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Bounded projections in weighted function spaces in a generalized unit disc

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Abstract. Let $M_{m,n}$ be the space of all complex $m \times n$ matrices. The *generalized unit disc* in $M_{m,n}$ is

$$R_{m,n} = \{Z \in M_{m,n} : I^{(m)} - ZZ^* \text{ is positive definite}\}.$$

Here $I^{(m)} \in M_{m,m}$ is the unit matrix. If $1 \le p < \infty$ and $\alpha > -1$, then $L^p_\alpha(R_{m,n})$ is defined to be the space $L^p\{R_{m,n}; [\det(I^{(m)} - ZZ^*)]^\alpha \, d\mu_{m,n}(Z)\}$, where $\mu_{m,n}$ is the Lebesgue measure in $M_{m,n}$, and $H^p_\alpha(R_{m,n}) \subset L^p_\alpha(R_{m,n})$ is the subspace of holomorphic functions. In [8, 9] M. M. Djrbashian and A. H. Karapetyan proved that, if $\operatorname{Re}\beta > (\alpha+1)/p - 1$ (for $1 < p < \infty$) and $\operatorname{Re}\beta \ge \alpha$ (for $p = 1$), then

$$f(\mathcal{Z}) = T^\beta_{m,n}(f)(\mathcal{Z}), \quad \mathcal{Z} \in R_{m,n},$$

where $T^\beta_{m,n}$ is the integral operator defined by (0.13)–(0.14). In the present paper, given $1 \le p < \infty$, we find conditions on α and β for $T^\beta_{m,n}$ to be a bounded projection of $L^p_\alpha(R_{m,n})$ onto $H^p_\alpha(R_{m,n})$. Some applications of this result are given.

0. Introduction

0.1. In the fourties M. M. Djrbashian [4, 5] introduced the classes $H^p(\alpha)$ ($1 \le p < \infty$, $\alpha > -1$) of functions $f(z)$ holomorphic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, with

$$\iint_{\mathbb{D}} |f(\zeta)|^p (1 - |\zeta|^2)^\alpha \, d\xi \, d\eta < \infty \quad (\zeta = \xi + i\eta). \tag{0.1}$$

In the same papers [4, 5] the following result was established.

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Theorem A. (i) *Let $1 \le p < \infty$ and $\alpha > -1$. Then for each $f \in H^p(\alpha)$ we have*

$$f(z) = \frac{\alpha+1}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)(1-|\zeta|^2)^\alpha}{(1-z\overline{\zeta})^{2+\alpha}}\, d\xi\, d\eta, \qquad z \in \mathbb{D}, \tag{0.2}$$

$$\overline{f(0)} = \frac{\alpha+1}{\pi} \iint_{\mathbb{D}} \frac{\overline{f(\zeta)}(1-|\zeta|^2)^\alpha}{(1-z\overline{\zeta})^{2+\alpha}}\, d\xi\, d\eta, \qquad z \in \mathbb{D}. \tag{0.3}$$

(ii) *The integral operator induced by the right hand side of* (0.2) *acts in $L^2\{\mathbb{D}; (1-|\zeta|^2)^\alpha\, d\xi\, d\eta\}$ as the orthogonal projection onto $H^2(\alpha)$, $\alpha > -1$.*

The classes $H^p(\alpha)$ began to play an important role in complex analysis. The integral representation (0.2) had numerous applications. For example, in the same papers [4, 5] by the use of (0.2)–(0.3) a canonical factorization was established for certain weighted classes of functions meromorphic in $\mathbb{D}$. For other applications of Theorem A see the surveys [6, 7] and the monograph [3].

0.2. Later on, in the fifties, the following problem arose: establish reasonable analogs of Theorem A for functions of several complex variables. To survey the relevant investigations we need first to introduce some notations.

