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Versal deformations of D_q-invariant 2-parameter families of planar vector fields

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Abstract. The paper deals with 2-parameter families of planar vector fields which are invariant under the group D_q for $q \geq 3$. The germs at $z = 0$ of such families are studied and versal families are found. We also give the phase portraits of the versal families.

1. Introduction and the statement of the result. In this work we solve the problem of classification of families of planar vector fields invariant under the group D_q for $q \geq 3$. The problem of classification of vector fields invariant under some subgroups of the group of isometries of $\mathbb{R}^2$ is quite natural; for example, some problems concerning multidimensional fields lead to this case. The general statement of the problem is given in [1], [2], [8], [11]. The main example of such fields are the ones invariant under C_q, the cyclic group of rotations by the angle $2k\pi/q$. The case of C_1 (no symmetry) with both eigenvalues at zero equal to zero is described in [4]. The cases of C_2 and C_3 appear in [9]. The fields invariant under C_4 are very complicated and are not completely investigated yet. [3], [10] and [13] deal with that case. Except the condition of rotation invariance there is also a natural additional condition of invariance under axial symmetry. This leads to the dihedral group D_q. The fields invariant under D_1 are described in [6], [12], [14] and the fields invariant under D_2 are found in [12], [15]. It turns out that invariance under symmetry allows us to avoid the problems appearing in the C_4 case and the versal families are simple. In this work we present a complete classification of 1- and 2-parameter families.

The author has made the calculations for 3- and more-parameter families, but in this case the phase portraits were very complicated and we were not able to find anything general. Recently the dihedral groups D_q draw some attention of specialists in bifurcation theory. For example, some bifurcations

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with D_3 and D_4 symmetries were studied in [7]. The present work deals with a specific problem with D_q symmetry for any q.

Every field invariant under rotation by $2\pi/q$ must have the form

$$\dot{z} = Az + Bz|z|^2 + C\overline{z}^{q-1} + Dz^{q+1} + E\overline{z}^{q-1}|z|^2 + O(|z|^{q+2}). \tag{1}$$

Our field must also be invariant under axial symmetry ($z \mapsto \overline{z}$), so $A, B, C, D, E \in \mathbb{R}$. In polar coordinates we obtain

$$\begin{aligned} \dot{r} &= r(A + Br^2 + Cr^{q-2})\cos(q\varphi) + (D+E)r^q\cos(q\varphi) + O(r^{q+1}), \\ \dot{\varphi} &= -r^{q-2}(C + (E-D)r^2 + O(r^3))\sin(q\varphi). \end{aligned} \tag{2}$$

2. The result. The main result of this work is the following

THEOREM 1. (a) *All* 1- *and* 2-*parameter families of germs at* $z = 0$ *of* D_q-*invariant planar vector fields* (1) *can be divided into non-degenerate and degenerate ones*, *the latter forming a finite union of positive codimension submanifolds in the space of all such families.*

(b) *The following main families are versal families* ($\varepsilon_{1,2}$ *are parameters of deformation*):

- $q = 3:$ $\quad \dot{z} = \varepsilon_1 z + \overline{z}^2,$

  $\dot{z} = \varepsilon_1 z + \varepsilon_2\overline{z}^2 + z|z|^2 + Dz^4 + E\overline{z}^2|z|^2, \quad D \neq E,\ D \neq 0,$
- $q = 4:$ $\quad \dot{z} = \varepsilon_1 z + Bz|z|^2 + \overline{z}^3, \quad |B| \neq 1,$

  $\dot{z} = \varepsilon_1 z + \varepsilon_2\overline{z}^3 + z|z|^2 + Dz^5 + E\overline{z}^3|z|^2, \quad D \neq E,\ D \neq 0,$

  $\dot{z} = \varepsilon_1 z + (1+\varepsilon_2)z|z|^2 + \overline{z}^3 + Dz^5 + E\overline{z}^3|z|^2, \quad |D| \neq |E|,$
- $q > 4:$ $\quad \dot{z} = \varepsilon_1 z + z|z|^2 + \overline{z}^{q-1},$

  $\dot{z} = \varepsilon_1 z + \varepsilon_2 z|z|^2 + \overline{z}^{q-1}.$

(c) *The bifurcational diagrams and phase portraits are given in Figures* 1–6, 8–10.

For the definition of versality, topological equivalence etc. see [1].

The remaining part of this work is devoted to the proof of Theorem 1. We shall see that the only bifurcations appearing in the 2-parameter families are bifurcations of critical points of saddle-node type. The analysis of such families is the same as the analysis of the main families (from Theorem 1(b)). Therefore in order to avoid unnecessary complications we shall study only the main families. Then in Section 5 we shall prove the conclusions (a) and (b) of the theorem. In fact, the result for 1-parameter families follows from the analysis of C_q-symmetric families in [1], [2].

In 3-parameter families other bifurcations (Hopf, saddle-connection) appear. But we do not study them here.

3. 1-parameter families

3.1. $q = 3$. We have

$$\dot{z} = A(\mu)z + C(\mu)\overline{z}^2, \qquad A(0) = 0.$$

We have omitted the terms of degree 3.

Conditions of genericity are

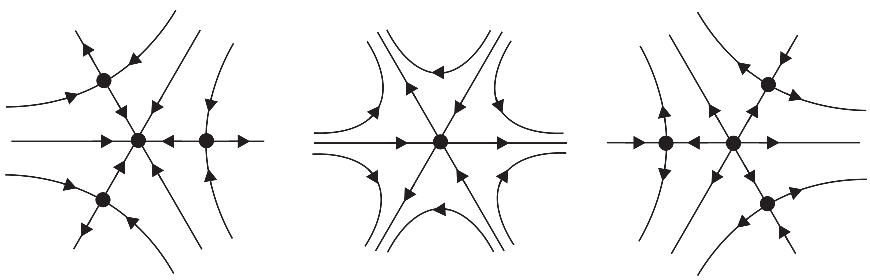
$$\frac{dA}{d\mu} \neq 0 \quad \text{and} \quad C(0) \neq 0.$$

By choosing a new parameter ε, rescaling z, writing our family in polar coordinates and dividing by r we obtain

$$\dot{r} = \varepsilon + r\cos 3\varphi, \qquad \dot{\varphi} = -\sin 3\varphi.$$

It follows from the equation for $\dot{\varphi}$ that all the critical points of this family lie on the lines $\sin 3\varphi = 0$. Since our field is D_3-invariant we can restrict ourselves to the axis $\varphi = 0$. We allow negative values of r, where we identify $(-r, \varphi)$ with $(r, \varphi + \pi)$.

