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On a nonlinear second order periodic boundary value problem with Carathéodory functions

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Abstract. The periodic boundary value problem $u''(t) = f(t, u(t), u'(t))$ with $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$ is studied using the generalized method of upper and lower solutions, where f is a Carathéodory function satisfying a Nagumo condition. The existence of solutions is obtained under suitable conditions on f. The results improve and generalize the work of M.-X. Wang *et al.* [5].

1. Introduction. In recent years, a number of authors have studied the following periodic boundary value problem of second order:

$$(1.1)\qquad \begin{aligned} -u''(t) &= f(t, u(t), u'(t)),\\ u(0) = u(2\pi), &\quad u'(0) = u'(2\pi). \end{aligned}$$

People mainly studied the problem for f continuous with respect to its variables (see [1–5] and the references therein).

In [5], M.-X. Wang, A. Cabada and J. Nieto studied (1.1) when f is a Carathéodory function, using a generalized upper and lower solution method. Also, they developed a monotone iterative technique for finding minimal and maximal solutions.

In this paper, we use a modified version of the method of [5] to study the existence of solutions to problem (1.1) and develop a monotone iterative technique for finding the minimal and maximal solutions. Our method substantially modifies that of [5] and part of our results improve and generalize the results obtained in [5]. With our method, it is possible to extend the result to a more general form.

For completeness, we include some of the results of [5] with their (or modified) proofs. We use the same definitions and notations as in [5]. We

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write $I = [0, 2\pi]$ and denote by $W^{2,1}(I)$ the set of functions defined in I with integrable second derivatives and define the sector $[\alpha, \beta]$ as the set $[\alpha, \beta] = \{u \in W^{2,1}(I) : \alpha(t) \le u(t) \le \beta(t) \text{ for } t \in I = [0, 2\pi]\}$.

We call a function $f : I \times \mathbb{R}^2 \to \mathbb{R}$ a *Carathéodory function* if the following conditions are satisfied:

(1) for almost all $t \in I$, the function $\mathbb{R}^2 \ni (u, v) \to f(t, u, v) \in \mathbb{R}$ is continuous;

(2) for every $(u, v) \in \mathbb{R}^2$, the function $I \ni t \to f(t, u, v)$ is measurable;

(3) for every $M > 0$, there exists a real-valued function $\phi(t) = \phi_M(t) \in L^1(I)$ such that

$$|f(t, u, v)| \le \phi(t) \tag{1.2}$$

for a.e. $t \in I$ and every $(u, v) \in \mathbb{R}^2$ satisfying $|u| \le M$ and $|v| \le M$.

We call a function $\alpha : I \to \mathbb{R}$ a *lower solution* of (1.1) if $\alpha \in W^{2,1}(I)$ and

$$\begin{aligned} &-\alpha''(t) \le f(t, \alpha(t), \alpha'(t)) \quad \text{for a.e. } t \in I, \\ &\alpha(0) = \alpha(2\pi), \quad \alpha'(0) \ge \alpha'(2\pi). \end{aligned} \tag{1.3}$$

Similarly, $\beta : I \to \mathbb{R}$ is called an *upper solution* of (1.1) if $\beta \in W^{2,1}(I)$ and

$$\begin{aligned} &-\beta''(t) \ge f(t, \beta(t), \beta'(t)) \quad \text{for a.e. } t \in I, \\ &\beta(0) = \beta(2\pi), \quad \beta'(0) \le \beta'(2\pi). \end{aligned} \tag{1.4}$$

The following hypothesis is adopted:

(H1) The nonlinear function f satisfies the Nagumo condition on the set

$$\Omega := \{(t, u, v) : 0 \le t \le 2\pi,\ \alpha(t) \le u \le \beta(t),\ v \in \mathbb{R}\},$$

i.e. there exist a real-valued function $h(t) \in L^\sigma(I)$, $1 \le \sigma \le \infty$, and a continuous function $g(v) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t, u, v)| \le h(t) g(|v|) \quad \text{on } \Omega, \tag{1.5}$$

and

$$\int_0^\infty \frac{u^{(\sigma-1)/\sigma}}{g(u)}\, du > \varrho^{(\sigma-1)/\sigma} \|h\|_\sigma, \tag{1.6}$$

where

$$\varrho = \max_{t \in I} \beta(t) - \min_{t \in I} \alpha(t) \tag{1.7}$$

and

$$\|h\|_\sigma = \begin{cases} (\int_0^{2\pi} (h(t))^\sigma\, dt)^{1/\sigma} & \text{for } \sigma \in (0, \infty), \\ \sup_{t \in [0, 2\pi]} |h(t)| & \text{for } \sigma = \infty. \end{cases} \tag{1.8}$$