For $m, n \ge 1$ we denote by $M_{m,n}$ the space of all complex $m \times n$ matrices. For each $Z \in M_{m,n}$, $Z^* \in M_{n,m}$ will denote the Hermitian conjugate of Z. Further, for $k \ge 1$, $I^{(k)} \in M_{k,k}$ denotes the unit matrix. The Lebesgue measure $\mu_{m,n}$ in $M_{m,n}$ can be written as

$$d\mu_{m,n}(Z) = \prod_{\substack{1 \le k \le m \\ 1 \le j \le n}} d\xi_{kj}\, d\eta_{kj}, \tag{0.4}$$

where $Z = (\zeta_{kj})_{1 \le k \le m,\, 1 \le j \le n} \in M_{m,n}$ with $\zeta_{kj} = \xi_{kj} + i\eta_{kj}$. Note that $M_{1,n}$ coincides with $\mathbb{C}^n$ and $\mu_{1,n}$ is $2n$-dimensional Lebesgue measure in $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

The *generalized unit disc* in $M_{m,n}$ is

$$R_{m,n} = \{Z \in M_{m,n} : I^{(m)} - ZZ^* \text{ is positive definite}\}. \tag{0.5}$$

It is easy to see that $R_{1,n}$ coincides with the unit ball $\mathbb{B}_n = \{\zeta \in \mathbb{C}^n : \zeta\zeta^* < 1\}$ in $M_{1,n} = \mathbb{C}^n$.

In Hua's monograph [12, Theorem 4.3.1] the following result was established.

Theorem B. (i) *Every holomorphic function $f(\mathcal{Z}) \in L^2\{R_{m,n}; d\mu_{m,n}\}$ admits an integral representation of the form*

$$f(\mathcal{Z}) = c_{m,n} \int_{R_{m,n}} \frac{f(Z)}{[\det(I^{(m)} - \mathcal{Z}Z^*)]^{m+n}}\, d\mu_{m,n}(Z), \qquad \mathcal{Z} \in R_{m,n}, \tag{0.6}$$

where

$$c_{m,n} = \pi^{-mn} \prod_{l=1}^{m+n} \Gamma(l) \prod_{k=1}^{m} \Gamma^{-1}(k) \prod_{j=1}^{n} \Gamma^{-1}(j). \tag{0.7}$$

(ii) *The integral operator induced by the right hand side of* (0.6) *acts in* $L^2\{R_{m,n}; d\mu_{m,n}\}$ *as the orthogonal projection onto the subspace of holomorphic functions.*

Note that for $m = 1$, Theorem B establishes the integral representation

$$f(z) = \frac{n!}{\pi^n} \int\limits_{\mathbb{B}_n} \frac{f(\zeta)}{(1 - z\zeta^*)^{1+n}} \, d\mu_{1,n}(\zeta), \qquad z \in \mathbb{B}_n, \tag{0.8}$$

for holomorphic functions $f \in L^2\{\mathbb{B}_n;\, d\mu_{1,n}\}$. Also, Theorem B is a generalization of Theorem A, but only for the particular values $p = 2$ and $\alpha = 0$.

0.3. In further investigations a multidimensional generalization of Theorem A (this time for arbitrary $1 \le p < \infty$ and $\alpha > -1$) was established. The result is

THEOREM C. (i) *Suppose that* $1 \le p < \infty$, $\alpha > -1$ *and the complex number* β *satisfies* $\operatorname{Re}\beta > (\alpha + 1)/p - 1$ (*if* $1 < p < \infty$) *and* $\operatorname{Re}\beta \ge \alpha$ (*if* $p = 1$). *Then every function* $f(z)$ *holomorphic in* $\mathbb{B}_n \subset \mathbb{C}^n$ *for which*

$$\int\limits_{\mathbb{B}_n} |f(\zeta)|^p (1 - \zeta\zeta^*)^\alpha d\mu_{1,n}(\zeta) < \infty \tag{0.9}$$

admits the integral representations

$$f(z) = \frac{(\beta+1)\ldots(\beta+n)}{\pi^n} \int\limits_{\mathbb{B}_n} \frac{f(\zeta)(1 - \zeta\zeta^*)^\beta}{(1 - z\zeta^*)^{1+n+\beta}} \, d\mu_{1,n}(\zeta), \qquad z \in \mathbb{B}_n, \tag{0.10}$$

$$\overline{f(0)} = \frac{(\beta+1)\ldots(\beta+n)}{\pi^n} \int\limits_{\mathbb{B}_n} \frac{\overline{f(\zeta)}(1 - \zeta\zeta^*)^\beta}{(1 - z\zeta^*)^{1+n+\beta}} \, d\mu_{1,n}(\zeta), \qquad z \in \mathbb{B}_n. \tag{0.11}$$

