac{|D^\alpha f\circ\mu(y)|}{h^\alpha M_\alpha}\le\sup_{\alpha\in\mathbb{N}_0}\frac{\widetilde{C}_\gamma L^\alpha}{h^\alpha}\|f\|_{\bar{I},h_1}^{(M_p)}<\frac{\varepsilon}{2}.$$

To end this section we quote the following version of

SEELEY EXTENSION THEOREM ([4], [7]). *Let $\Gamma=[v,\infty)$, $\Gamma_1=[v_1,\infty)$ with $v_1<v$ and $a\in\mathbb{R}$. Then there exists a linear continuous extension operator*

$$\mathcal{E}:L_a^{(M_p)}(\Gamma)\to L_a^{(M_p)}(\Gamma_1) \tag{12}$$

such that for every $\varphi\in L_a^{(M_p)}(\Gamma)$, $\operatorname{supp}\mathcal{E}\varphi\subset(v_1,\infty)$.

COROLLARY 1. *Let S be a linear functional on $L_{(\omega)}^{(M_p)}(\Gamma)$. Then $S\in L_{(\omega)}^{(M_p)\prime}(\Gamma)$ if and only if for every $a<\omega$ there exist $h>0$ and $C<\infty$ such that*

$$|S[\varphi]|\le C\|\varphi\|_{a,h}^{(M_p)}\quad \textit{for } \varphi\in L_a^{(M_p)}(\Gamma). \tag{13}$$

2. The Paley–Wiener type theorem for Laplace ultradistributions. We assume the conditions (M.1), (M.2) and (M.3). Let $\Gamma=[v,\infty)$ with $v\in\mathbb{R}$. By Example 2 the function $\exp_z$ belongs to $L_{(\omega)}^{(M_p)}(\Gamma)$ if and only if $\operatorname{Re}z<\omega$. Hence we can define the *Laplace transform* of $S\in L_{(\omega)}^{(M_p)\prime}(\Gamma)$ by

$$\mathcal{L}S(z)=S[\exp_z]\quad \text{for } \operatorname{Re}z<\omega.$$

Since the mapping

$$\{\operatorname{Re}z<\omega\}\ni z\to\exp_z\in L_{(\omega)}^{(M_p)}(\Gamma)$$

is holomorphic, $\mathcal{L}S$ is a holomorphic function on $\{\operatorname{Re}z<\omega\}$.

Define

$$\begin{aligned}\mathcal{O}_v^{(M_p)}&(\operatorname{Re}z<\omega)\\ &=\{F\in\mathcal{O}(\operatorname{Re}z<\omega):\\ &\qquad\text{for every } a<\omega \text{ there exist } h>0 \text{ and } C<\infty \text{ such that}\\ &\qquad|F(z)|\le C\exp\{v\operatorname{Re}z+M(|z|/h)\}\text{ for } \operatorname{Re}z\le a\}.\end{aligned} \tag{14}$$

Applying Corollary 1 with $\varphi=\exp_z$ and $\operatorname{Re}z\le a$, by Example 2 we get

THEOREM 1. *Let $S\in L_{(\omega)}^{(M_p)\prime}(\Gamma)$ and $F(z)=\mathcal{L}S(z)$ for $\operatorname{Re}z<\omega$. Then $F\in\mathcal{O}_v^{(M_p)}(\operatorname{Re}z<\omega)$.*

THEOREM 2. *Let $\omega_1\le\omega_2$, $S_1\in L_{(\omega_1)}^{(M_p)\prime}(\Gamma)$ and $S_2\in L_{(\omega_2)}^{(M_p)\prime}(\Gamma)$. If*

$$\mathcal{L}S_1(z)=\mathcal{L}S_2(z)\quad \textit{for } \operatorname{Re}z<\omega_1 \tag{15}$$

then $S_1=S_2$ in $L_{(\omega_1)}^{(M_p)\prime}(\Gamma)$.

P r o o f. We have to prove that for arbitrary $\varphi \in L^{(M_p)}_{(\omega_1)}(\Gamma)$, $S_1[\varphi] = S_2[\varphi]$. To this end fix $\varphi \in L^{(M_p)}_{(\omega_1)}(\Gamma)$, choose $b < \omega_1$ such that $\varphi \in L^{(M_p)}_{b}(\Gamma)$ and take $b < a < \omega_1$. Since by (15), $S_1[\exp_c] = S_2[\exp_c]$ for $c \le a$ the proof follows from Proposition 1.

To prove the converse of Theorem 1 we need two lemmas. The first one is a restatement of Lemma 9.1 of [11] (cf. also [12]).

LEMMA 2. *Let $a \in \mathbb{R}$. Suppose that G is holomorphic on the set $\{\operatorname{Re} z \le b\}$ and satisfies there the estimate*

$$|G(z)| \le \frac{C}{\langle z\rangle^2} e^{v \operatorname{Re} z}$$

with some $C < \infty$, $v \in \mathbb{R}$. Put

$$g(y) = \frac{1}{2\pi i} \int\limits_{c+i\mathbb{R}} G(z) e^{-zy}\, dz \qquad \text{for } y \in \mathbb{R}.$$

Then g does not depend on the choice of $c \le b$; it is a continuous function on $\mathbb{R}$ with support in $\Gamma = [v, \infty)$; the function $\Gamma \ni y \to e^{by} g(y)$ is bounded; $g \in L^{(M_p)\prime}_{(b)}(\Gamma)$ and $G(z) = \mathcal{L}g(z)$ for $\operatorname{Re} z < b$.