The critical points are $p_0 = \{r = 0\}$ and $p_1 = (-\varepsilon, 0)$. The point p_0 is a source for $\varepsilon > 0$ and a sink for $\varepsilon < 0$. The point p_1 is a saddle. For the bifurcational diagram see Figure 1.



$\varepsilon < 0$ $\qquad$ $\varepsilon = 0$ $\qquad$ $\varepsilon > 0$

Fig. 1

3.2. $q = 4$. We have

$$\dot{z} = A(\mu)z + B(\mu)z|z|^2 + C(\mu)\overline{z}^3, \qquad A(0) = 0.$$

We have omitted the terms of degree 4.

Conditions of genericity are

$$\frac{\partial A}{\partial \mu} \neq 0, \quad C(0) \neq 0, \quad |B(0)| \neq |C(0)|.$$

By choosing a new parameter ε, applying the change $z \mapsto \lambda z$ or $z \mapsto e^{\pi i/4}z$, possibly reversing the time, writing our family in polar coordinates and dividing by r we obtain,

$$\dot{r} = \varepsilon + (B + \cos 4\varphi)r^2, \quad \dot{\varphi} = -r\sin 4\varphi, \quad 0 \leq B \neq 1.$$

On the half-lines $\varphi = 0$ and $\varphi = \pi/4$ we have the following critical points:

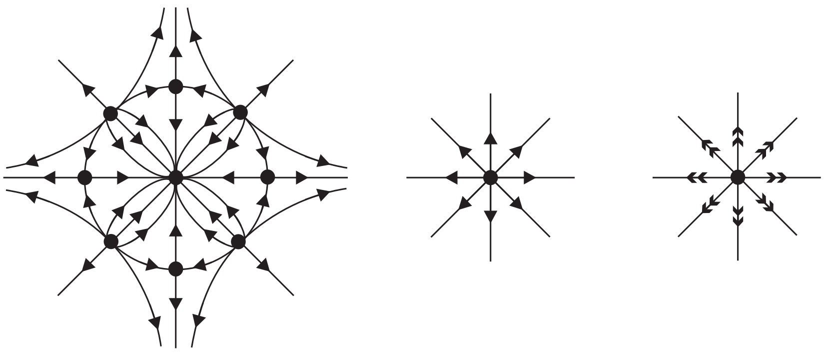
$$p_0 = \{r = 0\}, \quad p_1 = \left(\sqrt{\frac{-\varepsilon}{B+1}}, 0\right), \quad p_2 = \left(\sqrt{\frac{-\varepsilon}{B-1}}, \frac{\pi}{4}\right).$$

Denote by D_i the matrix of linearization of the field at p_i. We have

$$D_1 = \begin{pmatrix} 2\sqrt{-\varepsilon(B+1)} & 0 \\ 0 & -4\sqrt{\dfrac{-\varepsilon}{B+1}} \end{pmatrix},$$

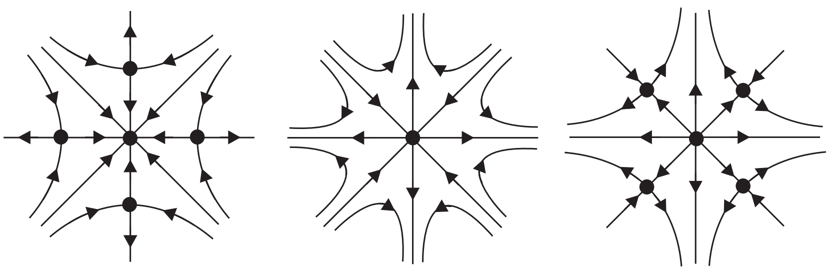
$$D_2 = \begin{pmatrix} 2(B-1)\sqrt{\dfrac{\varepsilon}{1-B}} & 0 \\ 0 & -4\sqrt{\dfrac{\varepsilon}{1-B}} \end{pmatrix}.$$

The point p_0 is a source for $\varepsilon > 0$ and a sink for $\varepsilon < 0$. Depending on the value of B we get the bifurcational diagrams given in Figure 2 for $B > 1$ and in Figure 3 for $B < 1$.



$\varepsilon < 0$ $\varepsilon = 0$ $\varepsilon > 0$

Fig. 2



$\varepsilon < 0$ $\varepsilon = 0$ $\varepsilon > 0$

Fig. 3

3.3. $q > 4$. We have

$$\dot{z} = A(\mu)z + B(\mu)z|z|^2 + C(\mu)\overline{z}^{q-1}, \qquad A(0) = 0.$$

We have omitted the terms of degree q.

Conditions of genericity are

$$\frac{\partial A}{\partial \mu} \neq 0, \quad B(0) \neq 0, \quad C(0) \neq 0.$$

As in the previous cases we obtain

$$\dot{z} = \varepsilon z + z|z|^2 + C\overline{z}^{q-1}, \qquad C = \pm 1.$$

For q odd, if $C < 0$ we can make a change of coordinates $z \mapsto -z$ to get $C = 1$. For q even we obtain the same result after the change $z \mapsto e^{\pi i/q} z$. In polar coordinates we get

$$\dot{r} = \varepsilon + r^2 + r^{q-2}\cos q\varphi, \qquad \dot{\varphi} = -r^{q-3}\sin q\varphi.$$

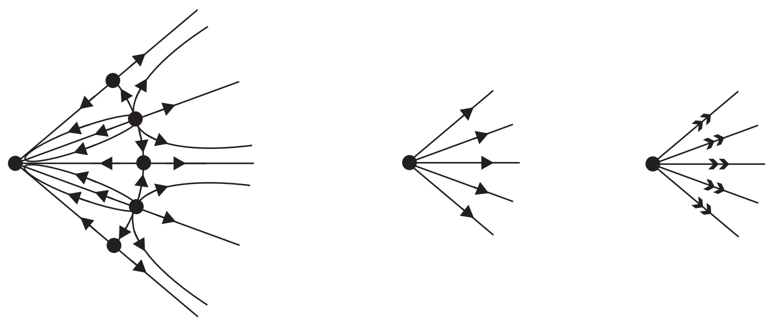
We can obtain the whole phase portrait by glueing together the q sectors $\{r \geq 0, 2\pi i/q \leq \varphi \leq 2\pi(i+1)/q\}$.