(ii) *For* $1 \le p < \infty$, $\alpha > -1$ *and* $\operatorname{Re}\beta > (\alpha+1)/p - 1$ *the integral operator induced by the right hand side of* (0.10) *is a bounded projection of* $L^p\{\mathbb{B}_n; (1 - \zeta\zeta^*)^\alpha\, d\mu_{1,n}(\zeta)\}$ *onto the subspace of holomorphic functions.*

As follows from the proof of Theorem A in [5], for $n = 1$ assertion (i) of Theorem C was actually established in [4, 5]. For $n \ge 1$ and $p = 2$, $\beta = \alpha = 0$, Theorem C follows from Theorem B (compare (0.8) and (0.10)). For $n \ge 1$ and $1 \le p < \infty$, $\alpha = 0$, $\operatorname{Re}\beta > 1/p - 1$, Theorem C was established by F. Forelli and W. Rudin [11] (see also [15, Theorem 7.1.4]). These conditions are exactly the same as in Theorem C(i) (for $\alpha = 0$) except the case $p = 1$, $\operatorname{Re}\beta = 0$ which is not considered in [11]. Finally, in the general form stated above, Theorem C(i) was proved in

M. M. Djrbashian's survey [7] by use of the methods developed in [4, 5]. Note that β was assumed to be real in [7], but this restriction is not essential. As to assertion (ii) of Theorem C, it was mentioned in [7] that the corresponding proof, given in [11] for $\alpha = 0$, can be easily adapted to the general case $\alpha > -1$.

0.4. Of course, Theorem C is a more or less satisfactory generalization of the main Theorem A. However, in the recent papers [8, 9] a much more general result was established. To be more precise, for the case of the generalized unit disc $R_{m,n}$ $(m, n \geq 1)$ similar weighted integral representations were obtained. To formulate the corresponding result we introduce some further notations.

Let $m, n \geq 1$ and $1 \leq p < \infty$, $\alpha > -1$. For an arbitrary complex measurable function $f(Z)$, $Z \in R_{m,n}$, set

$$\|f\|_{p,\alpha}^p := \int_{R_{m,n}} |f(Z)|^p [\det(I^{(m)} - ZZ^*)]^\alpha \, d\mu_{m,n}(Z). \tag{0.12}$$

Then we introduce the space $L^p_\alpha(R_{m,n}) := \{f : \|f\|_{p,\alpha} < \infty\}$. Next we define $H^p_\alpha(R_{m,n})$ to be the space of holomorphic functions in $L^p_\alpha(R_{m,n})$. Further, if $m, n \geq 1$ and $\operatorname{Re}\beta > -1$, then we set

$$c_{m,n}(\beta) = \pi^{-mn} \prod_{l=1}^{m+n} \Gamma(\beta + l) \prod_{k=1}^{m} \Gamma^{-1}(\beta + k) \prod_{j=1}^{n} \Gamma^{-1}(\beta + j) \tag{0.13}$$

and consider the integral operator

$$T^\beta_{m,n}(f)(\mathcal{Z}) = c_{m,n}(\beta) \int_{R_{m,n}} \frac{f(Z)[\det(I^{(m)} - ZZ^*)]^\beta}{[\det(I^{(m)} - \mathcal{Z}Z^*)]^{m+n+\beta}} \, d\mu_{m,n}(Z), \quad \mathcal{Z} \in R_{m,n}. \tag{0.14}$$