R e m a r k. In our paper, the Nagumo condition is defined in a slightly different way than in [5]. Our definition includes theirs as a special case. In fact, it is easy to see that under their definition the combination of their Carathéodory condition and the Nagumo condition implies that the function $h(t)$ in their paper must be bounded when $u \in [\alpha, \beta]$ and v is in a bounded set.

2. Existence of solutions. For any $u \in X = C^1(I)$, we define

$$p(t,u) = \begin{cases} \alpha(t), & u(t) < \alpha(t), \\ u(t), & \alpha(t) \le u(t) \le \beta(t), \\ \beta(t), & u(t) > \beta(t). \end{cases}$$

The following lemma is Lemma 2 of [5]:

LEMMA 1. *For $u \in X$, the following two properties hold*:

(1) $\frac{d}{dt}p(t,u(t))$ *exists for a.e.* $t \in I$.

(2) *If* $u, u_m \in X$ *and* $u_m \to u$ *in* X, *then*

$$\frac{d}{dt}p(t,u_m(t)) \to \frac{d}{dt}p(t,u(t)) \quad \textit{for a.e. } t \in I.$$

P r o o f. Note that $p(t,u) = [u-\alpha]^- - [u-\beta]^+ + u$, where $u^+(t) = \max\{u(t),0\}$ and $u^-(t) = \max\{-u(t),0\}$. The first assertion is obvious since u^+ and u^- are absolutely continuous for $u \in X$. To prove the second, we only have to show that if $u, u_m \in X$ and $u_m \to u$ in X, then

$$\lim_{m\to\infty} \frac{d}{dt}p(t,u_m^{\pm})(t) = \frac{d}{dt}p(t,u^{\pm})(t) \quad \text{for a.e. } t \in I.$$

We only need to check the limit at a point $t_0 \in I$ where $\frac{d}{dt}u_m^+$ and $\frac{d}{dt}u^+$ exist for all $m = 1, 2, \ldots$

If $u(t_0) > 0$, then $u(t_0) = u^+(t_0) > 0$. Therefore $\frac{d}{dt}u^+(t_0) = \frac{d}{dt}u(t_0)$ and there exists an $M > 0$ such that $u_m(t_0) = u_m^+(t_0) > 0$ for all $m > M$. Thus

$$\frac{d}{dt}u_m^+(t_0) = \frac{d}{dt}u_m(t_0) \to \frac{d}{dt}u(t_0).$$

If $u(t_0) < 0$, then $\frac{d}{dt}u^+(t_0) = 0$ and there exists an $M > 0$ such that $u_m^+(t) = 0$ on $(t_0 - \delta_m, t_0 + \delta_m)$ for some $\delta_m > 0$ for all $m > M$. Therefore $\frac{d}{dt}u^+(t_0) = 0 = \lim \frac{d}{dt}u_m^+(t_0)$.

If $u(t_0) = 0$, then $u^+(t_0) = 0$. Since $\frac{d}{dt}u^+(t_0)$ exists, we have $\frac{d}{dt}u^+(t_0) = 0$. It is obvious that $\frac{d}{dt}u(t_0) = 0$. Then

$$\left|\frac{d}{dt}u_m^+(t_0)\right| \le \left|\frac{d}{dt}u_m(t_0)\right| \to \left|\frac{d}{dt}u(t_0)\right| = 0 = \frac{d}{dt}u^+(t_0).$$

The proof for u^- is similar and thus the proof of Lemma 1 is complete.