In our search for critical points it is enough to consider only these two half-lines: $(r > 0, \varphi = 0)$ and $(r > 0, \varphi = \pi/q)$.

We have the following critical points:

$$p_1 = (\sim \sqrt{-\varepsilon}, 0), \qquad p_2 = (\sim \sqrt{-\varepsilon}, \pi/q).$$

The point $p_0 = \{r = 0\}$ is a source for $\varepsilon > 0$ and a sink for $\varepsilon < 0$. The point p_1 is a saddle, and p_2 is a source. For the bifurcational diagram see Figure 4.



$\varepsilon < 0$ $\varepsilon = 0$ $\varepsilon > 0$

Fig. 4

4. 2-parameter families

4.1. $q = 3$. We have

$$\dot{z} = A(\mu_1, \mu_2)z + C(\mu_1, \mu_2)\overline{z}^2 + B(\mu_1, \mu_2)z|z|^2 + D(\mu_1, \mu_2)z^4 + E(\mu_1, \mu_2)\overline{z}^2|z|^2.$$

For 1-parameter families we made only one nondegeneracy assumption $C(0) \neq 0$. In the 2-parameter case we have to allow $C(0,0)=0$. Therefore

$$A(0,0)=0, \qquad C(0,0)=0.$$

We have omitted the terms of degree 5.

Conditions of genericity are

$$\begin{vmatrix} \dfrac{\partial A}{\partial \mu_1} & \dfrac{\partial A}{\partial \mu_2} \\ \dfrac{\partial C}{\partial \mu_1} & \dfrac{\partial C}{\partial \mu_2} \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{array}{l} B(0,0) \neq 0, \\ D(0,0) \neq E(0,0), \\ D(0,0) \neq 0. \end{array}$$

By choosing new parameters ε_1 and ε_2, rescaling z and time and writing our family in polar coordinates we obtain

$$\begin{aligned} \dot r &= \varepsilon_1 + \varepsilon_2 r \cos 3\varphi + r^2 + K r^3 \cos 3\varphi, \\ \dot\varphi &= (\varepsilon_2 + r^2)\sin 3\varphi. \end{aligned}$$

Note that $D \neq 0$ implies $K \neq 1$.

It follows from the equation for $\dot\varphi$ that the critical points all lie on the lines $\sin 3\varphi = 0$ and on the circle $\varepsilon_2 + r^2 = 0$. We allow negative values of r, where we identify $(-r,\varphi)$ with $(r,\varphi+\pi)$.

We have the following critical points on the invariant half-lines:

$$p_0 = \{r=0\},$$
$$p_1 = \big(\sim -\tfrac12(\varepsilon_2 + \sqrt{\varepsilon_2^2 - 4\varepsilon_1}), 0\big), \qquad p_2 = \big(\sim -\tfrac12(\varepsilon_2 - \sqrt{\varepsilon_2^2 - 4\varepsilon_1}), 0\big).$$

There are two bifurcational curves $\varepsilon_1 = 0$ and the saddle-node (S-N) curve:

$$\Gamma_0 : \varepsilon_2^2 \approx 4\varepsilon_1.$$

On the circle $r = \sqrt{-\varepsilon_2}$ we have the equation

$$\varepsilon_1 - \varepsilon_2 + \varepsilon_2\sqrt{-\varepsilon_2}(1-K)\cos 3\varphi = 0.$$

That gives

$$\cos 3\varphi = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2\sqrt{-\varepsilon_2}(1-K)}.$$

There are two more bifurcational curves corresponding to the saddle-node bifurcations $|\cos 3\varphi| = 1$:

$$\varepsilon_1 = \varepsilon_2 \pm \varepsilon_2\sqrt{-\varepsilon_2}(1-K) = \varepsilon_2(1 \pm \sqrt{-\varepsilon_2}(1-K)), \qquad \varepsilon_2 \le 0.$$

Since $K \neq 1$, we have

$$\begin{aligned} \Gamma_1 &= \{(\varepsilon_1,\varepsilon_2) : \varepsilon_1 = \varepsilon_2(1 + \sqrt{-\varepsilon_2}|1-K|),\ \varepsilon_2 \le 0\}, \\ \Gamma_2 &= \{(\varepsilon_1,\varepsilon_2) : \varepsilon_1 = \varepsilon_2(1 - \sqrt{-\varepsilon_2}|1-K|),\ \varepsilon_2 \le 0\}. \end{aligned}$$

The point p_0 is a source for $\varepsilon_1 > 0$ and a sink for $\varepsilon_1 < 0$. The character of the remaining points changes when bifurcations take place. When the circle $r = \sqrt{-\varepsilon_2}$ goes through a critical point (that happens on the curves Γ_1 and Γ_2), a symmetric saddle-node bifurcation takes place in the direction transversal to the invariant line. For the bifurcational diagram see Figure 5.