LEMMA 3. *Let $\widetilde{\omega} \in \mathbb{R}$ and $k > 0$. Then there exists a symbol P of class (M_p) not vanishing on $\{\operatorname{Re} z < \widetilde{\omega} + 1\}$ such that*

$$\frac{\exp M(k|z|)}{P(z)} \le \frac{1}{\langle z\rangle^2} \qquad \text{for } \operatorname{Re} z \le \widetilde{\omega}. \tag{16}$$

P r o o f. Since $m_p \to \infty$ as $p \to \infty$ (by (M.3′)) we can find $p_0 \in \mathbb{N}$ such that $m_p > 2k|\widetilde{\omega}| + k$ and $|m_p - kz| \ge k|z|$ for $p \ge p_0$ and $\operatorname{Re} z \le \widetilde{\omega}$. Put

$$P(z) = (z - \widetilde{\omega} - 1)^{p_0+1} \prod_{p=p_0}^{\infty} \left(1 - \frac{kz}{m_p}\right) \qquad \text{for } z \in \mathbb{C}.$$

Then P does not vanish on $\{\operatorname{Re} z < \widetilde{\omega}\}$ and by the Hadamard factorization theorem (cf. [2], Propositions 4.5 and 4.6) it is a symbol of class (M_p). On the other hand, if $\operatorname{Re} z \le \widetilde{\omega}$ we estimate from below:

$$\begin{aligned}
\left| \prod_{p=p_0}^{\infty} \left(1 - \frac{kz}{m_p}\right) \right| &\ge \prod_{p=p_0}^{\infty} \left(1 - \frac{k|\widetilde{\omega}|}{m_p}\right) \sup_{q \ge p_0} \prod_{p=p_0}^{q} \left|1 - \frac{kz}{m_p}\right| \\
&\ge \prod_{p=p_0}^{\infty} \left(1 - \frac{k|\widetilde{\omega}|}{m_p}\right) \sup_{q \ge p_0} \prod_{p=p_0}^{q} \frac{k|z|}{m_p} \\
&= C|z|^{-p_0+1} \exp M(k|z|),
\end{aligned}$$

where

$$C = \frac{M_{p_0-1}}{M_0} \prod_{p=p_0}^{\infty} \left(1 - \frac{k|\widetilde{\omega}|}{m_p}\right) > 0.$$

Hence, possibly multiplying P by a suitable constant, we get (16).

THEOREM 3. *Let* $\omega \in \mathbb{R} \cup \{\infty\}$ *and let* $F \in \mathcal{O}_v^{(M_p)}(\operatorname{Re} z < \omega)$. *Then there exists a Laplace ultradistribution* $S \in L_{(\omega)}^{(M_p)\prime}(\Gamma)$ *such that*

$$F(z) = \mathcal{L}S(z) \quad \text{for } \operatorname{Re} z < \omega. \tag{17}$$

Proof. Fix $a < \omega$. Choose $\widetilde{\omega} \in \mathbb{R}$ such that $a < \widetilde{\omega} < \omega$ and assume (14). By Lemma 3 we can find a symbol P of class (M_p), not vanishing on $\{\operatorname{Re} z < \widetilde{\omega} + 1\}$ and satisfying (16). Next we apply Lemma 2 to the function

$$G(z) = \frac{F(z)}{P(z)}, \quad \operatorname{Re} z \le \widetilde{\omega}.$$

We get a continuous function g which belongs to $L_{(a)}^{(M_p)\prime}(\Gamma)$ and satisfies $\mathcal{L}g(z) = G(z)$ for $\operatorname{Re} z < a$. Put $S = P(-D)g$. Then $S \in L_{(a)}^{(M_p)\prime}(\Gamma)$ and $\mathcal{L}S(z) = P(z)\mathcal{L}g(z) = F(z)$ for $\operatorname{Re} z < a$.

Thus for every $a < \omega$ we can find $S_a \in L_{(a)}^{(M_p)\prime}(\Gamma)$ such that $\mathcal{L}S_a(z) = F(z)$ for $\operatorname{Re} z < a$. By Theorem 2 the definition $S = S_a$ on $L_{(a)}^{(M_p)}(\Gamma)$, $a < \omega$, defines correctly a functional $S \in L_{(\omega)}^{(M_p)\prime}(\Gamma)$ which satisfies (17).

It follows from the proof of Theorem 3 that Laplace ultradistributions can be characterized as follows.

THEOREM 4 (Structure theorem). *An ultradistribution* $S \in D^{(M_p)\prime}(\mathbb{R})$ *is in* $L_{(\omega)}^{(M_p)\prime}(\Gamma)$ *if and only if for every* $a < \omega$ *there exist an ultradifferential operator* P_a *of class* (M_p) *and a function* g_a *continuous on* $\mathbb{R}$ *with support in* Γ *such that*

$$|g_a(y)| \le Ce^{-ay} \qquad \text{for } y \in \Gamma,$$

$$|\mathcal{L}g_a(z)| \le \frac{C}{\langle z \rangle^2} e^{v \operatorname{Re} z} \qquad \text{for } \operatorname{Re} z < a$$

and

$$S = P_a(D)g_a \quad \text{in } L_{(a)}^{(M_p)\prime}(\Gamma). \tag{18}$$

3. Boundary value representation. In this section we use the following version of the Phragmén–Lindelöf theorem.

3-LINE THEOREM ([1]). *Let* $R > 0$ *and* $F \in \mathcal{O}(\Gamma_R) \cap C^0(\overline{\Gamma}_R)$. *Suppose that for some* $k > 0$ *the function*

$$\overline{\Gamma}_R \ni z \to \exp\{-k|z|\}F(z)$$

is bounded. If F is bounded on the boundary of $\overline{\varGamma}_R$ then it is also bounded on $\varGamma_R$.

Definition. Let $R>0$, $k>0$ and $a\in\mathbb{R}$. We define the spaces

$$\widetilde{L}_a(\varGamma_R)=\{\varphi\in\mathcal{O}(\varGamma_R)\cap C^0(\overline{\varGamma}_R):\|\varphi\|_{a,\varGamma_R}=\sup_{z\in\overline{\varGamma}_R}|\varphi(z)e^{az}|<\infty\},$$

$$\widetilde{L}^{(M_p)}_{a,k}(\varGamma_R\setminus\varGamma)=\{\varphi\in\mathcal{O}(\varGamma_R\setminus\varGamma):\varphi\cdot\exp\{-M^*(k/|\operatorname{Im}z|)\}\in C^0(\overline{\varGamma}_R),$$
$$\|\varphi\|^{(M_p)}_{a,k,\varGamma_R}=\sup_{z\in\overline{\varGamma}_R}|\varphi(z)\exp\{az-M^*(k/|\operatorname{Im}z|)\}|<\infty\}.$$