To study the problem (1.1), we first consider the following modified problem:

$$
\begin{aligned}
&-u'' + u = f^*\left(t, p(t,u), \frac{dp(t,u)}{dt}\right) + p(t,u),\\
&u(0) = u(2\pi), \quad u'(0) = u'(2\pi),
\end{aligned} \tag{2.1}
$$

where

$$
f^*(t,u,v) = \begin{cases} f(t,u,N) & \text{if } v > N,\\ f(t,u,v) & \text{if } |v| \le N,\\ f(t,u,-N) & \text{if } v < -N.\end{cases}
$$

We may choose N so large that

$$
N > \max\{\sup_{t\in I} |\beta'(t)|, \sup_{t\in I} |\alpha'(t)|\},
$$

and

$$
\int_0^N \frac{u^{(\sigma-1)/\sigma}}{g(u)}\, du > \varrho^{(\sigma-1)/\sigma} \|h\|_\sigma. \tag{2.2}
$$

(H1) assures the existence of such an N.

For each $q \in X$, we define

$$
\xi(q)(t) = \xi(t) = f^*\left(t, p(t,q(t)), \frac{dp(t,q(t))}{dt}\right) + p(t,q(t)),
$$

and consider the problem

$$
\begin{aligned}
&-u'' + u = \xi(t),\\
&u(0) = u(2\pi), \quad u'(0) = u'(2\pi).
\end{aligned} \tag{2.3}
$$

It is obvious that the solution of (2.3) can be written in the form

$$
u(t) = C_1 e^t + C_2 e^{-t} - \frac{e^t}{2} \int_0^t \xi(s) e^{-s}\, ds + \frac{e^{-t}}{2} \int_0^t \xi(s) e^s\, ds, \tag{2.4}
$$

where

$$
\begin{aligned}
C_1 &= \frac{1}{2(e^{2\pi}-1)} \int_0^{2\pi} \xi(s) e^{2\pi - s}\, ds,\\
C_2 &= \frac{1}{2(e^{2\pi}-1)} \int_0^{2\pi} \xi(s) e^{s}\, ds.
\end{aligned}
$$

Lemma 1 obviously implies that $\xi(t)$ is measurable and

$$
\left| f^*\left(t, p(t,q(t)), \frac{dp(t,q(t))}{dt}\right)\right| \le \phi(t) \in L^1(I).
$$

Hence, $\xi \in L^1(I)$. Differentiating (2.4) with respect to t, we obtain

$$u'(t) = C_1 e^t - C_2 e^{-t} - \frac{e^t}{2} \int_0^t \xi(s) e^{-s}\, ds + \frac{e^{-t}}{2} \int_0^t \xi(s) e^s\, ds, \tag{2.5}$$

which is obviously continuous. Therefore, the solution of (2.3) is in X for any $q \in X$.

Define the operator $T : X \to X$ by $T[q] = u$, with u defined by (2.4). As in [5], we have the following

LEMMA 2. *$T : X \to X$ is compact.*

Proof. Suppose that $\{q_m\} \subset X$ is such that $q_m \to q$ in X. By Lemma 1, $p(t, q_m) \to p(t, q)$ and $\frac{d}{dt} p(t, q_m) \to \frac{d}{dt} p(t, q)$ a.e. Then the properties of f and the Lebesgue dominated convergence theorem imply that

$$\lim_{m\to\infty} \int_0^t \xi_m(s) e^{\pm s}\, ds = \int_0^t \xi(s) e^{\pm s}\, ds,$$

where

$$\xi_m = f^*\left(t, p(t, q_m(t)), \frac{dp(t, q_m(t))}{dt}\right) + p(t, q_m(t)).$$

Therefore, (2.4) and (2.5) show that $T[q_m] \to T[q]$ in X, i.e., T is continuous from X to X.

Now, we only have to show that T maps every bounded sequence in X to a compact sequence in X. Since $|\xi_m(s)| \le h(s) g(N) + |\alpha(s)| + |\beta(s)| \in L^1(I)$, the sequence $\int_0^t \xi_m(s) e^{\pm s}\, ds$ is equicontinuous, and so are $T[q_m]$ and $\frac{d}{dt} T[q_m]$. The Arzelà–Ascoli Theorem implies that T is compact.