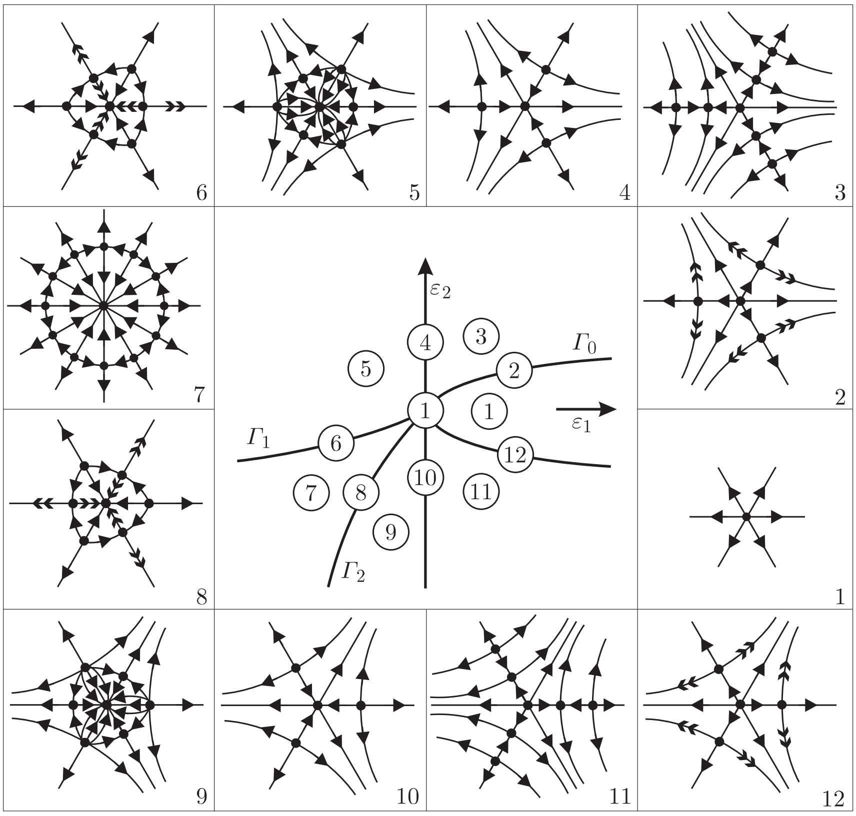


Fig. 5

4.2. $q = 4$**.** We have

$$\dot{z} = A(\mu_1, \mu_2)z + B(\mu_1, \mu_2)z|z|^2 + C(\mu_1, \mu_2)\bar{z}^3 + D(\mu_1, \mu_2)z^5 + E(\mu_1, \mu_2)\bar{z}^3|z|^2.$$

We have omitted the terms of degree 6.

The 1-parameter families satisfied two nondegeneracy conditions $C(0) \neq 0$ and $|B(0)| \neq | C(0) |$. In the 2-parameter case we have to allow $C(0,0) = 0$ or $B(0,0) = \pm C(0,0)$. Consequently, we have two cases:

I. $A(0,0)=0$ and $C(0,0)=0$. Then conditions of genericity are

$$\begin{vmatrix} \frac{\partial A}{\partial \mu_1} & \frac{\partial A}{\partial \mu_2} \\ \frac{\partial C}{\partial \mu_1} & \frac{\partial C}{\partial \mu_2} \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{array}{l} B(0,0)\neq 0, \\ D(0,0)\neq E(0,0), \\ D(0,0)\neq 0. \end{array}$$

II. $A(0,0)=0$ and $B(0,0)=C(0,0)$. Then conditions of genericity are

$$\begin{vmatrix} \frac{\partial A}{\partial \mu_1} & \frac{\partial A}{\partial \mu_2} \\ \frac{\partial B}{\partial \mu_1} & \frac{\partial B}{\partial \mu_2} \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{array}{l} C(0,0)\neq 0, \\ |D(0,0)|\neq |E(0,0)|. \end{array}$$

The third interesting situation for us is when

$$B(0,0)=-C(0,0).$$

We will show that by a suitable change of variables we can bring it to the case

$$B(0,0)=C(0,0).$$

4.2.I. $q=4$, *case I.* By choosing new parameters ε_1 and ε_2, rescaling z and writing our family in polar coordinates we obtain

$$\begin{aligned} \dot r &= \varepsilon_1+\varepsilon_2 r^2\cos 4\varphi + r^2 + Kr^4\cos 4\varphi, \\ \dot\varphi &= -r(\varepsilon_2+r^2)\sin 4\varphi. \end{aligned}$$

Here $D\neq 0$ implies $K\neq 1$.

It follows from the equation for $\dot\varphi$ that the critical points all lie on the lines $\sin 4\varphi=0$ and on the circle $\varepsilon_2+r^2=0$.

We have the following critical points on the invariant half-lines:

$$p_0=\{r=0\},\quad p_1=(\sim\sqrt{-\varepsilon_1},0),\quad p_2=(\sim\sqrt{-\varepsilon_1},\pi/4).$$

On the circle $r=\sqrt{-\varepsilon_2}$ we have the equation

$$\varepsilon_1-\varepsilon_2+\varepsilon_2^2(K-1)\cos 4\varphi=0.$$

That gives

$$\cos 4\varphi=\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2^2(K-1)}\,.$$

There are two bifurcational curves corresponding to the saddle-node bifurcations at $|\cos 4\varphi|=1$:

$$\varepsilon_1=\varepsilon_2\pm\varepsilon_2^2(K-1)=\varepsilon_2(1\pm\varepsilon_2(K-1)),\qquad \varepsilon_2\le 0.$$

Since $K\neq 1$, we have

$$\begin{aligned} \Gamma_1 &= \{(\varepsilon_1,\varepsilon_2):\varepsilon_1=\varepsilon_2(1-\varepsilon_2|1-K|),\ \varepsilon_2\le 0\}, \\ \Gamma_2 &= \{(\varepsilon_1,\varepsilon_2):\varepsilon_1=\varepsilon_2(1+\varepsilon_2|1-K|),\ \varepsilon_2\le 0\}. \end{aligned}$$

The point p_0 is a source for $\varepsilon_1 > 0$ and a sink for $\varepsilon_1 < 0$. The character of the remaining points changes when bifurcations take place. When the circle $r = \sqrt{-\varepsilon_2}$ goes through a critical point (that happens on the curves Γ_1 and Γ_2), a symmetric saddle-node bifurcation takes place in the direction transversal to the invariant line. For the bifurcational diagram see Figure 6.

