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Alexander's projective capacity for polydisks and ellipsoids in $\mathbb{C}^N$

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Abstract. Alexander's projective capacity for the polydisk and the ellipsoid in $\mathbb{C}^N$ is computed. Sharper versions of two inequalities concerning this capacity and some other capacities in $\mathbb{C}^N$ are given. A sequence of orthogonal polynomials with respect to an appropriately defined measure supported on a compact subset K in $\mathbb{C}^N$ is proved to have an asymptotic behaviour in $\mathbb{C}^N$ similar to that of the Siciak homogeneous extremal function associated with K.

1. Introduction. Let S be the unit sphere in $\mathbb{C}^N$. Let σ denote the Lebesgue surface area measure on S. Let

$$s_N := \int_S d\sigma.$$

Let $H_n = H_n(\mathbb{C}^N)$ denote the set of all homogeneous polynomials of degree n (with complex coefficients) of N complex variables.

Let K be a compact subset of $\mathbb{C}^N$. Let

$$\|f\|_K := \sup\{|f(z)| : z \in K\},$$

where $f : K \to \mathbb{C}$ is a continuous function.

Definition 1.1 (see [1], [6]). *Alexander's projective capacity* $\gamma(K)$ is

$$\gamma(K) := \lim_{n\to\infty} \gamma_n(K)^{1/n} = \inf_n \gamma_n(K)^{1/n},$$

where

$$\gamma_n(K) := \inf\{\|Q\|_K\},$$

the infimum being taken over all homogeneous polynomials $Q \in H_n$, nor-

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malized so that

$$\frac{1}{s_N} \int_S \log(|Q(z)|^{1/n})\, d\sigma(z) = \kappa_N,$$

where

$$\kappa_N := \frac{1}{s_N} \int_S \log|z_N|\, d\sigma(z).$$

It is known that

$$\kappa_N = -\frac{1}{2} \sum_{j=1}^{N-1} \frac{1}{j}.$$

Let $a_j > 0$ and $R_j > 0$ $(j = 1, \ldots, N)$. In this paper Alexander's projective capacity for the polydisk $\{(z_1, \ldots, z_N) \in \mathbb{C}^N : |z_1| \le R_1, \ldots, |z_N| \le R_N\}$ and the ellipsoid $\{(z_1, \ldots, z_N) \in \mathbb{C}^N : a_1|z_1|^2 + \ldots + a_N|z_N|^2 \le 1\}$ is computed.

2. Preliminaries. Let K be a compact subset of $\mathbb{C}^N$.

Definition 2.1 (see [4]–[6]). The *Siciak homogeneous extremal function* Ψ_K is

$$\Psi(z) = \Psi_K(z) := \lim_{n\to\infty} \Psi_n(z)^{1/n}, \qquad z \in \mathbb{C}^N,$$

where

$$\Psi_n(z) := \sup\{|Q(z)|\},$$

the supremum being taken over all $Q \in H_n$ normalized so that $\|Q\|_K = 1$.

Proposition 2.2 (see [4], p. 304). *If* $R_j > 0$ $(j = 1, \ldots, N)$ *and* P *is the polydisk*

$$P := \{(z_1, \ldots, z_N) \in \mathbb{C}^N : |z_1| \le R_1, \ldots, |z_N| \le R_N\},$$

then

$$\Psi_P(z_1, \ldots, z_N) = \max\{|z_1|/R_1, |z_2|/R_2, \ldots, |z_N|/R_N\}.$$

Proposition 2.3 (see [5], p. 342). *If* $a_j > 0$ $(j = 1, \ldots, N)$ *and* E *is the ellipsoid*

$$E := \{(z_1, \ldots, z_N) \in \mathbb{C}^N : a_1|z_1|^2 + \ldots + a_N|z_N|^2 \le 1\},$$

then

$$\Psi_E(z_1, \ldots, z_N) = (a_1|z_1|^2 + \ldots + a_N|z_N|^2)^{1/2}.$$

Definition 2.4 (see [6], p. 53). The constant $\tau(K)$ is

$$\tau(K) := \exp\Big(-\frac{1}{s_N} \int_S \log \Psi_K(z)\, d\sigma(z)\Big).$$

Theorem 2.5 (see [2]). *If* K *is a compact subset of* $\mathbb{C}^N$ *then*

$$\gamma(K) = e^{\kappa_N} \tau(K).$$

3. Main result

THEOREM 3.1. *If* $a_j > 0$ $(j = 1, \ldots, N)$ *and* E *is the ellipsoid*

$$E := \{(z_1, \ldots, z_N) \in \mathbb{C}^N : a_1|z_1|^2 + \ldots + a_N|z_N|^2 \le 1\},$$

then

$$\gamma(E) = \exp\Big(-\frac{1}{2} \cdot \frac{1}{2\pi i} \int_C \frac{z^{N-1} \operatorname{Log} z \, dz}{(z - a_1) \ldots (z - a_N)} \Big),$$

where C *is any contour in the right half-plane* $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ *enclosing all the points* $a_1, \ldots, a_N$ *and* $\operatorname{Log} z$ *is the principal branch of the logarithm.*

In particular, if $a_j \ne a_k$ *for* $j \ne k$, *then*

$$\gamma(E) = \exp\Big(-\frac{1}{2} \sum_{j=1}^{N} \frac{a_j^{N-1} \log a_j}{\prod_{k=1, k\ne j}^{N} (a_j - a_k)} \Big).$$

If $a_1 = \ldots = a_N = 1/R^2$ $(R > 0)$, *then*

$$\gamma(E) = Re^{\kappa_N} = R \exp\Big(-\frac{1}{2} \sum_{j=1}^{N-1} \frac{1}{j} \Big).$$

THEOREM 3.2. *If* $R_j > 0$ $(j = 1, \ldots, N)$ *and* P *is the polydisk*

$$P := \{(z_1, \ldots, z_N) \in \mathbb{C}^N : |z_1| \le R_1, \ldots, |z_N| \le R_N\}$$

then

$$\gamma(P) = \Big(\prod_{k=1}^{N} M_k^{(-1)^{k-1}} \Big)^{1/2},$$

where

$$M_k := \prod_{1 \le i_1 < \ldots < i_k \le N} \Big(\sum_{j=1}^{k} R_{i_j}^2 \Big).$$

We first prove some lemmas. Let

$$D = D^{N-1} := \Big\{ (r_1, \ldots, r_N) \in \mathbb{R}^N : r_1 \ge 0, \ldots, r_N \ge 0, \ \sum_{j=1}^{N} r_j^2 = 1 \Big\},$$

$$\Sigma = \Sigma^{N-1} := \Big\{ (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N : \theta_1 \ge 0, \ldots, \theta_N \ge 0, \ \sum_{j=1}^{N} \theta_j = 1 \Big\}.$$

LEMMA 3.3. *If* $f : D \to \mathbb{R}$ *is a continuous function then*

$$\frac{1}{s_N} \int_S f(|z_1|, \ldots, |z_N|) \, d\sigma(z) = \frac{1}{\operatorname{vol}(\Sigma)} \int_\Sigma f(\sqrt{\theta_1}, \ldots, \sqrt{\theta_N}) \, d\omega(\theta),$$

where ω *is the Lebesgue surface area measure on the hyperplane* $\{(\theta_1, \ldots, \theta_N) \in \mathbb{R}^N : \sum_{j=1}^{N} \theta_j = 1\}$ *and* $\operatorname{vol}(\Sigma) := \int_\Sigma d\omega(\theta)$.