By the 3-line theorem $\widetilde{L}_a(\varGamma_R)$ is a closed subspace of the Banach space $\widetilde{L}^{(M_p)}_{a,k}(\varGamma_R\setminus\varGamma)$ and we can define

$$H^{(M_p)}_{a,k}(\varGamma_R,\varGamma)=\widetilde{L}^{(M_p)}_{a,k}(\varGamma_R\setminus\varGamma)/\widetilde{L}_a(\varGamma_R).$$

Further, we define

$$\widetilde{L}_a(\mathbb{C})=\varprojlim_{R\to\infty}\widetilde{L}_a(\varGamma_R),\qquad \widetilde{L}^{(M_p)}_{a,k}(\mathbb{C}\setminus\varGamma)=\varprojlim_{R\to\infty}\widetilde{L}^{(M_p)}_{a,k}(\varGamma_R\setminus\varGamma),$$
$$\widetilde{L}^{(M_p)}_{a}(\mathbb{C}\setminus\varGamma)=\varinjlim_{k\to\infty}\widetilde{L}^{(M_p)}_{a,k}(\mathbb{C}\setminus\varGamma),$$
$$\widetilde{H}^{(M_p)}_{a}(\mathbb{C},\varGamma)=\widetilde{L}^{(M_p)}_{a}(\mathbb{C}\setminus\varGamma)/\widetilde{L}_a(\mathbb{C}),\qquad \underset{\sim}{H}^{(M_p)}_{a}(\varGamma)=\varinjlim_{\substack{R\to0\\ k\to\infty}}H^{(M_p)}_{a,k}(\varGamma_R,\varGamma).$$

Let $a<b$. Then the natural mappings

$$\widetilde{H}^{(M_p)}_{b}(\mathbb{C},\varGamma)\to\widetilde{H}^{(M_p)}_{a}(\mathbb{C},\varGamma),\qquad \underset{\sim}{H}^{(M_p)}_{b}(\varGamma)\to\underset{\sim}{H}^{(M_p)}_{a}(\varGamma)$$

are well defined and by the 3-line theorem they are injections. Thus, for $\omega\in\mathbb{R}\cup\{\infty\}$, we can define

$$\widetilde{H}^{(M_p)}_{(\omega)}(\mathbb{C},\varGamma)=\varprojlim_{a<\omega}\widetilde{H}^{(M_p)}_{a}(\mathbb{C},\varGamma),\qquad \underset{\sim}{H}^{(M_p)}_{(\omega)}(\varGamma)=\varprojlim_{a<\omega}\underset{\sim}{H}^{(M_p)}_{a}(\varGamma).$$

An element $f\in\widetilde{H}^{(M_p)}_{(\omega)}(\mathbb{C},\varGamma)$ is given by a set $\{F_a\}_{a<\omega}$ of functions such that for $a<\omega$, $F_a\in\widetilde{L}^{(M_p)}_{a}(\mathbb{C}\setminus\varGamma)$ and for $a<b<\omega$, $F_a-F_b\in\widetilde{L}_a(\mathbb{C})$. On the other hand, an element $g\in\underset{\sim}{H}^{(M_p)}_{(\omega)}(\varGamma)$ is given by a set $\{G_a\}_{a<\omega}$ of functions such that for $a<\omega$ there exist $R_a>0$ and $k_a<\infty$ such that $G_a\in\widetilde{L}^{(M_p)}_{a,k_a}(\varGamma_{R_a}\setminus\varGamma)$ and for $a<b<\omega$, $G_a-G_b\in\widetilde{L}_a(\varGamma_{R_a}\cap\varGamma_{R_b},\varGamma)$.

The natural mapping

$$i:\widetilde{H}^{(M_p)}_{(\omega)}(\mathbb{C},\varGamma)\to\underset{\sim}{H}^{(M_p)}_{(\omega)}(\varGamma)\tag{19}$$

is defined by retaining the same set of defining functions. Obviously it is an injection.

LEMMA 4. *Let* $S\in L^{(M_p)\prime}_{(\omega)}(\Gamma)$ *and* $a<\omega$. *Put*

$$\mathcal{C}_aS(z)=\frac{1}{2\pi i}S\left[\frac{e^{a(\cdot-z)}}{z-\cdot}\right]\quad \text{for } z\in\mathbb{C}\setminus\Gamma.$$

Then $\mathcal{C}_aS\in\widetilde{L}^{(M_p)}_a(\mathbb{C}\setminus\Gamma)$. *Furthermore, if* $a<b<\omega$ *then* $\mathcal{C}_aS-\mathcal{C}_bS\in\widetilde{L}_a(\mathbb{C})$.

Proof. Take $a<c<\omega$. By Theorem 4 we can find an ultradifferential operator P_c of class (M_p) and a continuous function g_c with support in Γ satisfying $|g_c(y)|\le Ce^{-cy}$ for $y\in\Gamma$ and $S=P_c(D)g_c$ in $L^{(M_p)\prime}_{(c)}(\Gamma)$. Since for fixed $z\in\mathbb{C}\setminus\Gamma$ the function

$$\Gamma\ni y\to\frac{e^{a(y-z)}}{z-y}$$

belongs to $L^{(M_p)}_{(c)}(\Gamma)$ we have

$$\mathcal{C}_a(z)=\frac{1}{2\pi i}e^{-az}\int_{\Gamma}g_c(y)P^*_c(D_y)\left(\frac{e^{ay}}{z-y}\right)dy.$$