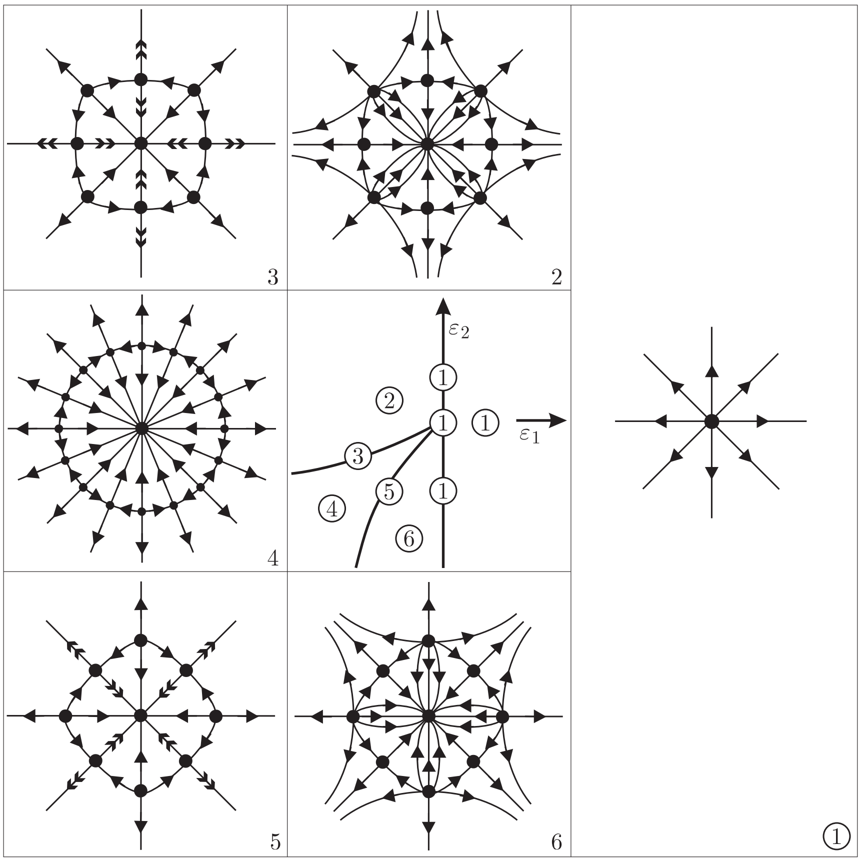


Fig. 6

4.2.II. $q = 4$, *case II*. We have

$$\dot{z} = \varepsilon_1 z + (B + \varepsilon_2) z|z|^2 + B\overline{z}^3 + Dz^5 + E\overline{z}^3|z|^2,$$

where we can put $B = 1,\ E - D = 1$. In polar coordinates,

$$\dot{r} = \varepsilon_1 + (1 + \varepsilon_2) r^2 + r^2 \cos 4\varphi + K r^4 \cos 4\varphi,$$
$$\dot{\varphi} = -r(1 + r^2) \sin 4\varphi.$$

Here $|D| \neq |E|$ implies $K \neq 0$.

We also get a versal family when $B(0,0) = -C(0,0)$. We need not deal with this because it can be obtained from the above family by the following change of variables:

$$\tau = -t, \quad \varphi = \Phi + \pi/4 \quad \text{or} \quad z \to e^{\pi i/4} z; \quad {}' = \frac{d}{d\tau}.$$

It follows from the equation for $\dot\varphi$ that the critical points all lie on the lines $\sin 4\varphi = 0$.

We have the following critical points on the invariant half-lines:

$$p_0 = \{r = 0\}, \qquad p_1 = \left(\sqrt{\frac{-\varepsilon_1}{\varepsilon_2 + 2}}, 0\right),$$

$$p_2 = \left(\frac{\varepsilon_2 - \sqrt{\varepsilon_2^2 + 4K\varepsilon_1}}{2K}, \frac{\pi}{4}\right), \qquad p_3 = \left(\frac{\varepsilon_2 + \sqrt{\varepsilon_2^2 + 4K\varepsilon_1}}{2K}, \frac{\pi}{4}\right).$$

The bifurcational curves are

$$\{\varepsilon_1 = 0\}, \qquad \Gamma = \{4K\varepsilon_1 = -\varepsilon_2^2,\ K\varepsilon_2 > 0\}.$$

Consider the case $K > 0$, $\varepsilon_1 < 0$, $\varepsilon_2 > 0$, after the S-N bifurcation. We have one critical point p_1 on the line $\varphi = 0$ (a saddle), and two points p_2 (a source) and p_3 (a saddle) on the line $\varphi = \pi/4$. We now prove that there is no saddle-connection bifurcation between p_1 and p_3. Consider the fragments of the circles O_1, O_2, O_3 with their ends at p_1, p_2, p_3 and contained between the lines $\varphi = 0$ and $\varphi = \pi/4$ (see Figure 7).

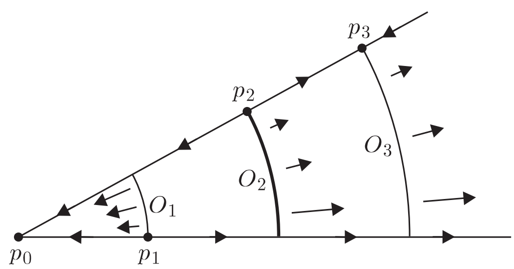


Fig. 7

There cannot exist any trajectory from p_3 to p_1, because it would have to cross O_2 in the direction opposite to the direction of the vector field. So we get two bifurcational diagrams depending on the value of K.

For the bifurcational diagram corresponding to $K < 0$ see Figure 8, and for $K > 0$ see Figure 9.

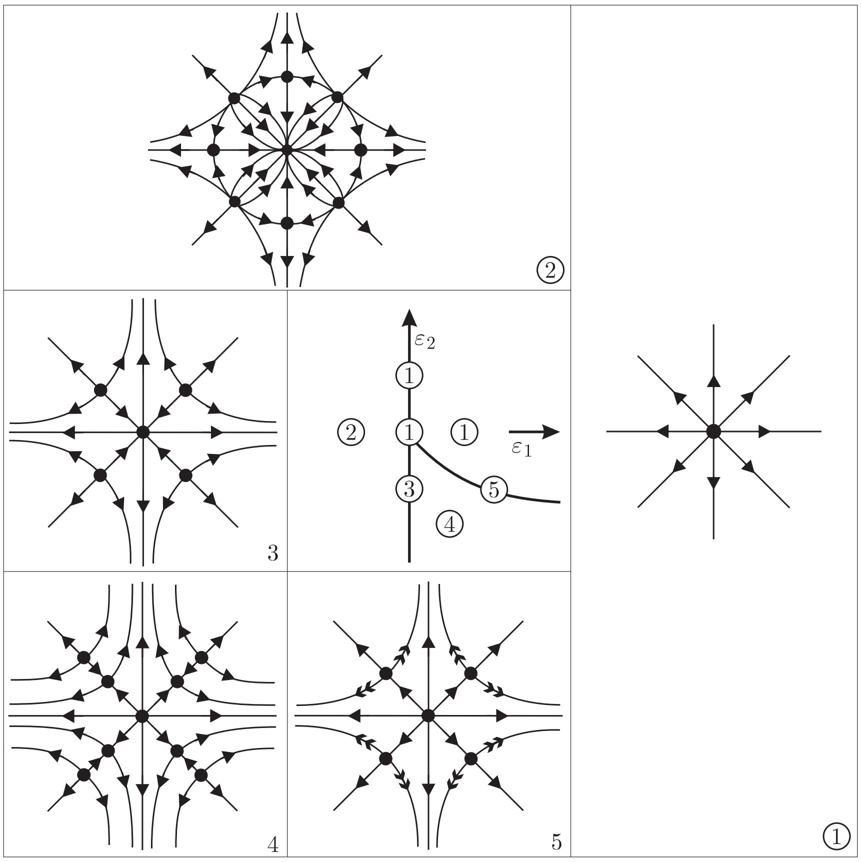


Fig. 8

4.3. $q > 4$. We have

$$\dot{z} = A(\mu_1, \mu_2)z + B(\mu_1, \mu_2)z|z|^2 + C(\mu_1, \mu_2)\bar{z}^{q-1},$$
$$A(0,0) = 0, \quad B(0,0) = 0.$$

We have omitted the terms of degree q.