LEMMA 3. *Let $u \in W^{2,1}(I)$ with $u''(t) \ge M(t) u(t)$ for a.e. $t \in I$, $u(0) = u(2\pi)$ and $u'(0) \ge u'(2\pi)$, where $M(t) \in L^1(I)$ and $M(t) > 0$. Then $u(t) \le 0$ fro every $t \in I$.*

Proof. Set $G = \{t \in I : u(t) > 0\}$. Then $u''(t) > 0$ on G. If $G \supset (0, 2\pi)$, then

$$u'(2\pi) \ge u'(0) + \int_0^{2\pi} M(t) u(t)\, dt > u'(0),$$

which is impossible. Hence, there exists at least one $\tau \in I$ with $u(\tau) \le 0$. If $u(0) > 0$, then there exist $0 < s_1 \le s_2 < 2\pi$ with $u(s_1) = u(s_2) = 0$ and $u(s) > 0$ for $s \in J = [0, s_1) \cup (s_2, 2\pi]$. Therefore, u' is nondecreasing in $[0, s_1)$ and $(s_2, 2\pi]$. But

$$u'(0) < u'(s_1) \le 0 \le u'(s_2) < u'(2\pi),$$

a contradiction.

If $u(0) \le 0$ and $\max\{u(s) : s \in I\} = u(t_0) > 0$ then there exist $t_1, t_2 \in (0, 2\pi)$ such that $t_1 < t_0 < t_2$, $u(t_1) = u(t_2) = 0$ and $u(s) > 0$ for $s \in (t_1, t_2)$. This implies that u is convex on $[t_1, t_2]$ and hence $u(t) \le 0$ on $[t_1, t_2]$, which is impossible. Therefore $u(s) \le 0$, and the proof is complete.

Now, we are ready to show the existence of solutions for the problem (1.1). We have

THEOREM 1. *Suppose that* $\alpha(t)$, $\beta(t)$ *are lower and upper solutions of problem* (1.1) *respectively*, *and* $\alpha(t) \le \beta(t)$ *on* I. *If* (H1) *holds*, *then there exists a solution* u *of* (1.1) *such that* $u \in [\alpha, \beta]$.

Proof. We first consider the operator T defined as above. It is easy to verify from (2.4) and (2.5) that T maps X to a bounded subset of X. Hence, by the compactness of the operator and the Schauder fixed point principle, we know that there exists a function $u \in X$ such that $u = T[u]$. Such a u is obviously a solution of problem (2.1), therefore, it suffices to show that $u \in [\alpha, \beta]$ and $|u'| \le N$.

We first show that $u \in [\alpha, \beta]$. Indeed, if $u > \beta$ on I, then $p(t, u) = \beta$. Therefore,

$$-u'' + u = f(t, \beta, \beta') \le -\beta'' + \beta \tag{2.6}$$

by the definition of f^* and the choice of N. Lemma 3 then implies that $u \le \beta$ on I, a contradiction. Therefore there must be a point $s \in I$ with $u(s) \le \beta(s)$. If $u(0) \le \beta(0)$ and there exists $s_1 \in (0, 2\pi)$ with $u(s_1) > \beta(s_1)$, then by the continuity of u, we know that there would be $t_1 < s_1 < t_2$ in $(0, 2\pi)$ such that $u > \beta$ on (t_1, t_2) with $(u - \beta)(t_1) = (u - \beta)(t_2) = 0$. Then (2.6) holds in the interval (t_1, t_2). This and the boundary conditions imply that $u \le \beta$ on (t_1, t_2), which is again a contradiction.

If $u(0) > \beta(0)$, then there exist $t_1 < t_2$ in I such that $u > \beta$ on $[0, t_1) \cup (t_2, 2\pi]$ with $(u - \beta)(t_1) = (u - \beta)(t_2) = 0$ and hence $(u - \beta)'(t_1) \le 0$ and $(u - \beta)'(t_2) \ge 0$. In both intervals, $(u - \beta)'' \ge u - \beta > 0$. Hence, $(u - \beta)'$ is increasing, which implies that $(u - \beta)'(0) < (u - \beta)'(t_1) \le 0$ and $(u - \beta)'(2\pi) > (u - \beta)'(t_2) \ge 0$, contrary to the boundary conditions.

To sum up, we know that $u \le \beta$ on I. Analogously we can prove that $u \ge \alpha$.

All that remains to be proved is that $|u'| \le N$.