Proof. Our proof starts with the observation that

$$\frac{1}{s_N}\int\limits_S f(|z_1|,|z_2|,\ldots,|z_N|)\,d\sigma(z) = \frac{1}{\widetilde{s}_N}\int\limits_D f(r_1,\ldots,r_N)r_1\ldots r_N\,d\widetilde{\sigma}(r), \tag{3.1}$$

where $\widetilde{\sigma}$ is the Lebesgue surface area measure on the unit sphere in $\mathbb{R}^N$ and

$$\widetilde{s}_N := \int\limits_D r_1\ldots r_N\,d\widetilde{\sigma}(r).$$

Indeed, consider the following coordinates in $\mathbb{C}^N$ ($z_j = x_j + iy_j$ for $j = 1,\ldots,N$):

$$x_j = r_j\cos\psi_j,\quad y_j = r_j\sin\psi_j\quad (r_j\ge 0,\ 0\le\psi_j\le 2\pi,\ j=1,\ldots,N),$$

where

$$\begin{aligned}
r_1 &= r\cos\phi_1,\\
r_2 &= r\sin\phi_1\cos\phi_2,\\
r_3 &= r\sin\phi_1\sin\phi_2\cos\phi_3,\\
&\ \vdots\\
r_{N-1} &= r\sin\phi_1\sin\phi_2\ldots\sin\phi_{N-2}\cos\phi_{N-1},\\
r_N &= r\sin\phi_1\sin\phi_2\ldots\sin\phi_{N-2}\sin\phi_{N-1}\\
&\qquad (r\ge 0,\ 0\le\phi_j\le\pi/2,\ j=1,\ldots,N-1).
\end{aligned}$$

It is easy to see that

$$dx_1\ldots dx_N\,dy_1\ldots dy_N = r_1\ldots r_N\,dr_1\ldots dr_N\,d\psi_1\ldots d\psi_N$$

and

$$dr_1\ldots dr_N = r^{N-1}\sin^{N-2}\phi_1\sin^{N-3}\phi_2\ldots\sin\phi_{N-2}\,dr\,d\phi_1\ldots d\phi_{N-1}.$$

Therefore

$$\begin{aligned}
dx_1\ldots dx_N\,dy_1\ldots dy_N &= r^{2N-1}\cos\phi_1\sin^{2N-3}\phi_1\cos\phi_2\sin^{2N-5}\phi_2\\
&\quad\ldots\cos\phi_{N-1}\sin\phi_{N-1}dr\,d\phi_1\ldots d\phi_{N-1}\,d\psi_1\ldots d\psi_N,
\end{aligned}$$

where

$$r\ge 0,\quad 0\le\phi_j\le\pi/2\quad\text{for } j=1,\ldots,N-1,$$
$$0\le\psi_j\le 2\pi\quad\text{for } j=1,\ldots,N.$$

Hence

$$d\sigma = \cos\phi_1 \sin^{2N-3}\phi_1 \cos\phi_2 \sin^{2N-5}\phi_2 \dots \cos\phi_{N-1} \sin\phi_{N-1} \, d\phi_1 \dots d\phi_{N-1}\, d\psi_1 \dots d\psi_N.$$

In the same manner we can see that

$$d\widetilde{\sigma} = \sin^{N-2}\phi_1 \sin^{N-3}\phi_2 \dots \sin\phi_{N-2} d\phi_1 \dots d\phi_{N-1},$$

where $0 \le \phi_j \le \pi/2$, $j = 1, \dots, N-1$.

Now it is easy to see that (3.1) is true.

It remains to prove that

$$\text{(3.2)} \qquad \frac{1}{\widetilde{s}_N} \int_D f(r_1, \dots, r_N) r_1 \dots r_N \, d\widetilde{\sigma}(r) = \frac{1}{\operatorname{vol}(\Sigma)} \int_\Sigma f(\sqrt{\theta_1}, \dots, \sqrt{\theta_N})\, d\omega(\theta),$$

which is equivalent to the equality

$$\text{(3.3)} \qquad \widetilde{A}_1 = \widetilde{A}_2,$$

where

$$\widetilde{A}_1 := \frac{1}{s_N^*} \int_{D_*} f\Big(r_1, \dots, r_{N-1}, \Big(1 - \sum_{j=1}^{N-1} r_j^2\Big)^{1/2}\Big) r_1 \dots r_{N-1}\, dr_1 \dots dr_{N-1},$$

$$\widetilde{A}_2 := \frac{1}{\operatorname{vol}(\Sigma_*)} \int_{\Sigma_*} f\Big(\sqrt{\theta_1}, \dots, \sqrt{\theta_{N-1}}, \Big(1 - \sum_{j=1}^{N-1} \theta_j\Big)^{1/2}\Big) d\theta_1 \dots d\theta_{N-1},$$

$$D_* := \Big\{(r_1, \dots, r_{N-1}) \in \mathbb{R}^{N-1} : r_1 \ge 0, \dots, r_{N-1} \ge 0, \sum_{j=1}^{N-1} r_j^2 \le 1\Big\},$$

$$\Sigma_* := \Big\{(\theta_1, \dots, \theta_{N-1}) \in \mathbb{R}^{N-1} : \theta_1 \ge 0, \dots, \theta_{N-1} \ge 0, \sum_{j=1}^{N-1} \theta_j \le 1\Big\},$$

$$s_N^* := \int_{D_*} r_1 \dots r_{N-1}\, dr_1 \dots dr_{N-1},$$

and $\operatorname{vol}(\Sigma_*) := \int_{\Sigma_*} d\theta_1 \dots d\theta_{N-1}$. Indeed, consider the following parametrization of D:

$$r_j = r_j \quad \text{for } j = 1, \dots, N-1, \quad r_N = \Big(1 - \sum_{j=1}^{N-1} r_j^2\Big)^{1/2},$$

and the parametrization of the simplex Σ:

$$\theta_j = \theta_j \quad \text{for } j = 1, \dots, N-1, \quad \theta_N = 1 - \sum_{j=1}^{N-1} \theta_j.$$

Obviously,

$$d\widetilde{\sigma} = \left(1 + \sum_{j=1}^{N-1} \left(\frac{\partial r_N}{\partial r_j}\right)^2\right)^{1/2} dr_1 \ldots dr_{N-1} = \frac{dr_1 \ldots dr_{N-1}}{\sqrt{1 - r_1^2 - \ldots - r_{N-1}^2}}$$

and

$$d\omega = \left(1 + \sum_{j=1}^{N-1} \left(\frac{\partial \theta_N}{\partial \theta_j}\right)^2\right)^{1/2} d\theta_1 \ldots d\theta_{N-1} = \sqrt{N} d\theta_1 \ldots d\theta_{N-1}.$$

Now it follows easily that (3.2) is equivalent to (3.3). Therefore it suffices to prove (3.3). We change the variables:

$$r_j = \sqrt{\theta_j} \quad \text{for } j = 1, \ldots, N-1.$$

Clearly,

$$dr_1 \ldots dr_{N-1} = \frac{d\theta_1 \ldots d\theta_{N-1}}{2^{N-1}\sqrt{\theta_1 \ldots \theta_{N-1}}}.$$

We see at once that (3.3) is true, which completes the proof.