The conditions of genericity are

$$\begin{vmatrix} \dfrac{\partial A}{\partial \mu_1} & \dfrac{\partial A}{\partial \mu_2} \\ \dfrac{\partial B}{\partial \mu_1} & \dfrac{\partial B}{\partial \mu_2} \end{vmatrix} \neq 0 \quad \text{and} \quad C(0,0) \neq 0.$$

By choosing new parameters $\varepsilon_1, \varepsilon_2$ and rescaling z we obtain

$$\dot{z} = \varepsilon_1 z + \varepsilon_2 z|z|^2 + C\bar{z}^{q-1}, \qquad C = \pm 1.$$

For q odd, if $C < 0$ we can make the change of variables $z \mapsto -z$ to get $C = 1$. For q even we obtain the same result by the change $z \mapsto e^{\pi i/q} z$.

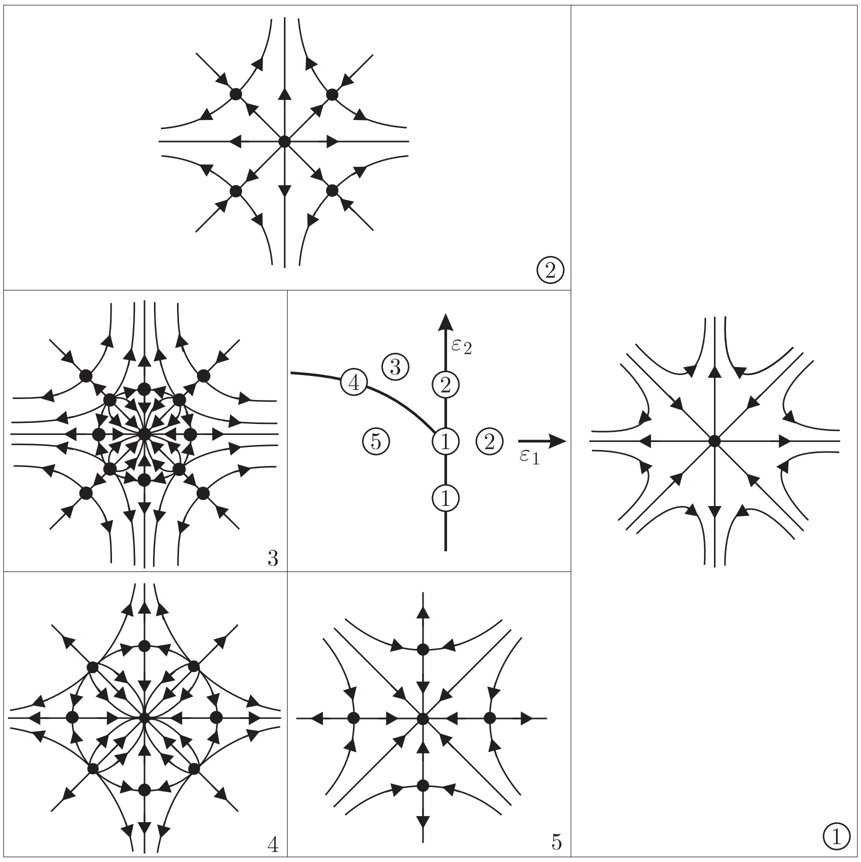


Fig. 9

In polar coordinates we get

$$\dot{r} = \varepsilon_1 + \varepsilon_2 r^2 + r^{q-2} \cos q\varphi, \qquad \dot{\varphi} = -r^{q-3} \sin q\varphi.$$

We can obtain the whole phase portrait by glueing together the q sectors $\{r \geq 0,\ 2\pi i/q \leq \varphi \leq 2\pi(i+1)/q\}$.

In our search for critical points it is enough to consider only the two half-lines:

$$R = \{(r,\varphi) : \varphi = 0\} \quad \text{and} \quad S = \{(r,\varphi) : \varphi = 2\pi/q\}.$$

The half-line R is attracting, while S is repelling. Simple calculations show that there are at most two points at each of this lines, but there can be maximally three critical points on $R \cup S$. Arguments similar to those we used in the case $q = 4$(II) show that there can be no saddle-connection bifurcation in this case. For the bifurcational diagram see Figure 10.