The mean value theorem asserts that there exists a point $t_0 \in I$ such that $u'(t_0) = 0$. Assume that $|u'| \le N$ is not true. Then there exists an interval $[t_1, t_2] \subset I$ such that one of the following cases holds:

(i) $u'(t_1) = 0$, $u'(t_2) = N$ and $0 < u'(t) < N$ on (t_1, t_2),
(ii) $u'(t_1) = N$, $u'(t_2) = 0$ and $0 < u'(t) < N$ on (t_1, t_2),
(iii) $u'(t_1) = 0$, $u'(t_2) = -N$ and $-N < u'(t) < 0$ on (t_1, t_2),
(iv) $u'(t_1) = -N$, $u'(t_2) = 0$ and $-N < u'(t) < 0$ on (t_1, t_2).

Let us consider the case (i). By (2.1),

$$|u''(t)| = |f^*(t,u(t),u'(t))| \le h(t)g(|u'(t)|) \quad \text{on } [t_1,t_2]$$

and as a result

$$\begin{aligned}\int_0^N \frac{|u|^{(\sigma-1)/\sigma}}{g(|u|)}\,du &= \int_{t_1}^{t_2} \frac{|u'(t)|^{(\sigma-1)/\sigma}u''(t)}{g(|u'(t)|)}\,dt \\ &\le \int_{t_1}^{t_2} \frac{|u'(t)|^{(\sigma-1)/\sigma}|u''(t)|}{g(|u'(t)|)}\,dt \\ &\le \int_{t_1}^{t_2} h(t)|u'(t)|^{(\sigma-1)/\sigma}\,dt \\ &\le \Big(\int_{t_1}^{t_2} |h(t)|^\sigma\,dt\Big)^{1/\sigma}(u(t_2)-u(t_1))^{(\sigma-1)/\sigma} \\ &\le \|h\|_\sigma \varrho^{(\sigma-1)/\sigma} \quad \text{if } 1<\sigma\le\infty\end{aligned}$$

and

$$\int_0^N \frac{du}{g(|u|)} = \int_{t_1}^{t_2} \frac{u''(t)}{g(|u'(t)|)}\,dt \le \int_{t_1}^{t_2} h(t)\,dt \le \|h\|_1 \quad \text{if } \sigma = 1.$$

This contradicts (2.2). The other cases are dealt with similarly. This completes the proof of Theorem 1.

3. Monotone iterative technique. In this section, we develop a monotone iterative technique for our equation, the method being similar to that of [5]. Our conditions are more precise and applicable.

In addition to the hypotheses of the first two sections, we introduce the following hypotheses:

(H2) There exists an $M \in L^1(I)$ such that $M(t) > 0$ for a.e. $t \in I$ and

$$f(t,p,s) - f(t,q,s) \ge -M(t)(p-q) \tag{3.1}$$

for a.e. $t \in I$ and every $\alpha \le q \le p \le \beta$, $s \in \mathbb{R}$.

(H3) There exists a $U \in L^1(I)$ such that $U(t) > 0$ for a.e. $t \in I$ and

$$f(t,p,s) - f(t,p,y) \ge -U(t)(s-y) \tag{3.2}$$

for a.e. $t \in I$ and every $\alpha \le p \le \beta$, $s \ge y$, $s,y \in \mathbb{R}$.

(H1*) Define

$$g^*(v) = \max\{g(v), \max|\alpha| + \max|\beta|\}, \quad h^*(t) = h(t) + 2M(t),$$

where $g(v)$ and $h(t)$ are as in (H1). Then

$$\int_0^\infty \frac{u^{(\sigma-1)/\sigma}}{g^*(u)}\,du > \varrho^{(\sigma-1)/\sigma}\|h^*\|_\sigma.$$

We have

THEOREM 2. *Suppose that* (H1*)–(H3) *hold. Then there exist monotone sequences* $\alpha_n \nearrow x$ *and* $\beta_n \searrow z$ *as* $n \to \infty$, *uniformly on* I, *with* $\alpha_0 = \alpha$ *and* $\beta_0 = \beta$. *Here*, x *and* z *are the minimal and maximal solutions of* (1.1) *respectively on* $[\alpha, \beta]$, *that is*, *if* $u \in [\alpha, \beta]$ *is a solution of* (1.1), *then* $u \in [x, z]$. *Moreover*, *the sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy* $\alpha = \alpha_0 \le \ldots \le \alpha_n \le \beta_n \le \ldots \le \beta_0 = \beta$.