LEMMA 3.4 (see [3], Lemma 6.3). *If* $a_1, \ldots, a_N \in \mathbb{R}$ *and* $a_j \neq a_k$ *for* $j \neq k$, *then*

$$\sum_{j=1}^{N} \frac{1}{\prod_{k=1,\, k\neq j}^{N} (a_j - a_k)} = 0, \tag{3.4}$$

$$\sum_{j=1}^{N} \frac{a_j^{N-1}}{\prod_{k=1,\, k\neq j}^{N} (a_j - a_k)} = 1. \tag{3.5}$$

Proof. Let $W(x) := -1 + \sum_{j=1}^{N} W_j(x)$ and

$$V(x) := -x^{N-1} + \sum_{j=1}^{N} a_j^{N-1} W_j(x), \quad \text{where} \quad W_j(x) := \prod_{\substack{k=1\\ k\neq j}}^{N} \frac{x - a_k}{a_j - a_k}.$$

It is clear that $W(x) = \sum_{i=0}^{N-1} B_i x^i$ and $V(x) = \sum_{j=0}^{N-1} C_j x^j$ (where $B_i, C_j \in \mathbb{R}$). We see that $W_j(a_j) = 1$ and $W_j(a_k) = 0$ for $k \neq j$. Hence $W(a_j) = 0$ for $j = 1, \ldots, N$ and $V(a_j) = 0$ for $j = 1, \ldots, N$, which implies $W \equiv 0$ and $V \equiv 0$. Therefore $B_{N-1} = 0$ and $C_{N-1} = 0$, which is the desired conclusion.

LEMMA 3.5 (see [3], Lemma 6.2). *If* $f : \Sigma^{N-1} \to \mathbb{R}$ *is a continuous function*, *then*

$$\frac{1}{\mathrm{vol}(\Sigma^{N-1})} \int_{\Sigma^{N-1}} f(\theta_1, \ldots, \theta_N)\, d\omega(\theta) = (N-1) \int_0^1 x^{N-2} L(x)\, dx,$$

where

$$L(x) := \frac{1}{\operatorname{vol}(\Sigma^{N-2})} \int\limits_{\Sigma^{N-2}} f(\xi_1 x, \ldots, \xi_{N-1} x, 1-x)\, d\widetilde{\omega}(\xi),$$

$\widetilde{\omega}$ *is the Lebesgue surface area measure on the simplex*

$$\Sigma^{N-2} = \Big\{(\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1} : \xi_1 \ge 0, \ldots, \xi_{N-1} \ge 0, \sum_{j=1}^{N-1} \xi_j = 1\Big\}$$

and $\operatorname{vol}(\Sigma^{N-2}) = \int_{\Sigma^{N-2}} d\widetilde{\omega}(\xi)$.

Proof. As in the proof of Lemma 3.3, we obtain

$$d\omega(\theta) = \sqrt{N}\, d\theta_1 \ldots d\theta_{N-1}.$$

Therefore

$$\frac{1}{\operatorname{vol}(\Sigma^{N-1})} \int\limits_{\Sigma^{N-1}} f(\theta_1, \ldots, \theta_N)\, d\omega(\theta) = \frac{1}{\operatorname{vol}(\Sigma_*^{N-1})} \int\limits_{\Sigma_*^{N-1}} f\Big(\theta_1, \ldots, \theta_{N-1}, 1 - \sum_{j=1}^{N-1} \theta_j\Big) d\theta_1 \ldots d\theta_{N-1},$$

where $\operatorname{vol}(\Sigma_*^{N-1}) = \int_{\Sigma_*^{N-1}} d\theta_1 \ldots d\theta_{N-1}$. We change the variables:

$$\theta_j = \xi_j x \quad \text{for } j = 1, \ldots, N-2, \quad \theta_{N-1} = \Big(1 - \sum_{j=1}^{N-2} \xi_j\Big) x,$$

where $0 \le x \le 1$ and $(\xi_1, \ldots, \xi_{N-2}) \in \Sigma_*^{N-2}$. It is easy to check that $d\theta_1 \ldots d\theta_{N-1} = x^{N-2} dx d\xi_1 \ldots d\xi_{N-2}$. Therefore

$$\begin{aligned}
&\frac{1}{\operatorname{vol}(\Sigma_*^{N-1})} \int\limits_{\Sigma_*^{N-1}} f\Big(\theta_1, \ldots, \theta_{N-1}, 1 - \sum_{j=1}^{N-1} \theta_j\Big)\, d\theta_1 \ldots d\theta_{N-1} \\
&\quad = \frac{1}{\operatorname{vol}(\Sigma_*^{N-1})} \int\limits_0^1 x^{N-2} \\
&\qquad \times \int\limits_{\Sigma_*^{N-2}} f\Big(\xi_1 x, \ldots, \xi_{N-2} x, \Big(1 - \sum_{j=1}^{N-2} \xi_j\Big) x, 1 - x\Big)\, d\xi_1 \ldots d\xi_{N-2}\, dx \\
&\quad = \frac{\operatorname{vol}(\Sigma_*^{N-2})}{\operatorname{vol}(\Sigma_*^{N-1})} \int\limits_0^1 x^{N-2} \Big(\frac{1}{\operatorname{vol}(\Sigma_*^{N-2})} \\
&\qquad \times \int\limits_{\Sigma_*^{N-2}} f\Big(\xi_1 x, \ldots, \xi_{N-2} x, \Big(1 - \sum_{j=1}^{N-2} \xi_j\Big) x, 1 - x\Big)\, d\xi_1 \ldots d\xi_{N-2}\Big)\, dx
\end{aligned}$$

$$= \frac{((N-2)!)^{-1}}{((N-1)!)^{-1}} \int_0^1 x^{N-2}$$

$$\times \left(\frac{1}{\operatorname{vol}(\Sigma^{N-2})} \int_{\Sigma^{N-2}} f(\xi_1 x, \ldots, \xi_{N-2}x, \xi_{N-1}x, 1-x)\, d\widetilde{\omega}(\xi) \right) dx$$

$$= (N-1) \int_0^1 x^{N-2} L(x)\, dx,$$

which completes the proof.

LEMMA 3.6. *If* $a_j > 0$ *for* $j = 1, \ldots, N$, *then*

$$(3.6) \quad \frac{1}{\operatorname{vol}(\Sigma^{N-1})} \int_{\Sigma^{N-1}} \log(a_1\theta_1 + \ldots + a_N\theta_N)\, d\omega(\theta) = -\sum_{j=1}^{N-1} \frac{1}{j} + \frac{1}{2\pi i} \int_C \frac{z^{N-1} \operatorname{Log} z\, dz}{(z-a_1)\ldots(z-a_N)},$$

where C *is any contour in the right half-plane enclosing all the points* $a_1, \ldots, a_N$ *and* $\operatorname{Log} z$ *is the principal branch of the logarithm. In particular, if* $a_j \neq a_k$ *for* $j \neq k$, *then*

$$(3.7) \quad \frac{1}{\operatorname{vol}(\Sigma^{N-1})} \int_{\Sigma^{N-1}} \log(a_1\theta_1 + \ldots + a_N\theta_N)\, d\omega(\theta) = -\sum_{j=1}^{N-1} \frac{1}{j} + \sum_{j=1}^{N} \frac{a_j^{N-1} \log a_j}{\prod_{k=1,\, k\neq j}^{N} (a_j - a_k)}.$$

If $a_1 = \ldots = a_N = 1/R^2$ $(R > 0)$, *then the above integral is equal to* $-2\log R$.