Let $P^*_c(D)=\sum_{\alpha=0}^{\infty}(-1)^{\alpha}a_{\alpha}D^{\alpha}$ with a_α satisfying (5) and let $R>0$. Then for $z\in\Gamma_R\setminus\Gamma$ we estimate

$$\begin{aligned}\left|P^*_c(D)\left(\frac{e^{ay}}{z-y}\right)\right|&\le\sum_{\alpha=0}^{\infty}|a_\alpha|\left|D^{\alpha}\left(\frac{e^{ay}}{z-y}\right)\right|\le e^{ay}\sum_{\beta=0}^{\infty}\frac{|a|^{\beta}}{\beta!}\sum_{\alpha=\beta}^{\infty}\frac{|a_\alpha|\alpha!}{|\operatorname{Im}z|^{\alpha-\beta+1}}\\&\le e^{ay}\sum_{\beta=0}^{\infty}\frac{|a|^{\beta}R^{\beta}}{\beta!}\sum_{\alpha=0}^{\infty}\frac{|a_\alpha|\alpha!}{|\operatorname{Im}z|^{\alpha+1}}\\&\le e^{ay+|a|R}\cdot\frac{AC}{HK}\exp M^*\left(\frac{2HK}{|\operatorname{Im}z|}\right)\end{aligned}$$

since by (5) and (M.2′) we have

$$\begin{aligned}\sum_{\alpha=0}^{\infty}\frac{|a_\alpha|\alpha!}{|\operatorname{Im}z|^{\alpha+1}}&\le C\sum_{\alpha=0}^{\infty}\frac{K^{\alpha}\alpha!}{|\operatorname{Im}z|^{\alpha+1}M_\alpha}\le 2C\sup_{\alpha\in\mathbb{N}_0}\frac{(2K)^{\alpha}\alpha!}{|\operatorname{Im}z|^{\alpha+1}M_\alpha}\\&\le\frac{2AC}{2HK}\sup_{\alpha\in\mathbb{N}_0}\frac{(2HK)^{\alpha+1}\alpha!}{|\operatorname{Im}z|^{\alpha+1}M_{\alpha+1}}\le\frac{AC}{HK}\exp M^*\left(\frac{2HK}{|\operatorname{Im}z|}\right).\end{aligned}$$

Put $k=2HK$. Then for every $R>0$ there exists $C<\infty$ such that

$$|\mathcal{C}_aS(z)|\le C\exp\{-a\operatorname{Re}z+M^*(k/|\operatorname{Im}z|)\}\quad\text{for } z\in\Gamma_R\setminus\Gamma.$$

Thus, $\mathcal{C}_aS\in\widetilde{L}^{(M_p)}_a(\mathbb{C}\setminus\Gamma)$. If $a<b<\omega$ we take $c<\omega$ such that $b<c$ and

note that for $z \in \mathbb{C}$ the function

$$\Gamma \ni y \to \frac{e^{a(y-z)} - e^{b(y-z)}}{z-y}$$

belongs to $L_{(c)}^{(M_p)}(\Gamma)$. Thus, the holomorphic extension of $\mathcal{C}_a S - \mathcal{C}_b S$ is given by

$$(\mathcal{C}_a S - \mathcal{C}_b S)(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} g_c(y) P_c^*(D) \left(\frac{e^{a(y-z)} - e^{b(y-z)}}{z-y} \right) dy$$

and we easily find that $\mathcal{C}_a - \mathcal{C}_b \in \widetilde{L}_a(\mathbb{C})$.

DEFINITION. Let $S \in L_{(\omega)}^{(M_p)\prime}(\Gamma)$. Then by Lemma 4 the set $\{\mathcal{C}_a S\}_{a<\omega}$ of functions defines an element $f \in \widetilde{H}_{(\omega)}^{(M_p)}(\Gamma)$. We call f the *Cauchy transform* of S and write $f = \mathcal{C}S$.

PROPOSITION 2. *Let $F_a \in \widetilde{L}_{a,k}^{(M_p)}(\Gamma_R \backslash \Gamma)$ with $a \le 0$. Then there exist an ultradifferential operator $P_a(D)$ of class (M_p) and functions $H_a^{\pm} \in \mathcal{O}(\Gamma_R \cap \{\pm \operatorname{Im} z > 0$ or $\operatorname{Re} z < v\})$ such that*

1° $P_a(D) H_a^{\pm} = F_a$;

2° *For every $0 < R' < R$ and $a' < a$ there exists $C < \infty$ such that*

$$|H_a^{\pm}(z)| \le C \exp\{-a' \operatorname{Re} z\} \quad \text{for } z \in \Gamma_{R'} \cap \{\pm \operatorname{Im} z > 0\};$$

3° *$H_a^{\pm}(x+iy)$ converges uniformly as $y \to 0+$ to a function $h_a^{\pm}$ continuous on $(v-R, \infty)$ and analytic on $(v-R, v)$ satisfying*

$$|h_a^{\pm}(x)| \le C \exp\{-a' x\} \quad \text{for } x \in (v-R', \infty).$$

Furthermore, if we put

$$S_a = F_a^{+}(x+i0) - F_a^{-}(x-i0), \quad \text{where } F_a^{\pm}(x \pm i0) = P_a(D) h_a^{\pm},$$

then S_a extends to a Laplace ultradistribution $\widetilde{S}_a \in L_{(a)}^{(M_p)\prime}(\Gamma)$ defined by

$$(20) \quad \widetilde{S}_a[\varphi] = \int\limits_{v-R}^{\infty} (h_a^{+}(x) - h_a^{-}(x)) P_a^*(D) \mathcal{E}\varphi(x)\, dx \quad \text{for } \varphi \in L_{(a)}^{(M_p)}(\Gamma).$$

In (20), $\mathcal{E}$ is a linear continuous extension mapping $\mathcal{E} : L_{(a)}^{(M_p)}(\Gamma) \to L_{(a)}^{(M_p)}([v-R, \infty))$, which exists by the Seeley extension theorem.

Proof. Put

$$P(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} \left(1 + \frac{k\zeta}{m_p} \right) \quad \text{for } \zeta \in \mathbb{C}.$$

Then P is a symbol of class (M_p). Define the Green kernel for P by

$$G(z)=\frac{1}{2\pi i}\int_0^\infty \frac{e^{z\zeta}}{P(\zeta)}\,d\zeta \quad \text{for } \operatorname{Re} z<0.$$