Proof. For any $q \in [\alpha, \beta] \cap X$, consider the following quasilinear periodic boundary value problem:

$$\text{(3.3)} \qquad \begin{aligned} &-u''(t) = f(t, q(t), u'(t)) + M(t)(q(t) - u(t)),\\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{aligned}$$

It is easy to verify that α and β are also lower and upper solutions of (3.3) respectively and

$$\begin{aligned} |f(t, q(t), u'(t)) &+ M(t)(q(t) - u(t))|\\ &\le h(t)g(|u'(t)|) + 2M(t)(\max|\alpha| + \max|\beta|)\\ &\le [h(t) + 2M(t)]g^*(|u'(t)|) = h^*(t)g^*(|u'(t)|). \end{aligned}$$

Then, by Theorem 1, there exists a solution u of the problem (3.3) with $u \in [\alpha, \beta]$. It is not difficult to show that this solution is unique by using the argument for Lemma 3. Now, define the operator $T : X \to X$ by $T[q] = u$, where u is the solution of (3.3).

We shall prove:

CLAIM. *If* $\alpha \le q_1 \le q_2 \le \beta$, $q_1, q_2 \in X$, *then* $u_1 = T[q_1] \le u_2 = T[q_2]$.

Indeed, let $y = u_2 - u_1$. Then

$$\text{(3.4)} \qquad \begin{aligned} -y'' &= f(t, q_2(t), u_2'(t)) - f(t, q_1(t), u_1'(t)) + M(t)[(q_2 - q_1)(t) - y(t)]\\ &\ge -U(t)y'(t) - M(t)y(t). \end{aligned}$$

Assume that t_0 is such that $y(t_0) = \min\{y(t) : t \in I\}$. We only need to prove that $y(t_0) \ge 0$.

In fact, if $t_0 \in (0, 2\pi)$ and $y(t_0) < 0$, then there would be $0 \le t_1 < t_0 < t_2 \le 2\pi$ such that $y(t) < 0$ on (t_1, t_2), $y'(t_1) \le 0$ and $y'(t_2) \ge 0$. Now (3.4) implies that $y'' - U(t)y' < 0$ on (t_1, t_2). Solving the differential inequality, we obtain

$$y'(t_2)\exp\Big\{-\int_{t_1}^{t_2} U(t)\,dt\Big\} < y'(t_1) \le 0,$$

which is impossible. If $t_0 = 0$ or $t_0 = 2\pi$ and $y(0) = y(2\pi) < 0$, then there would be $t_1, t_2 \in (0, 2\pi)$ such that $y'(t_1) \geq 0 \geq y'(t_2)$, $y''(t) - U(t)y' < 0$ on $[0, t_1) \cap (t_2, 2\pi]$ and hence

$$0 \leq y'(t_1) \exp\Big\{ -\int_0^{t_1} U(t)\,dt \Big\} < y'(0),$$

$$y'(2\pi) \exp\Big\{ -\int_{t_2}^{2\pi} U(t)\,dt \Big\} < y'(t_2) \leq 0,$$

again a contradiction. This proves the claim.

Now, define sequences $\alpha_0 = \alpha$, $\alpha_n = T[\alpha_{n-1}]$, $\beta_0 = \beta$ and $\beta_n = T[\beta_{n-1}]$. Since the solution u of (3.3) satisfies $u \in [\alpha, \beta]$, using the monotonicity of T, we see that $\alpha = \alpha_0 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_0 = \beta$. Hence, the limits $\lim_{n\to\infty} \alpha_n(t) = x(t)$ and $\lim_{n\to\infty} \beta_n(t) = z(t)$ exist. From the previous proof, we know that $|\alpha_n'|, |\beta_n'| \leq N$ uniformly in n. Using the argument for Theorem 1, we know that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous and uniformly bounded and hence converge to x and z in X. By the definitions, we know that $T[x] = x$ and $T[z] = z$. Then it is obvious by formulas similar to (2.4) and (2.5) that x and z satisfy (1.1).

Furthermore, if $u \in X \cap [\alpha, \beta]$ solves (1.1), then since $T[u] = u$, we have $\alpha_n \leq u \leq \beta_n$ for any $n = 1, 2, \ldots$ and hence $u \in [x, z]$ in I.

This completes the proof of the theorem.

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