Proof. A trivial verification shows that the lemma is true if $a_1 = a_2 = \ldots = a_N = 1/R^2$ $(R > 0)$. This result can also be obtained from (3.6) (by the residue theorem), because $g^{(N-1)}(z) = (N-1)!(\operatorname{Log} z + \sum_{j=1}^{N-1} 1/j)$ if $g(z) := z^{N-1} \operatorname{Log} z$.

Let us prove (3.6). Both its sides are continuous functions of the parameters a_j. Therefore it suffices to prove (3.7). The proof is by induction on N. It is easy to check the case $N = 2$:

$$\frac{1}{\operatorname{vol}(\Sigma^1)} \int_{\Sigma^1} \log(a_1\theta_1 + a_2\theta_2)\, d\omega(\theta) = -1 + \frac{a_1 \log a_1}{a_1 - a_2} + \frac{a_2 \log a_2}{a_2 - a_1}.$$

Assuming (3.7) to hold for $N - 1$ $(N \geq 3)$, we will prove it for N, applying

Lemma 3.5. We first compute (with $f(\theta) := \log(\sum_{j=1}^N a_j\theta_j)$):

$$\operatorname{vol}(\Sigma^{N-2})L(x) = \int\limits_{\Sigma^{N-2}} f(\xi_1 x, \ldots, \xi_{N-1}x, 1-x)\, d\widetilde{\omega}(\xi)$$

$$= \int\limits_{\Sigma^{N-2}} \log\Big(\sum_{j=1}^{N-1} a_j\xi_j x + a_N(1-x)\Big)\, d\widetilde{\omega}(\xi)$$

We have $a_N(1-x) \equiv a_N(1-x)\sum_{j=1}^{N-1}\xi_j$ on Σ^{N-2}. Therefore

$$\operatorname{vol}(\Sigma^{N-2})L(x) = \int\limits_{\Sigma^{N-2}} \log\Big(\sum_{j=1}^{N-1} A_j\xi_j\Big)\, d\widetilde{\omega}(\xi),$$

where $A_j = A_j(x) := a_N + (a_j - a_N)x$ for $j = 1, \ldots, N-1$. By assumption,

$$L(x) = \frac{1}{\operatorname{vol}(\Sigma^{N-2})}\int\limits_{\Sigma^{N-2}} \log\Big(\sum_{j=1}^{N-1} A_j\xi_j\Big) d\widetilde{\omega}(\xi)$$

$$= -\sum_{j=1}^{N-2}\frac{1}{j} + \sum_{j=1}^{N-1}\frac{(A_j(x))^{N-2}\log A_j(x)}{\prod_{k=1,\,k\neq j}^{N-1}(A_j(x) - A_k(x))}$$

$$= -\sum_{j=1}^{N-2}\frac{1}{j} + \sum_{j=1}^{N-1}\frac{(a_N + (a_j - a_N)x)^{N-2}\log(a_N + (a_j - a_N)x)}{\prod_{k=1,\,k\neq j}^{N-1}(a_j - a_k)x}$$

Applying Lemma 3.5 we obtain

$$\frac{1}{\operatorname{vol}(\Sigma^{N-1})}\int\limits_{\Sigma^{N-1}} \log\Big(\sum_{j=1}^{N} a_j\theta_j\Big)\, d\omega(\theta) = (N-1)\int\limits_0^1 x^{N-2}L(x)dx = G_1 + G_2,$$

where

$$G_1 := (N-1)\int\limits_0^1 x^{N-2}\Big(-\sum_{j=1}^{N-2}\frac{1}{j}\Big)\, dx = -\sum_{j=1}^{N-2}\frac{1}{j},$$

$$G_2 := (N-1)\sum_{j=1}^{N-1}\frac{\int_0^1 (a_N + (a_j - a_N)x)^{N-2}\log(a_N + (a_j - a_N)x)\, dx}{\prod_{k=1,\,k\neq j}^{N-1}(a_j - a_k)}.$$

Integrating by parts the integral in G_2, we obtain $G_2 = Z_1 + Z_2$, where

$$Z_1 := -\frac{1}{N-1}\sum_{j=1}^{N-1}\frac{a_j^{N-1} - a_N^{N-1}}{\prod_{k=1,\,k\neq j}^{N}(a_j - a_k)},$$

$$Z_2 := \sum_{j=1}^{N-1}\frac{a_j^{N-1}\log a_j - a_N^{N-1}\log a_N}{\prod_{k=1,\,k\neq j}^{N}(a_j - a_k)}.$$

Applying (3.4) and (3.5) (see Lemma 3.4) we get

$$\begin{aligned} Z_1 &= -\frac{1}{N-1}\Big(\sum_{j=1}^{N-1}\frac{a_j^{N-1}}{\prod_{k=1,\,k\neq j}^{N}(a_j-a_k)} - a_N^{N-1}\sum_{j=1}^{N-1}\frac{1}{\prod_{k=1,\,k\neq j}^{N}(a_j-a_k)}\Big)\\ &= -\frac{1}{N-1}\Big(\sum_{j=1}^{N-1}\frac{a_j^{N-1}}{\prod_{k=1,\,k\neq j}^{N}(a_j-a_k)} + \frac{a_N^{N-1}}{\prod_{k=1}^{N-1}(a_N-a_k)}\Big)\\ &= -\frac{1}{N-1}\sum_{j=1}^{N}\frac{a_j^{N-1}}{\prod_{k=1,\,k\neq j}^{N}(a_j-a_k)} = -\frac{1}{N-1}. \end{aligned}$$

By (3.4),

$$\begin{aligned} Z_2 &= \sum_{j=1}^{N-1}\frac{a_j^{N-1}\log a_j}{\prod_{k=1,\,k\neq j}^{N}(a_j-a_k)} - a_N^{N-1}\log a_N\sum_{j=1}^{N-1}\frac{1}{\prod_{k=1,\,k\neq j}^{N}(a_j-a_k)}\\ &= \sum_{j=1}^{N-1}\frac{a_j^{N-1}\log a_j}{\prod_{k=1,\,k\neq j}^{N}(a_j-a_k)} + \frac{a_N^{N-1}\log a_N}{\prod_{k=1}^{N-1}(a_N-a_k)}\\ &= \sum_{j=1}^{N}\frac{a_j^{N-1}\log a_j}{\prod_{k=1,\,k\neq j}^{N}(a_j-a_k)}. \end{aligned}$$

Thus $G_1+Z_1+Z_2$ is equal to the right-hand side of (3.7), which proves the lemma.