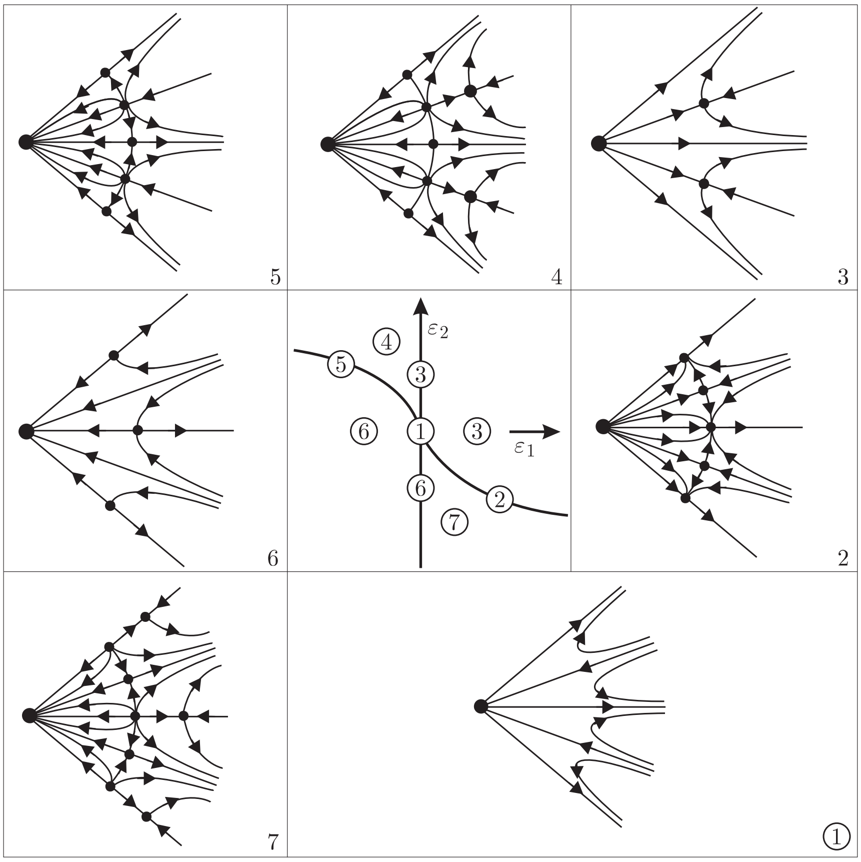


Fig. 10

5. Proof of the conclusions (a) and (b) of Theorem 1

5.1. *Proof of* (a). We only consider 2-parameter families. The proof for the 1-parameter case is given in [1], [2].

Consider a family v of vector fields invariant under D_3. By Section 1, it must have the form

$$\dot{z} = A(\mu_1, \mu_2)z + C(\mu_1, \mu_2)\overline{z}^2 + B(\mu_1, \mu_2)z|z|^2 + D(\mu_1, \mu_2)z^2 + E(\mu_1, \mu_2)\overline{z}^2|z|^2 + O(|z|^5).$$

We are dealing with germs at $z = 0$ of families of planar vector fields, so we may consider it as a small perturbation of the main family V.

We call our family *nondegenerate* if it satisfies the following conditions:

$$\begin{vmatrix} \dfrac{\partial A}{\partial \mu_1} & \dfrac{\partial A}{\partial \mu_2} \\ \dfrac{\partial C}{\partial \mu_1} & \dfrac{\partial C}{\partial \mu_2} \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{array}{l} B(0,0) \neq 0, \\ D(0,0) \neq 0, \\ D(0,0) \neq E(0,0). \end{array}$$

Other families are *degenerate.* Denote by Ξ the set of all germs of D_3-invariant 2-parameter planar vector fields. Families not satisfying the first nondegeneracy condition are solutions of the equation $F(v)=0$, where

$$F:\Xi\to\mathbb{R},\qquad F(v)=\begin{vmatrix}\dfrac{\partial A}{\partial\mu_1} & \dfrac{\partial A}{\partial\mu_2}\\ \dfrac{\partial C}{\partial\mu_1} & \dfrac{\partial C}{\partial\mu_2}\end{vmatrix}.$$

The derivative of F is nonsingular, so it follows from the Implicit Function Theorem (IFT) that the solutions of this equation form a submanifold of codimension 1. The same arguments hold for the functions not satisfying the remaining conditions. This proves the conclusion (a) of Theorem 1 for D_3-invariant 2-parameter families.

The same arguments give the proof in all the other cases considered in this paper.

5.2. *Proof of the versality of the main families* (*conclusion* (b))

5.2.1. *The D_3-invariant case.* To prove the versality of the main family we must construct a homeomorphism φ of the parameter spaces, and a family h_μ of homeomorphisms of the plane which transform the trajectories of the family $v(\mu)$ into the trajectories of the family $V[\varphi(\mu)]$ in a small neighbourhood of $z=0$.

The proof of versality for 1-parameter families can be found in [1] and [2].

Consider a D_3-invariant 2-parameter family. We have seen that the neighbourhood of the origin of the parameter space for the main family V is divided into six domains by the bifurcational curves Γ_0,Γ_1 and Γ_2 (see Figure 5). We shall prove that this holds for any nondegenerate 2-parameter family v for both the parameters and $|z|$ sufficiently small. We denote by F_1 (F_2) the first (second) coordinate of V written in polar coordinates and by f_1 (f_2) the first (second) coordinate of v in polar coordinates.

Let us first deal with Γ_0 on which an S-N bifurcation in the direction of r takes place on the invariant lines $\{\operatorname{Im} z^3=0\}$. We consider the map

$$\Phi(\varepsilon_1,\varepsilon_2,r,F)=\begin{pmatrix}F(\varepsilon_1,\varepsilon_2,r,0)\\ \dfrac{\partial F}{\partial r}(\varepsilon_1,\varepsilon_2,r,0)\end{pmatrix},\qquad \Phi:\mathbb{R}^3\times C^1(\mathbb{R}^2,\mathbb{R})\to\mathbb{R}^2.$$

We know that for $F=F_1$ and for every ε_2 there exist ε_1 and r such that

$$\Phi(\varepsilon_1,\varepsilon_2,r,F_1)=\begin{pmatrix}0\\0\end{pmatrix}.$$

We also have

$$D_{(\varepsilon_1,r)}\Phi(\varepsilon_1,\varepsilon_2,r,F_1)=\begin{pmatrix}1 & \varepsilon_2+2r+3Kr^2\\ 0 & 2+6Kr\end{pmatrix}\cong\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}.$$

We see that for small r and $\varepsilon_2, D_{(\varepsilon_1,r)}\Phi(\varepsilon_1,\varepsilon_2,r,F_1)$ is invertible, so it follows from IFT that for sufficiently small perturbations F of F_1 there also exist r and ε_1 such that

$$\Phi(\varepsilon_1,\varepsilon_2,r,F)=\begin{pmatrix}0\\0\end{pmatrix}.$$

We also know that ε_1 is a continuous function of ε_2 and that for fixed ε_2 it is a locally unique solution. As f_1 is a small perturbation of F_1 we know that for v there exists a bifurcational curve Δ_0 corresponding to Γ_0.

The same arguments hold for the bifurcational curve $\varepsilon_1=0$ on which an S-N bifurcation in the direction of r takes place (at the point $r=0$).