Then by Lemma 11.4 of [2], G can be holomorphically continued to the Riemann domain $\{z:-\pi/2<\arg z<5\pi/2\}$, on which we have

$$P(D)G(z)=-\frac{1}{2\pi i}\frac{1}{z}\,.$$

Furthermore, since for any $0\le\psi<\pi/2$,

$$\frac{|z|}{|1-z|}\le\frac{1}{\cos\psi} \quad\text{and}\quad \frac{1}{|1-z|}\le\frac{1}{\cos\psi}$$

for $z\in\mathbb{C}$ with $|\arg z|\le\psi+\pi/2$, following the proof of the above-mentioned lemma we conclude that on the domain $\{-\psi\le\arg z\le 2\pi+\psi\}$, G is bounded by $C/\cos\psi$ with $C<\infty$ not depending on ψ. We also have

$$|g(x)|\le A\sqrt{x}\exp\{-M^*(k/x)\} \quad \text{for } x>0,$$

where

$$g(z)=G_+(z)-G_-(e^{2\pi i}z) \quad \text{for } \operatorname{Re} z>0,$$

G_+ being the branch of G on $\{-\pi/2<\arg z<\pi/2\}$ and G_- that on $\{3\pi/2<\arg z<5\pi/2\}$. Put

$$H_a^\pm(z)=\pm i\int_{\gamma_\pm} G(\pm i(z-w))F_a(w)\,dw,$$

where $\gamma_\pm$ is a closed curve encircling z once, in the anticlockwise direction, such that $-\pi/2<\arg(\pm i(z-w))<5\pi/2$ for $w\in\gamma_\pm$. We choose a starting point $\mathring{z}_\pm$ of $\gamma_\pm$ in such a way that $|\arg\{\pm i(z-\mathring{z}_\pm)\}|<\pi/2$. Then $H_a^\pm$ is a holomorphic function on $\varGamma_R\cap\{\pm\operatorname{Im} z>0 \text{ or } \operatorname{Re} z<v\}$ and does not depend on the choice of $\gamma_\pm$ with a fixed starting point $\mathring{z}_\pm$. For a fixed γ and z changing in a compact set in the domain bounded by γ we have

$$\begin{aligned}P(D)H_a^\pm(z)&=\pm i\int_\gamma P(D_z)G(\pm i(z-w))F_a(w)\,dw\\&=\frac{-1}{2\pi i}\int_\gamma\frac{F_a(w)}{z-w}\,dw=F_a(z).\end{aligned}$$

Let $0<R'<R$ and $z\in\varGamma_{R'}\cap\{\operatorname{Im} z>0\}$. Fix $\mathring{z}\in\varGamma_R\cap\{\operatorname{Im} z>r'\}\cap\{\operatorname{Re} z<v-R'\}$ and take $\gamma_+=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$, where $\gamma_1=[\mathring{x}+i\mathring{y},x+i\mathring{y}]$, $\gamma_2=[x+i\mathring{y},x+iy]$, $\gamma_3=[x+iy,x+i\mathring{y}]$ and $\gamma_4=[x+i\mathring{y},\mathring{x}+i\mathring{y}]$. Since $0\le\arg(i(z-w))\le\psi$ for $w\in\gamma_1$ and $2\pi\le\arg(i(z-w))\le 2\pi+\psi$ for $w\in\gamma_4$, where $0\le\psi<\pi/2$ is such that $\tan\psi=(x-\mathring{x})/(\mathring{y}-y)$, by the boundedness of G on $\{0 \le \arg\psi \le 2\pi + \psi\}$ we have $|G(i(z-w))| \le Cx$ for $w \in \gamma_1 \cup \gamma_4$, where C does not depend on x. So

$$\Big|\int\limits_{\gamma_1\cup\gamma_4} G(i(z-w))F_a(w)\,dw\Big| \le Cx^2e^{-ax} \qquad \text{for } z \in \Gamma_{R'} \cap \{\operatorname{Im} z > 0\}.$$

On the other hand,

$$\begin{aligned}\Big|\int\limits_{\gamma_2\cup\gamma_3} G(i(z-w))F_a(w)\,dw\Big| &= \Big| -i\int\limits_0^{\mathring{y}-y} g(t)F(x+i(y+t))\,dt\Big| \\ &\le AC\int\limits_0^{\mathring{y}-y} \sqrt{t}\exp\Big\{M^*\Big(\frac{k}{y+t}\Big) - M^*\Big(\frac{k}{t}\Big) - ax\Big\}dt \\ &\le AC(\mathring{y}-y)^{3/2}e^{-ax} \qquad \text{for } z \in \Gamma_{R'} \cap \{\operatorname{Im} z > 0\}.\end{aligned}$$

Thus, for any $a' < a$ one can find $C < \infty$ such that

$$|H_a^+(z)| \le Ce^{-a'x} \qquad \text{for } z \in \Gamma_{R'} \cap \{\operatorname{Im} z > 0\}.$$

The estimate of H_a^- is obtained in an analogous way.

The assertion 3° is clear from the above estimates.

Let $\psi \in D^{(M_p)}((v-R,\infty))$. By 1° and 2° we derive

$$\begin{aligned}F_a(x \pm i0)[\psi] = P_a(D)h_a^{\pm}[\psi] &= \int\limits_{v-R}^{\infty} h_a^{\pm}(x)P_a^*(D)\psi(x)\,dx \\ &= \lim_{y\to 0+}\int\limits_{v-R}^{\infty} H_a(x\pm iy)P_a^*(D)\psi(x)\,dx \\ &= \lim_{y\to 0+}\int\limits_{v-R}^{\infty} F_a(x\pm iy)\psi(x)\,dx.\end{aligned}$$

Since for $\psi \in D^{(M_p)}((v-R,v))$,

$$S_a[\psi] = \lim_{y\to 0+}\int\limits_{v-R}^{v} P_a(D)(H_a^+(x+iy) - H_a^-(x-iy))\psi(x)\,dx = 0,$$

S_a has support in Γ and we can define the extension of S_a by (20).