Lemma 3.7. *Let* $\{g_1,\ldots,g_l\}$ *and* $\{h_1,\ldots,h_l\}$ *be two sets of integers. Let* $b_{h_j}>0$ *for* $j=1,\ldots,l$. *Let* $S((g),(h))$ *be the set of all bijections* $\{g_1,\ldots,g_l\}\to\{h_1,\ldots,h_l\}$. *Then*

$$\sum_{\varrho\in S((g),(h))}\frac{\prod_{j=1}^{l}b_{\varrho(g_j)}}{\prod_{j=1}^{l}(b_{\varrho(g_1)}+b_{\varrho(g_2)}+\ldots+b_{\varrho(g_j)})}=1. \tag{3.8}$$

Proof. We proceed by induction on l. The lemma is true for $l=1$. Assuming (3.8) to hold for $l-1$ ($l\ge 2$), we will prove it for l. Let $k\in\{1,\ldots,l\}$. Let

$$S_k((g),(h)) := \{\varrho\in S((g),(h)) : \varrho(g_l)=h_k\}.$$

Let

$$T_k := \sum_{\varrho\in S_k((g),(h))}\frac{\prod_{j=1}^{l}b_{\varrho(g_j)}}{\prod_{j=1}^{l}(b_{\varrho(g_1)}+b_{\varrho(g_2)}+\ldots+b_{\varrho(g_j)})}.$$

It is easy to check that

$$T_k = \frac{b_{h_k}}{b_{h_1}+\ldots+b_{h_l}}\sum_{\mu}\frac{\prod_{j=1}^{l-1}b_{\mu(g_j)}}{\prod_{j=1}^{l-1}(b_{\mu(g_1)}+\ldots+b_{\mu(g_j)})},$$

where the summation extends over all bijections $\mu : \{g_1, \ldots, g_{l-1}\} \to \{h_1, \ldots, h_l\} \setminus \{h_k\}$. By the induction hypothesis,

$$T_k = \frac{b_{h_k}}{b_{h_1} + \ldots + b_{h_l}}.$$

Obviously, $S((g),(h)) = \bigcup_{k=1}^{l} S_k((g),(h))$ and

$$S_i((g),(h)) \cap S_j((g),(h)) = \emptyset \quad \text{for } i \neq j.$$

Therefore the left-hand side of (3.8) is equal to $\sum_{k=1}^{l} T_k = 1$, which completes the proof.

Proof of Theorem 3.1. Applying Theorem 2.5, Proposition 2.3, Lemma 3.3 and Lemma 3.6 gives

$$\begin{aligned}
\log \gamma(E) &= \kappa_N + \log \tau(E) = \kappa_N - \frac{1}{s_N} \int\limits_S \log \Psi_E(z)\, d\sigma(z) \\
&= \kappa_N - \frac{1}{2} \cdot \frac{1}{s_N} \int\limits_S \log(a_1|z_1|^2 + \ldots + a_N|z_N|^2)\, d\sigma(z) \\
&= \kappa_N - \frac{1}{2} \cdot \frac{1}{\mathrm{vol}(\Sigma)} \int\limits_\Sigma \log(a_1\theta_1 + \ldots + a_N\theta_N)\, d\omega(\theta) \\
&= -\frac{1}{2} \sum_{j=1}^{N-1} \frac{1}{j} - \frac{1}{2}\bigg(- \sum_{j=1}^{N-1} \frac{1}{j} + \frac{1}{2\pi i} \int\limits_C \frac{z^{N-1} \operatorname{Log} z\, dz}{(z-a_1)\ldots(z-a_N)}\bigg) \\
&= -\frac{1}{2} \cdot \frac{1}{2\pi i} \int\limits_C \frac{z^{N-1} \operatorname{Log} z\, dz}{(z-a_1)\ldots(z-a_N)}.
\end{aligned}$$

The particular case $a_j \neq a_k$ for $j \neq k$ is an easy consequence of the above formula (it suffices to apply the residue theorem).

Proof of Theorem 3.2. We have to show that

$$\text{(3.9)} \qquad \log \gamma(P) = \frac{1}{2} \sum_{j=1}^{N} \sum_{1 \le k_1 < \ldots < k_j \le N} (-1)^{j-1} \log(b_{k_1} + \ldots + b_{k_j}),$$

where $b_j := R_j^2$ for $j = 1, \ldots, N$. We first observe that

$$\text{(3.10)} \qquad \log \gamma(P) = \frac{1}{2}\bigg(- \sum_{j=1}^{N-1} \frac{1}{j} - \frac{1}{\mathrm{vol}(\Sigma)} \int\limits_\Sigma \max\bigg(\log \frac{\theta_1}{b_1}, \ldots, \log \frac{\theta_N}{b_N}\bigg)\, d\omega(\theta)\bigg).$$

Indeed, combining Theorem 2.5 with Proposition 2.2 and Lemma 3.3 gives

$$\begin{aligned}\log\gamma(P) &= \kappa_N + \log\tau(P) = \kappa_N - \frac{1}{s_N}\int\limits_S \log\Psi_P(z)\,d\sigma(z)\\ &= \kappa_N - \frac{1}{s_N}\int\limits_S \log\Big(\max\Big(\frac{|z_1|}{R_1},\ldots,\frac{|z_N|}{R_N}\Big)\Big)d\sigma(z)\\ &= \kappa_N - \frac{1}{s_N}\int\limits_S \max\Big(\log\frac{|z_1|}{R_1},\ldots,\log\frac{|z_N|}{R_N}\Big)\,d\sigma(z)\\ &= -\frac{1}{2}\sum_{j=1}^{N-1}\frac{1}{j} - \frac{1}{2}\cdot\frac{1}{s_N}\int\limits_S \max\Big(\log\frac{|z_1|^2}{R_1^2},\ldots,\log\frac{|z_N|^2}{R_N^2}\Big)\,d\sigma(z)\\ &= \frac{1}{2}\Big(-\sum_{j=1}^{N-1}\frac{1}{j} - \frac{1}{\mathrm{vol}(\Sigma)}\int\limits_\Sigma \max\Big(\log\frac{\theta_1}{b_1},\ldots,\log\frac{\theta_N}{b_N}\Big)d\omega(\theta)\Big).\end{aligned}$$

Let $S(N)$ denote the set of all permutations of $\{1,\ldots,N\}$. For $\varrho\in S(N)$ define

$$\begin{aligned}\Lambda_\varrho &:= \Big\{(\theta_1,\ldots,\theta_N)\in\Sigma^{N-1} : \frac{\theta_{\varrho(1)}}{b_{\varrho(1)}}\ge\ldots\ge\frac{\theta_{\varrho(N)}}{b_{\varrho(N)}}\Big\},\\ t(\varrho(1),\ldots,\varrho(N)) &:= \frac{1}{\mathrm{vol}(\Sigma)}\int\limits_{\Lambda_\varrho}\max\Big(\log\frac{\theta_1}{b_1},\ldots,\log\frac{\theta_N}{b_N}\Big)\,d\omega(\theta)\\ &= \frac{1}{\mathrm{vol}(\Sigma)}\int\limits_{\Lambda_\varrho}\log\frac{\theta_{\varrho(1)}}{b_{\varrho(1)}}\,d\omega(\theta).\end{aligned}$$

Obviously,

$$\frac{1}{\mathrm{vol}(\Sigma)}\int\limits_\Sigma \max\Big(\log\frac{\theta_1}{b_1},\ldots,\log\frac{\theta_N}{b_N}\Big)d\omega(\theta) = \sum_{\varrho\in S(N)} t(\varrho(1),\ldots,\varrho(N)). \tag{3.11}$$

We next show that

$$\begin{aligned}&t(\varrho(1),\ldots,\varrho(N))\\ &= \frac{\prod_{i=1}^N b_{\varrho(i)}}{\prod_{i=1}^N(b_{\varrho(1)}+\ldots+b_{\varrho(i)})}\Big(-\sum_{j=1}^{N-1}\frac{1}{j}\Big)\\ &\quad+\sum_{j=1}^N(-1)^j\frac{(\prod_{i=1}^N b_{\varrho(i)})\log(b_{\varrho(1)}+\ldots+b_{\varrho(j)})}{(\prod_{i=1}^j(b_{\varrho(i)}+\ldots+b_{\varrho(j)}))(\prod_{i=j+1}^N(b_{\varrho(j+1)}+\ldots+b_{\varrho(i)}))},\end{aligned}\tag{3.12}$$

where $\prod_{i=j+1}^N(b_{\varrho(j+1)}+\ldots+b_{\varrho(i)}) := 1$ for $j=N$.