Now we deal with Γ_1 on which an S-N bifurcation in the direction of φ takes place. We consider the map

$$\Psi(\varepsilon_1,\varepsilon_2,r,G,F)=\begin{pmatrix}G(\varepsilon_1,\varepsilon_2,r,0)\\ \dfrac{\partial F}{\partial\varphi}(\varepsilon_1,\varepsilon_2,r,0)\end{pmatrix},\qquad \Psi:\mathbb{R}^3\times C^1(\mathbb{R}^2,\mathbb{R}^2)\to\mathbb{R}^2.$$

We know that for $F=F_2, G=F_1$ and for every ε_2 there exist ε_1 and r such that

$$\Psi(\varepsilon_1,\varepsilon_2,r,F_1,F_2)=\begin{pmatrix}0\\0\end{pmatrix}.$$

We also have

$$D_{(\varepsilon_1,r)}\Psi(\varepsilon_1,\varepsilon_2,r,F_1,F_2)=\begin{pmatrix}1 & \varepsilon_2+2r+3Kr^2\\ 0 & 6r\cos 3\varphi\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 6r\end{pmatrix}.$$

For small r and $\varepsilon_2, D_{(\varepsilon_1,r)}\Psi(\varepsilon_1,\varepsilon_2,r,F_1,F_2)$ is invertible, so by IFT for sufficiently small perturbations G,F of F_1 and F_2 there also exist r and ε_1 such that

$$\Psi(\varepsilon_1,\varepsilon_2,r,G,F)=\begin{pmatrix}0\\0\end{pmatrix}.$$

We also know that ε_1 is a continuous function of ε_2 and that for fixed ε_2 it is a locally unique solution. As f_1 is a small perturbation of F_1 we know that for v there exists a bifurcational curve Δ_1 corresponding to Γ_1.

The same arguments hold for Γ_2.

Of course for r and parameters small enough there cannot happen any bifurcations for parameters lying outside any fixed neighbourhood of the mentioned curves. This follows from the fact that all the critical points are hyperbolic (by IFT). We can choose sufficiently small neighbourhoods of the curves to provide uniqueness of these curves in these neighbourhoods. This also follows from IFT.

The above arguments show that the families V and v have the same bifurcational diagrams and their phase portraits in the corresponding areas of the diagrams are the same. For the construction of φ and homeomorphisms

h_μ see [4] and [5]. It is rather cumbersome and we do not include it in the present work.

This ends the proof of the versality of the 2-parameter main family in the D_3 case. Of course the same arguments hold for 1-parameter families.

5.2.2. *The D_q-invariant case* ($q > 3$). The proof of the existence and uniqueness of the bifurcation curves on which an S-N bifurcation takes place is the same as in the D_3 case. To complete the proof of Theorem 1 we only have to prove that there cannot happen a saddle-connection bifurcation in the case $q = 4$ (case II) and in the case $q > 4$. In both cases the proofs are similar, so we only consider case II for D_4.

The proof of nonexistence of a saddle-connection bifurcation is based on the construction of a curve O_2. We shall prove that such a curve must also exist for the family v.

Consider the map

$$\Phi(r,\varphi,F) = F(r,\varphi) - r^2\cos 4\varphi - Kr^4\cos 4\varphi,$$
$$\Phi : \mathbb{R}^2 \times C^1(\mathbb{R}^2,\mathbb{R}) \to \mathbb{R}.$$

For $F := F_1$ and for every φ there exists r such that

$$\Phi(r,\varphi,F_1) \equiv 0.$$

We also have

$$\frac{\partial\Phi}{\partial r}(r,\varphi,F_1) = 2r(1+\varepsilon_2 - \cos 4\varphi) + O(r^3).$$

Of course, $\varepsilon_2 > 0$ so by IFT for sufficiently small perturbations f_1 of F_1 and for every φ there also exists r such that

$$\Phi(r,\varphi,f_1) = 0.$$

Denote by o_2 the obtained curve $r = r(\varphi)$. We have

$$\dot r|_{o_2} = r^2(1+Kr^2)\cos 4\varphi > 0.$$

There cannot exist any trajectory from p_3 to p_1 because it would have to cross the curve o_2 in the direction opposite to the direction of the vector field v. This completes the proof of Theorem 1.

References

[1] V. I. Arnold, *On the loss of stability of oscillations near resonance and deformations of equivariant vector fields*, Funktsional. Anal. i Prilozhen. 6 (2) (1977), 1–11 (in Russian).

[2] —, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer, New York, 1983.

[3] F. S. Berezovskaya and A. I. Khibnik, *On bifurcations of separatrices in the problem of loss of stability of self-oscillations near the* 1 : 4 *resonance*, Prikl. Mat. Mekh. 44 (1980), 938–943 (in Russian).

[4] R. I. Bogdanov, *Versal deformations of singular points of vector fields on the plane in the case of zero eigenvalues*, Trudy Sem. Petrovsk. 2 (1976), 37–65 (in Russian).

[5] F. Dumortier and R. Roussarie, *On the saddle loop bifurcation*, in: Bifurcations of Planar Vector Fields (Luminy 1989), Lecture Notes in Math. 1455, Springer, New York, 1990, 44–73.

[6] N. K. Gavrilov, *On bifurcations of equilibrium state with one zero and one pair of pure imaginary roots*, in: Methods of Qualitative Theory of Differential Equations, Gorki Univ., 1980, 33–40 (in Russian).

[7] M. Golubitsky, I. Stewart and D. Shaefer, *Singularities and Groups in Bifurcation Theory*, Vol. 2, Appl. Math. Sci. 69, Springer, New York, 1988.

[8] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations*, *Dynamical Systems and Bifurcations of Planar Vector Fields*, Springer, New York, 1983.

[9] E. I. Khorozov, *Versal deformations of equivariant vector fields in the case of symmetry of order* 2 *and* 3, Trudy Sem. Petrovsk. 5 (1979), 163–192 (in Russian).

[10] A. I. Nieĭshtadt, *Bifurcations of the phase portrait of a certain system of equations arising in the problem of loss of stability of self-oscillations near the* 1 : 4 *resonance*, Prikl. Mat. Mekh. 42 (1978), 830–840 (in Russian).

[11] F. Takens, *Forced oscillations and bifurcations*, in: Applications of Global Analysis I, Comm. Math. Inst. Rijksuniv. Utrecht 3 (1974).

[12] A. Zegeling and R. E. Kooij, *Uniqueness of limit cycles in polynomial systems with algebraic invariants*, Bull. Austral. Math. Soc. 49 (1994), 7–20.

[13] —, *Equivariant unfoldings in the case of symmetry of order* 4, preprint TU Delft, 1992.

[14] H. Żołądek, *On versality of a certain family of symmetric vector fields on the plane*, Mat. Sb. 120 (1983), 473–499 (in Russian).

[15] —, *Bifurcations of a certain family of planar vector fields tangent to axes*, J. Differential Equations 67 (1987), 1–55.

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