Let $f \in \underset{\sim}{H}{}_a^{(M_p)}(\Gamma)$. Then there exist $R > 0$, $k < \infty$ and a function $F_a \in \underset{\sim}{\widetilde{L}}{}_{a,k}^{(M_p)}(\Gamma_R \setminus \Gamma)$ such that $f = [F_a]$. If $a \le 0$ we can apply Proposition 2 to F_a. If $a > 0$ then we apply Proposition 2 to $F_a^* = e^{az}F_a$ instead of F_a. In this case denote by $\widetilde{S}_a^*$ the element of $L_{(0)}^{(M_p)\prime}(\Gamma)$ given by (20) and define $\widetilde{S}_a = e^{-ax}\widetilde{S}_a^*$. In both cases $\widetilde{S}_a \in L_{(a)}^{(M_p)\prime}(\Gamma)$ does not depend on the choice of a defining function F_a for f. Thus, the assignment $f \to \widetilde{S}_a$ defines a mapping

$$b : \underset{\sim}{H}_a^{(M_p)}(\Gamma) \to L_{(a)}^{(M_p)\prime}(\Gamma). \tag{21}$$

Since (21) holds for every $a < \omega$ we have

$$b : \underset{\sim}{H}_{(\omega)}^{(M_p)} \to \varprojlim_{a<\omega} L_{(a)}^{(M_p)\prime}(\Gamma) \simeq (\varinjlim_{a<\omega} L_{(a)}^{(M_p)}(\Gamma))' = L_{(\omega)}^{(M_p)\prime}(\Gamma),$$

where the isomorphism $\simeq$ follows by the formula (1.2) of [2].

THEOREM 5. *The mapping*

$$\mathcal{C} : L_{(\omega)}^{(M_p)\prime}(\Gamma) \to \widetilde{H}_{(\omega)}^{(M_p)}(\mathbb{C}, \Gamma)$$

is a topological isomorphism with inverse $b \circ i$.

Proof. Let $S \in L_{(\omega)}^{(M_p)\prime}(\Gamma)$ and $\mathcal{C}S = f \in \widetilde{H}_{(\omega)}^{(M_p)}(\Gamma)$. Then

$$f = [\{F_a\}_{a<\omega}] \quad \text{with} \quad F_a(z) = \frac{1}{2\pi i} S\left[\frac{e^{a(\cdot - z)}}{z - \cdot}\right] \quad \text{for } z \in \mathbb{C} \setminus \Gamma.$$

Treat f as an element of $\underset{\sim}{H}_{(\omega)}^{(M_p)}(\Gamma)$ and put $\widetilde{S} = b(f) \in L_{(\omega)}^{(M_p)\prime}(\Gamma)$. Fix $a < \omega$. Then for $\varphi \in Y_{(a)}$ we have

$$\widetilde{S}[\varphi] = -\int_{\partial\Gamma_\varepsilon} F_a(z)\varphi(z)\,dz = S\left[-\frac{1}{2\pi i}\int_{\partial\Gamma_\varepsilon} \frac{e^{a(\cdot - z)}}{z - \cdot}\varphi(z)\,dz\right] = S[\varphi],$$

by the Cauchy integral formula. Since $Y_{(a)}$ is dense in $L_{(a)}^{(M_p)}(\Gamma)$ and $a < \omega$ is arbitrary, we have $\widetilde{S}[\varphi] = S[\varphi]$ for $\varphi \in L_{(\omega)}^{(M_p)}(\Gamma)$. Thus $b \circ i \circ \mathcal{C} = \mathrm{id}$.

Let $f \in \widetilde{H}_{(\omega)}^{(M_p)}(\mathbb{C}, \Gamma)$, $f = [\{F_a\}_{a<\omega}]$ with $F_a \in \widetilde{L}_a^{(M_p)}(\mathbb{C} \setminus \Gamma)$ and $F_a - F_b \in \widetilde{L}_a(\mathbb{C})$ for $a < b < \omega$. Put $\underset{\sim}{f} = i(f)$ and fix $a < \omega$. Then $\underset{\sim}{f} = [F_a]$ in $\underset{\sim}{H}_a^{(M_p)}(\Gamma)$ and by (21), $\widetilde{S}_a = b(\underset{\sim}{f}) \in L_{(a)}^{(M_p)\prime}(\Gamma)$. Observe that for $\varepsilon > 0$ we have

$$\widetilde{S}_a[\varphi] = -\int_{\partial\Gamma_\varepsilon} F_a(z)\varphi(z)\,dz \quad \text{for } \varphi \in Y_{(a)}.$$

On the other hand, by the part of the theorem just proved,

$$\widetilde{S}_a[\varphi] = -\int_{\partial\Gamma_\varepsilon} \frac{1}{2\pi i}\widetilde{S}_a\left[\frac{e^{a(\cdot - z)}}{z - \cdot}\right]\varphi(z)\,dz \quad \text{for } \varphi \in Y_{(a)}.$$

So for $\varphi \in Y_{(a)}$,

$$\int_{\partial\Gamma_\varepsilon} \psi_a(z)\varphi(z)\,dz = 0, \quad \text{where} \quad \psi_a(z) = \frac{1}{2\pi i}\widetilde{S}_a\left[\frac{e^{a(\cdot - z)}}{z - \cdot}\right] - F_a(z). \tag{22}$$

Then $\psi_a \in \widetilde{L}^{(M_p)}_{a,k}(\mathbb{C}\setminus\Gamma)$ and we shall show that ψ_a extends holomorphically to a function $\widetilde{\psi}_a \in \widetilde{L}_a(\mathbb{C})$, which proves that $\mathcal{C}\circ b\circ i = \mathrm{id}$. To this end observe that (22) holds also for $\varphi \in L^{(M_p)}_{(a)}(\Gamma)\cap\mathcal{O}(\Gamma_\varepsilon)$ and put for any $b < a$, $R > \varepsilon$,

$$G_b(z) = \int\limits_{\partial\Gamma_R} \psi_a(\zeta)\frac{e^{b(\zeta-z)}}{z-\zeta}\,d\zeta \qquad \text{for } z\in\Gamma_R.$$

Then $|G_b(z)| \le C\exp\{-b\operatorname{Re} z\}$ for $z \in \overline{\Gamma}_{R'}$ with $R' < R$. Using (22) with $\varphi(\zeta) = \exp\{b(\zeta - z)\}/(z-\zeta)$, $z \in \Gamma_R\setminus\overline{\Gamma}_\varepsilon$, we get $G_b(z) = \psi_a(z)$ for $z\in\Gamma_R\setminus\overline{\Gamma}_\varepsilon$. Put

$$\widetilde{\psi}_a(z) = \begin{cases} \psi_a(z) & \text{for } z\in\Gamma_R\setminus\Gamma,\\ G_b(z) & \text{for } z\in\Gamma_R.\end{cases}$$

Then $\widetilde{\psi}_a \in \mathcal{O}(\Gamma_R)$ and by the 3-line theorem $\widetilde{\psi}_a\in\widetilde{L}_a(\Gamma_R)$. Since $R>\varepsilon$ was arbitrary we have $\widetilde{\psi}\in\widetilde{L}_a(\mathbb{C})$.