Without loss of generality we can assume that $\varrho = \varrho_0$, where $\varrho_0(j) := j$ for $j = 1, \ldots, N$. Clearly,

$$t(1, \ldots, N) = \frac{1}{\mathrm{vol}(\Sigma)} \int\limits_{\Lambda} \log \frac{\theta_1}{b_1}\, d\omega(\theta), \tag{3.13}$$

where $\Lambda := \Lambda_{\varrho_0}$.

We first prove that

$$\begin{aligned}\frac{1}{\mathrm{vol}(\Sigma)} \int\limits_{\Lambda} \log \frac{\theta_1}{b_1}\, d\omega(\theta) \\ = \frac{\prod_{i=1}^N b_i}{\prod_{i=1}^N (b_1 + \ldots + b_i)} \frac{1}{\mathrm{vol}(\Sigma)} \int\limits_{\Sigma} \log\Big(\sum_{j=1}^N \frac{\eta_j}{b_1 + \ldots + b_j}\Big) d\omega(\eta).\end{aligned} \tag{3.14}$$

Let

$$\Sigma_{(*)} := \Big\{(\eta_2, \ldots, \eta_N) \in \mathbb{R}^{N-1} : \eta_2 \ge 0, \ldots, \eta_N \ge 0,\ \sum_{j=2}^N \eta_j \le 1\Big\},$$

$$\Lambda_{(*)} := \Big\{(\theta_2, \ldots, \theta_N) \in \mathbb{R}^{N-1} : \frac{1 - \sum_{j=2}^N \theta_j}{b_1} \ge \frac{\theta_2}{b_2} \ge \ldots \ge \frac{\theta_N}{b_N} \ge 0\Big\}.$$

Analysis similar to that in the proof of Lemma 3.3 shows that (3.14) is equivalent to

$$\begin{aligned}\frac{1}{\mathrm{vol}(\Sigma_{(*)})} \int\limits_{\Lambda_{(*)}} \log \frac{1 - \sum_{j=2}^N \theta_j}{b_1}\, d\theta_2 \ldots d\theta_N \\ = \frac{\prod_{i=1}^N b_i}{\prod_{i=1}^N (b_1 + \ldots + b_i)} \cdot \frac{1}{\mathrm{vol}(\Sigma_{(*)})} \\ \times \int\limits_{\Sigma_{(*)}} \log\Big(\sum_{j=1}^N \frac{\eta_j}{b_1 + b_2 + \ldots + b_j}\Big) d\eta_2 \ldots d\eta_N,\end{aligned} \tag{3.15}$$

where $\eta_1 := 1 - \sum_{j=2}^N \eta_j$.

The proof of (3.15) is immediate. We change the variables:

$$\theta_k = \sum_{j=k}^N \frac{b_k}{b_1 + \ldots + b_j} \eta_j, \qquad \text{for } k = 2, \ldots, N,$$

where $(\theta_2, \ldots, \theta_N) \in \Lambda_{(*)}$ and $(\eta_2, \ldots, \eta_N) \in \Sigma_{(*)}$. Obviously,

$$\frac{1 - \sum_{j=2}^N \theta_j}{b_1} = \sum_{j=1}^N \frac{\eta_j}{b_1 + \ldots + b_j}$$

and

$$d\theta_2 \dots d\theta_N = \frac{\prod_{i=1}^N b_i}{\prod_{i=1}^N (b_1 + \dots + b_i)} d\eta_2 \dots d\eta_N.$$

This establishes the formula (3.15). Hence (3.14) is also true.

Let us observe that the simplex Λ has the vertices

$$A_k = \left(\frac{b_1}{b_1 + \dots + b_k}, \frac{b_2}{b_1 + \dots + b_k}, \dots, \frac{b_k}{b_1 + \dots + b_k}, 0, \dots, 0 \right)$$

$(k = 1, \dots, N)$ and $\eta_1, \dots, \eta_N$ are the barycentric coordinates on Λ.

Let $c_j := b_1 + \dots + b_j$ and $a_j := 1/c_j$ (for $j = 1, \dots, N$). Applying (3.13), (3.14) and (3.7) (see Lemma 3.6) gives

$$t(1, \dots, N) = \frac{\prod_{i=1}^N b_i}{\prod_{i=1}^N (b_1 + \dots + b_i)} \times \left(-\sum_{j=1}^{N-1} \frac{1}{j} + \sum_{j=1}^{N} \frac{a_j^{N-1} \log a_j}{\prod_{k=1,\, k\neq j}^N (a_j - a_k)} \right). \tag{3.16}$$

We have, for $j = 1, \dots, N$,

$$\begin{aligned}
&\frac{a_j^{N-1} \cdot \log a_j}{\prod_{k=1,\, k\neq j}^N (a_j - a_k)} \\
&\qquad = \frac{(1/c_j)^{N-1} \log(1/c_j)}{\prod_{k=1,\, k\neq j}^N (1/c_j - 1/c_k)} \\
&\qquad = -\frac{c_j^{N-1} \prod_{i=1}^N c_i}{c_j^{N-1} \cdot \prod_{i=1}^N c_i} \cdot \frac{(1/c_j)^{N-1} \log c_j}{\prod_{k=1,\, k\neq j}^N (1/c_j - 1/c_k)} \\
&\qquad = -\frac{(\prod_{i=1}^N c_i) \log c_j}{c_j \cdot \prod_{i=1,\, i\neq j}^N (c_i - c_j)} \\
&\qquad = -\frac{(\prod_{i=1}^N (b_1 + \dots + b_i)) \log(b_1 + \dots + b_j)}{(-1)^{j-1} (\prod_{i=1}^j (b_i + \dots + b_j)) (\prod_{i=j+1}^N (b_{j+1} + \dots + b_i))},
\end{aligned}$$

where $\prod_{i=j+1}^N (b_{j+1} + \dots + b_i) := 1$ for $j = N$. On substituting the above expression into (3.16) we obtain

$$\begin{aligned}
t(1, \dots, N) = \frac{\prod_{i=1}^N b_i}{\prod_{i=1}^N (b_1 + \dots + b_i)} &\left(-\sum_{j=1}^{N-1} \frac{1}{j} \right) \\
&+ \sum_{j=1}^{N} (-1)^j \frac{(\prod_{i=1}^N b_i) \log(b_1 + \dots + b_j)}{(\prod_{i=1}^j (b_i + \dots + b_j)) (\prod_{i=j+1}^N (b_{j+1} + \dots + b_i))},
\end{aligned}$$

and (3.12) is proved for $\varrho = \varrho_0$. In the same manner we can see that (3.12) is true for each $\varrho \in S(N)$.

By Lemma 3.7,

$$\sum_{\varrho \in S(N)} \frac{\prod_{i=1}^N b_{\varrho(i)}}{\prod_{i=1}^N (b_{\varrho(1)} + \ldots + b_{\varrho(i)})} = 1. \tag{3.17}$$

Combining (3.10) with (3.11), (3.12) and (3.17) gives

$$\begin{aligned}
\log \gamma(P) = & -\frac{1}{2}\Big(\sum_{j=1}^{N-1} \frac{1}{j} + \sum_{\varrho \in S(N)} t(\varrho(1), \ldots, \varrho(N))\Big) \\
= & -\frac{1}{2}\Big(\sum_{j=1}^{N-1} \frac{1}{j}\Big)\Big(1 - \sum_{\varrho \in S(N)} \frac{\prod_{i=1}^N b_{\varrho(i)}}{\prod_{i=1}^N (b_{\varrho(1)} + \ldots + b_{\varrho(i)})}\Big) \\
& + \frac{1}{2}\sum_{j=1}^{N} \sum_{\varrho \in S(N)} (-1)^{j-1} \\
& \times \frac{(\prod_{i=1}^N b_{\varrho(i)}) \log(b_{\varrho(1)} + \ldots + b_{\varrho(j)})}{(\prod_{i=1}^j (b_{\varrho(i)} + \ldots + b_{\varrho(j)}))(\prod_{i=j+1}^N (b_{\varrho(j+1)} + \ldots + b_{\varrho(i)}))} \\
= & \frac{1}{2}\sum_{j=1}^{N} \sum_{\varrho \in S(N)} (-1)^{j-1} \\
& \times \frac{(\prod_{i=1}^N b_{\varrho(i)}) \log(b_{\varrho(1)} + \ldots + b_{\varrho(j)})}{(\prod_{i=1}^j (b_{\varrho(i)} + \ldots + b_{\varrho(j)}))(\prod_{i=j+1}^N (b_{\varrho(j+1)} + \ldots + b_{\varrho(i)}))} \\
= & \frac{1}{2}\sum_{j=1}^{N} \sum_{1 \le k_1 < \ldots < k_j \le N} (-1)^{j-1} B(k_1, \ldots, k_j) \log(b_{k_1} + \ldots + b_{k_j}),
\end{aligned}$$

where

$$B(k_1, \ldots, k_j) := \sum_{\varrho} \frac{\prod_{i=1}^N b_{\varrho(i)}}{(\prod_{i=1}^j (b_{\varrho(i)} + \ldots + b_{\varrho(j)}))(\prod_{i=j+1}^N (b_{\varrho(j+1)} + \ldots + b_{\varrho(i)}))}$$

and the summation runs over all permutations $\varrho \in S(N)$ such that $\varrho(\{1, \ldots, j\}) = \{k_1, \ldots, k_j\}$. Every such permutation is a combination of a bijection

$$\mu : \{1, \ldots, j\} \to \{k_1, \ldots, k_j\}$$

and a bijection

$$\nu : \{j+1, \ldots, N\} \to \{1, \ldots, N\} \setminus \{k_1, \ldots, k_j\}.$$

Therefore

$$B(k_1,\ldots,k_j)$$
$$=\Big(\sum_{\mu}\frac{\prod_{i=1}^{j} b_{\mu(i)}}{\prod_{i=1}^{j}(b_{\mu(i)}+\ldots+b_{\mu(j)})}\Big)\Big(\sum_{\nu}\frac{\prod_{i=j+1}^{N} b_{\nu(i)}}{\prod_{i=j+1}^{N}(b_{\nu(j+1)}+\ldots+b_{\nu(i)})}\Big),$$

where the summation extends over all bijections μ, ν as above. By Lemma 3.7, both the factors in the above product are equal to 1. Hence $B(k_1,\ldots,k_j) = 1$. This establishes the formula (3.9), and the proof of Theorem 3.2 is complete.

4. The capacities D and ϱ. Let K be a compact subset of $\mathbb{C}^N$. Let $\|f\|_K$ denote the supremum norm of a function $f: K\to\mathbb{C}$. For a nonnegative integer s put

$$h_s := \binom{s+N-1}{N-1} \quad\text{and}\quad m_s := \binom{s+N}{N}.$$

Let $e_1(z), e_2(z), \ldots$ be all monomials $z^\alpha := z_1^{\alpha_1}\ldots z_N^{\alpha_N}$ ordered so that the degrees of the $e_j(z)$ are nondecreasing and the monomials of a fixed degree are ordered lexicographically. For an integer $j\geq 1$ let $\alpha(j) := (\alpha_1,\ldots,\alpha_N)$, where $z_1^{\alpha_1}\ldots z_N^{\alpha_N} = e_j(z)$. Put $|\alpha(j)| := \alpha_1+\ldots+\alpha_N$. For an integer k ($1\leq k\leq h_s$) let $\beta(s,k) := \alpha(m_{s-1}+k)$, where $m_{-1} := 0$, and put $e_{s,k}(z) := e_j(z)$, where $j = m_{s-1}+k$. It is easy to check that $e_{s,1},\ldots,e_{s,h_s}$ are all monomials z^α of degree s ordered lexicographically. Put

$$M_{s,k} := \inf\{\|q\|_K : q\in H_{s,k}\},$$

where

$$H_{s,k} := \Big\{e_{s,k}(z)+\sum_{j<k} c_j e_{s,j}(z) : c_j\in\mathbb{C}\Big\}.$$

Let $\tau_{s,k} := (M_{s,k})^{1/s}$. Put

$$\Sigma = \Sigma^{N-1}$$
$$:= \Big\{\theta = (\theta_1,\ldots,\theta_N)\in\mathbb{R}^N : \theta_j\geq 0 \text{ for } j = 1,\ldots,N;\ \sum_{j=1}^{N}\theta_j = 1\Big\},$$
$$\Sigma_0 = \Sigma_0^{N-1} := \{\theta\in\Sigma^{N-1} : \theta_j > 0 \text{ for } j = 1,\ldots,N\}.$$

For $\theta\in\Sigma$ let us define the *directional Chebyshev constants*:

$$\widetilde{\tau}(K,\theta) := \limsup\{\tau_{s,k} : s\to\infty,\ \beta(s,k)/s\to\theta\},$$
$$\widetilde{\tau}_-(K,\theta) := \liminf\{\tau_{s,k} : s\to\infty,\ \beta(s,k)/s\to\theta\}.$$

It is known (see [3]) that $\widetilde{\tau}(K,\theta) = \widetilde{\tau}_-(K,\theta)$ for each $\theta\in\Sigma_0$ and that $\log\widetilde{\tau}(K,\theta)$ is a convex function on Σ_0.

DEFINITION 4.1 (see [3]). The *homogeneous transfinite diameter* $D(K)$ is

$$D(K) := \exp\left\{\frac{1}{\operatorname{vol}\Sigma}\int_{\Sigma}\log\widetilde{\tau}(K,\theta)\,d\omega(\theta)\right\},$$

where $\operatorname{vol}\Sigma := \int_\Sigma d\omega(\theta)$ and ω denotes the Lebesgue surface area measure on the hyperplane $\{\theta = (\theta_1,\ldots,\theta_N) \in \mathbb{R}^N : \sum_{j=1}^N \theta_j = 1\}$.

DEFINITION 4.2 (see [6]). The *Chebyshev constant* $\varrho(K)$ is

$$\varrho(K) := \lim_{s\to\infty}(\varrho_s(K))^{1/s},$$

where $\varrho_s(K) := \inf\{\|Q\|_K\}$, the infimum being taken over all polynomials $Q \in H_s$ normalized so that $\|Q\|_S = 1$.