4. Mellin ultradistributions

Definition. Let $\omega\in\mathbb{R}\cup\{\infty\}$, $v\in\mathbb{R}$ and $I=(0,e^{-v}]$. We define the space $M^{(M_p)\prime}_{(\omega)}(I)$ of *Mellin ultradistributions* as the dual space of

$$M^{(M_p)}_{(\omega)}(I) = \varinjlim_{a<\omega}\varprojlim_{h>0} M^{(M_p)}_{a,h}(I),$$

where for any $a\in\mathbb{R}$ and $h>0$,

$$M^{(M_p)}_{a,h}(I) = \Big\{\psi\in C^\infty(I) : \varrho^{(M_p)}_{a,h}(\psi) = \sup_{x\in I}\sup_{\alpha\in\mathbb{N}_0}\frac{|x^{a+1}(Dx)^\alpha\psi(x)|}{h^\alpha M_\alpha} < \infty\Big\}.$$

Lemma 5. *Let $a\in\mathbb{R}$, $h>0$, $\psi\in M^{(M_p)}_{a,h}(I)$ and $\varphi=\mu\cdot\psi\circ\mu$. Then $\varphi\in L^{(M_p)}_{a,h}(\Gamma)$ and $\|\varphi\|^{(M_p)}_{a,h} = \varrho^{(M_p)}_{a,h}(\psi)$. Thus, the mapping*

$$M^{(M_p)}_{(\omega)}(I)\ni\psi\to\mu\cdot\psi\circ\mu\in L^{(M_p)}_{(\omega)}(\Gamma)$$

is a continuous isomorphism with inverse

$$L^{(M_p)}_{(\omega)}(\Gamma)\ni\varphi\to\exp_1\circ\mu^{-1}\cdot\varphi\circ\mu^{-1}\in M^{(M_p)}_{(\omega)}(I).$$

Proof. The proof follows easily from the formula

$$D^\alpha_y(\mu(y)\psi\circ\mu(y)) = (-1)^\alpha x(D_x x)^\alpha\psi(x), \qquad \text{for } \alpha\in\mathbb{N}_0,\ x=\mu(y),$$

which can be proved by induction.

Let $S\in L^{(M_p)\prime}_{(\omega)}(\Gamma)$. Put

$$S\circ\mu^{-1}[\psi] = S[\mu\cdot\psi\circ\mu] \qquad \text{for } \psi\in M^{(M_p)}_{(\omega)}(I).$$

Then by Lemma 5, $S\circ\mu^{-1}$ is a well defined element of $M_{(\omega)}^{(M_p)\prime}(I)$ and the mapping

$$L_{(\omega)}^{(M_p)\prime}(\Gamma)\ni S\to S\circ\mu^{-1}\in M_{(\omega)}^{(M_p)\prime}(I)$$

is continuous.

Observe that the function

$$I\ni x\to x^{-z-1}=\exp_{z+1}\circ\mu^{-1}(x)$$

belongs to $M_{(\omega)}^{(M_p)}(I)$ if and only if $\operatorname{Re} z<\omega$. Thus, we can define the *Mellin transform* of $T\in M_{(\omega)}^{(M_p)\prime}(I)$ by

$$\mathcal{M}T(z)=T[\exp_{z+1}\circ\mu^{-1}]\quad \text{ for } \operatorname{Re} z<\omega.$$

Let $S\in L_{(\omega)}^{(M_p)\prime}(\Gamma)$ and $T=S\circ\mu^{-1}$. Then for $\operatorname{Re} z<\omega$ we have

$$\mathcal{M}T(z)=S\circ\mu^{-1}[\exp_{z+1}\circ\mu^{-1}]=S[\exp_z]=\mathcal{L}S(z).$$

5. Strong quasi-analyticity principle

Definition. Let $S\in L_{(\omega)}^{(M_p)\prime}(\Gamma)$. We define the *Taylor transform* of S by

$$\mathcal{T}S(x)=\mathcal{L}S(\ln x)\quad \text{ for } x\in\widetilde{B}(e^{\omega}).$$

We also define

$$\begin{aligned}&\mathcal{O}_v^{(M_p)}(\widetilde{B}(e^{\omega}))\\ &\quad=\{u\in\mathcal{O}(\widetilde{B}(e^{\omega})):\\ &\qquad \text{for every } t<e^{\omega} \text{ there exist } k<\infty \text{ and } C<\infty \text{ such that}\\ &\qquad |u(x)|\le C\exp\{M(k(\omega-\ln|x|+|\arg x|))\}\cdot|x|^v \text{ for } |x|\le t\}.\end{aligned}$$

By Theorems 1 and 3 we get

Theorem 6. *The Taylor transformation is an isomorphism of* $L_{(\omega)}^{(M_p)\prime}(\Gamma)$ *onto* $\mathcal{O}_v^{(M_p)}(\widetilde{B}(e^{\omega}))$.

Let $u\in\mathcal{O}_v^{(M_p)}(\widetilde{B}(e^{\omega}))$. Then for any $t<e^{\omega}$, $u_{|(0,t]}\in M_{(v)}^{(M_p)\prime}((0,t])$ and

$$\mathcal{M}_t u(z)=\int_0^t u(x)x^{-z-1}\,dx\quad \text{ for } \operatorname{Re} z<v.$$

By Theorem 6, $u(x)=S[x^{\cdot}]$ for $x\in\widetilde{B}(e^{\omega})$ with $S=\mathcal{T}^{-1}u\in L_{(\omega)}^{(M_p)\prime}(\Gamma)$, $\Gamma=[v,\infty)$. For $\operatorname{Re} z<v$ we derive

$$\mathcal{M}_t u(z)=S\Big[\int_0^t x^{\cdot-z-1}\,dx\Big]=S\bigg[\frac{t^{\cdot-z}}{\cdot-z}\bigg]=-2\pi i\mathcal{C}_{\ln t}S(z).$$

Thus, by Lemma 4, $\mathcal{M}_t u$ extends holomorphically to a function $\mathcal{M}_t u \in \widetilde{L}^{(M_p)}_{\ln t}(\mathbb{C} \setminus \Gamma)$ and the set of functions $\{\mathcal{M}_t u\}_{t<e^\omega}$ defines an element of $\widetilde{H}^{(M_p)}_{(\omega)}(\mathbb{C}, \Gamma)$, which will be denoted by $\mathcal{M}u$ and called the *Mellin transform* of u.