DEFINITION 4.3. Let μ be a positive Radon measure supported on K. The pair (K,μ) is said to have the *Bernstein–Markov property* if for every $\lambda > 1$ there exists an $M > 0$ such that for all polynomials p

$$\|p\|_K \le M\lambda^{\deg p}\|p\|_2, \quad \text{where} \quad \|p\|_2 := \Big(\int_K |p|^2 d\mu\Big)^{1/2}.$$

A few examples of pairs with the Bernstein–Markov property can be found e.g. in [7] and [8].

DEFINITION 4.4. K is called *unisolvent with respect to homogeneous polynomials* if no nonzero homogeneous polynomial vanishes identically on K.

5. Orthogonal polynomials associated with K. Suppose that the pair (K,μ) has the Bernstein–Markov property. For two integers s,k ($s \ge 0$, $1 \le k \le h_s$) put

$$C_{s,k} := \inf\{\|p\|_2 : p \in H_{s,k}\}.$$

It is easy to check that there exists at least one polynomial $A_{s,k}(z) \in H_{s,k}$ attaining the infimum. For each s, $\{A_{s,k}(z)\}_{k=1,\ldots,h_s}$ is obviously a sequence of orthogonal polynomials in the space $L^2(K,\mu)$. If $\|A_{s,k}\|_2 > 0$ then put

$$B_{s,k}(z) := A_{s,k}(z)/\|A_{s,k}\|_2, \quad z \in \mathbb{C}^N.$$

For an integer $j \ge 1$, let $A_j := A_{s,k}$ and $B_j := B_{s,k}$, where the integers s,k ($s \ge 0$, $1 \le k \le h_s$) are chosen so that $j = m_{s-1} + k$.

PROPOSITION 5.1. *For each* $\theta \in \Sigma_0$,

$$\widetilde{\tau}(K,\theta) = \lim\{(\|A_j\|_2)^{1/|\alpha(j)|} : j \to \infty,\ \alpha(j)/|\alpha(j)| \to \theta\}.$$

Proof. This is an easy consequence of the Bernstein–Markov property and the equality $\widetilde{\tau}(K,\theta) = \widetilde{\tau}_-(K,\theta)$ for $\theta \in \Sigma_0$ ([3], Lemma 4.1).

THEOREM 5.2. *Let K be unisolvent with respect to homogeneous polynomials. Then*:

(a) $\|A_j\|_2 > 0$ *for each* j,
(b) $\lim_{j\to\infty}(\|B_j\|_K)^{1/|\alpha(j)|} = 1$,
(c) $\Psi_K(z) = \limsup_{j\to\infty} |B_j(z)|^{1/|\alpha(j)|}$, $z \in \mathbb{C}^N$.

Proof. The theorem can be proved in the same manner as Theorem 1 of [8] (it suffices to replace $e_j(z) + \sum_{i<j} c_i e_i(z)$ by $e_{s,k}(z) + \sum_{i<k} b_i e_{s,i}(z)$, where $e_{s,k} = e_j$). Zériahi's result is true for $z \in \mathbb{C}^N \setminus \widehat{K}$, where $\widehat{K}$ is the polynomially convex hull of K, because for $z \in \widehat{K}$ one cannot prove that the sequence (j_d) in the proof of Theorem 1 of [8] is not bounded. In our proof the sequence (j_d) must be unbounded for each $z \in \mathbb{C}^N$, because we deal with the homogeneous polynomials and their degrees must increase.

6. Comparison of the capacities D, ϱ and γ. Let K be a compact subset of the unit ball $B := \{z \in \mathbb{C}^N : \sum_{j=1}^N |z_j|^2 \le 1\}$. It is known that

$$\varrho(K)/\sqrt{N} \le D(K) \le \varrho(K)^{1/N}$$

and

$$\gamma(K)/\sqrt{N} \le D(K) \le \exp(-\kappa_N/N)\gamma(K)^{1/N}$$

(see [3], Theorem 5.7 and Corollary 5.8). It turns out that the exponents in the estimates above cannot be improved. This is an easy consequence of (6.1), (6.3), (6.4) and (6.6) in the following theorem:

THEOREM 6.1. *If $\varepsilon > 0$ and $C > 0$, then there exist compact subsets K_1, K_2, K_4, K_5 of the unit sphere S and compact subsets K_3, K_6 of the unit ball B such that*

$$D(K_1) > C\varrho(K_1)^{1/N+\varepsilon}, \tag{6.1}$$
$$D(K_2) < C\varrho(K_2)^{1-1/N-\varepsilon}, \tag{6.2}$$
$$D(K_3) < C\varrho(K_3)^{1-\varepsilon}, \tag{6.3}$$
$$D(K_4) > C\gamma(K_4)^{1/N+\varepsilon}, \tag{6.4}$$
$$D(K_5) < C\gamma(K_5)^{1-1/N-\varepsilon}, \tag{6.5}$$
$$D(K_6) < C\gamma(K_6)^{1-\varepsilon}. \tag{6.6}$$

Proof. It is known that for every compact subset K of $\mathbb{C}^N$,

$$\gamma(K) \le \varrho(K) \le \gamma(K)\exp(-\kappa_N)$$

(see [6], Proposition 12.1). Therefore it suffices to prove (6.1)–(6.3). Let

$$P = P(r_1, \ldots, r_N) := \{z \in \mathbb{C}^N : |z_1| = r_1, \ldots, |z_N| = r_N\},$$

where $r_1, \ldots, r_N$ are real positive numbers. It is known that

$$\Psi_P(z) = \max(|z_1|/r_1, \ldots, |z_N|/r_N), \quad z \in \mathbb{C}^N$$

(see [4], p. 304). For every compact set K we have $\varrho(K)^{-1} = \sup\{\Psi_K(z) : z \in B\}$ (see [6], Theorem 8.2). Hence $\varrho(P) = \min(r_1, \ldots, r_N)$.

It is also known that

$$D(P) = \Big(\prod_{j=1}^{N} r_j\Big)^{1/N}$$

(see e.g. [3], Corollary 6.4). Therefore we can take

$$K_1 := P(r_1, \ldots, r_N)$$

(where $r_1 = \ldots = r_{N-1} = (1/(N-1) - \delta)^{1/2}$ and $r_N = ((N-1)\delta)^{1/2}$, $0 < \delta < 1/(N(N-1))$),

$$K_2 := P(R_1, \ldots, R_N)$$

(where $R_1 = \ldots = R_{N-1} = \delta^{1/2}$ and $R_N = (1-(N-1)\delta)^{1/2}$, $0 < \delta < 1/N$) and

$$K_3 := P(s_1, \ldots, s_N)$$

(where $s_1 = \ldots = s_{N-1} = s_N = \delta^{1/2}$, $0 < \delta < 1/N$). It is easy to check that (6.1)–(6.3) are true if $\delta > 0$ is sufficiently small. This proves the theorem.

It is known that $D(K) = \gamma(K)^{1/2}$ for every compact set K such that

$$K \subset \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

(see [6], Theorem 12.3). We can ask whether the following conjecture is true for the unit sphere $S \subset \mathbb{C}^N$ $(N \geq 3)$:

There exist positive numbers δ, m and M such that for every compact subset K of S,

$$m\gamma(K)^\delta \leq D(K) \leq M\gamma(K)^\delta.$$

COROLLARY 6.2. *The above conjecture is false for each* $N \geq 3$.

Proof. Suppose that this conjecture is true for some N. Then $\delta \geq 1 - 1/N$ by (6.5) and $\delta \leq 1/N$ by (6.4). This is impossible for $N \geq 3$.

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