We can summarize Theorems 1, 3, 5 and 6 as follows:

COROLLARY 2. *We have the following diagram of linear topological isomorphisms*:

$$\begin{array}{ccc} M^{(M_p)\prime}_{(\omega)}(I) & \xrightarrow{\ \mathcal{M}\ } & \mathcal{O}^{(M_p)}_v(\operatorname{Re} z < \omega) \\ {\scriptstyle\circ\mu}\downarrow\uparrow{\scriptstyle\circ\mu^{-1}} & \nearrow{\scriptstyle\mathcal{L}} & \downarrow{\scriptstyle\circ-\mu^{-1}} \\ L^{(M_p)\prime}_{(\omega)}(\Gamma) & \xrightarrow{\ \mathcal{T}\ } & \mathcal{O}^{(M_p)}_v(\widetilde{B}(e^\omega)) \\ {\scriptstyle b}\uparrow & \searrow{\scriptstyle\mathcal{C}} & \downarrow{\scriptstyle\mathcal{M}} \\ \underset{\sim}{H}^{(M_p)}_{(\omega)}(\Gamma) & \xleftarrow{\ i\ } & \widetilde{H}^{(M_p)}_{(\omega)}(\mathbb{C}, \Gamma). \end{array}$$

Following [13] we call the elements of $\mathcal{O}^{(M_p)}_v(\operatorname{Re} z < \omega)$ *generalized analytic functions* determined by $L^{(M_p)\prime}_{(\omega)}(\Gamma)$. Generalized analytic functions have the following quasi-analyticity property:

THEOREM 7. *Let* $u \in \mathcal{O}^{(M_p)}_v(\widetilde{B}(e^\omega))$. *Suppose that for some* $t < e^\omega$ *and every* $m \in \mathbb{N}$ *there exist* C_m *such that*

$$|u(x)| \le C_m x^m \quad \text{for } 0 < x \le t.$$

Then $u \equiv 0$.

Proof. By Theorem 6, $u(x) = \mathcal{T}S(x)$ for $x \in \widetilde{B}(e^\omega)$ with some $S \in L^{(M_p)\prime}_{(\omega)}(\Gamma)$. The assumption that u is flat of arbitrary order $m \in \mathbb{N}$ on $(0, t)$ implies that $\mathcal{M}_t u \in \mathcal{O}(\mathbb{C})$. Since for every $R > 0$, $L^{(M_p)}_{a,k}(\Gamma_R) \cap \mathcal{O}(\Gamma_R) = L_a(\Gamma_R)$, $\mathcal{M}u$ defines the zero element in $\widetilde{H}^{(M_p)}_{(\omega)}(\Gamma)$. Thus $S = 0$ and $u \equiv 0$.

THEOREM 8 (Strong quasi-analyticity principle). *Let* $-\pi/2 < \theta < \pi/2$, $l_\theta = \{z = re^{i\theta} : r > 0\}$ *and* $F \in \mathcal{O}(\operatorname{Re} z > 0)$. *Suppose that for some* $v \in \mathbb{R}$ *and every* $\kappa > 0$ *there exist* $k < \infty$ *and* $C < \infty$ *such that*

$$|F(z)| \le C \exp\{v \operatorname{Re} z + M(k|z|)\} \quad \text{for } \operatorname{Re} z \ge \kappa. \tag{23}$$

If for some $\tau > 0$ *and every* $m \in \mathbb{N}$ *there exists* $C_m < \infty$ *such that*

$$|F(z)| \le C_m e^{-m \operatorname{Re} z} \quad \text{for } z \in l_\theta,\ \operatorname{Re} z \ge \tau, \tag{24}$$

then $F \equiv 0$.

Proof. Put $u(x) = F \circ \mu^{-1}(x)$ for $x \in \widetilde{B}(1)$. Then $u \in \mathcal{O}_v^{(M_p)}(\widetilde{B}(1))$. Set $t = e^{-\tau}$, let $\gamma_{t,\theta}$ be the set of $x \in \widetilde{B}(1)$ that satisfy

$$x = \begin{cases} t\exp\{-ir\sin\theta\} & \text{for } 0 \le r \le \tau/\cos\theta, \\ \exp\{-r(\cos\theta + i\sin\theta)\} & \text{for } r \ge \tau/\cos\theta, \end{cases}$$

and observe that

$$\mathcal{M}_t u(z) = \int_{\gamma_{t,\theta}} u(x)x^{-z-1} \quad \text{for } z \in \Omega_{v,\theta}, \tag{25}$$

where $\Omega_{v,\theta} = \{z \in \mathbb{C} : \operatorname{Re} z < v \text{ and } \sin\theta \operatorname{Im} z > \cos\theta(\operatorname{Re} z - v)\}$. Using (24) we infer that the right hand side of (25) is defined for $z \in \mathbb{C}$. Thus, $\mathcal{M}_t u \in \mathcal{O}(\mathbb{C})$. As in the proof of Theorem 7 this implies that $u \equiv 0$ and hence $F \equiv 0$.

Remark 3. The conclusion of Theorem 8 does not hold if instead of (23) we assume that for every $\varepsilon > 0$ and $\kappa > 0$ there exists $C_{\varepsilon,\kappa}$ such that

$$|F(z)| \le C_{\varepsilon,\kappa} \exp\{v \operatorname{Re} z + \varepsilon|z|\} \quad \text{for } \operatorname{Re} z \ge \kappa.$$

In this case the function $u = F \circ \mu^{-1}$ is the Taylor transform of an analytic functional with carrier at $\{\infty\}$ and need not be equal to zero.

Remark 4. The results of the paper can be easily extended to the n-dimensional case if Γ is a cone of product type. The case of an arbitrary convex, proper cone in $\mathbb{R}^n$ is more difficult and will be studied in a subsequent paper.

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