The result established in [8, 9] is

THEOREM D. *Suppose that $m, n \geq 1$, $1 \leq p < \infty$, $\alpha > -1$ and the complex number β satisfies $\operatorname{Re}\beta > (\alpha + 1)/p - 1$ (if $1 < p < \infty$) and $\operatorname{Re}\beta \geq \alpha$ (if $p = 1$). Then for each $f \in H^p_\alpha(R_{m,n})$ the following integral representations hold*:

$$f(\mathcal{Z}) = T^\beta_{m,n}(f)(\mathcal{Z}), \quad \mathcal{Z} \in R_{m,n}, \tag{0.15}$$

$$\overline{f(0)} = T^\beta_{m,n}(\bar{f})(\mathcal{Z}), \quad \mathcal{Z} \in R_{m,n}. \tag{0.16}$$

Remark 0.1. In [8, 9] only the formula (0.15) was written down. But it is easy to see that (0.16) can be directly deduced from (0.15).

For $m = 1$, Theorem D coincides with assertion (i) of Theorem C. Moreover, for all $m, n \geq 1$ and the particular values $p = 2$, $\beta = \alpha = 0$, Theorem D gives assertion (i) of Theorem B. In connection with Theorem D we have to mention the paper [16] by M. Stoll, published earlier than [8, 9]. In [16] weighted integral representations were established for all bounded symmetric domains, including $R_{m,n}$, but only for holomorphic functions in L^p-spaces *without weights*. Theorem D can be deduced from the results of [16] only for $\alpha = 0$ and real $\beta \geq 0$.

0.5. In [8], in addition to the establishment of Theorem D the following problem was posed: for $m, n \geq 1$ and $1 \leq p < \infty$, under what conditions on α and β is $T^{\beta}_{m,n}$ (see (0.14)) a bounded projection of $L^p_\alpha(R_{m,n})$ onto its subspace $H^p_\alpha(R_{m,n})$? A similar problem was also raised in [16]. Theorems 3.1 and 3.2 of the present paper give a solution of these problems. The technique of the proof of the main Theorem 3.1 goes back to [11]. However, in our case we had to overcome some additional computational difficulties. For instance, we had to compute the determinant (see [13])

$$\det |B(l_i + j, t+1)|^n_{i,j=1}, \qquad \mathrm{Re}\, l_k > -1 \ (1 \leq k \leq n),\ \mathrm{Re}\, t > -1, \tag{0.17}$$

where B is the Euler beta function. (When $t = 0$ in (0.17), we get a special case of the Cauchy determinant $\det |(l_i + j)^{-1}|^n_{i,j=1}$.)

Concluding the paper we give some applications of Theorems D and 3.1, 3.2. To be more precise, we establish integral representations and integral inequalities for functions pluriharmonic in $R_{m,n}$.

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1. Preliminaries and auxiliary facts

1.1. We recall that for $A = (a_{ij})^n_{i,j=1} \in M_{n,n}$,

$$\begin{aligned} \det(A) &= \sum_i \delta_{i_1 i_2 \ldots i_n} a_{i_1 1} a_{i_2 2} \ldots a_{i_n n} \\ &= \sum_i \delta_{i_1 i_2 \ldots i_n} a_{1 i_1} a_{2 i_2} \ldots a_{n i_n}, \end{aligned} \tag{1.1}$$

where the summation is over all permutations $i = (i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ and $\delta_{i_1 i_2 \ldots i_n}$ is the signature of the permutation. We denote by $M^*_{n,n}$ the set of all invertible $n \times n$ matrices.

Further, for every $A = (a_{ij})^n_{i,j=1} \in M_{n,n}$ we set

$$(A)^\wedge = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) \in \mathbb{C}^{n^2}, \tag{1.2}$$

$$\mathrm{sp}(A) = a_{11} + a_{22} + \ldots + a_{nn}. \tag{1.3}$$

It is easy to verify that

$$\mathrm{sp}(A^*) = \overline{\mathrm{sp}(A)}, \quad \mathrm{sp}(AB) = \mathrm{sp}(BA), \quad \mathrm{sp}(XAX^{-1}) = \mathrm{sp}(A). \tag{1.4}$$

We denote by H_n the set of all Hermitian $n \times n$ matrices. For $A \in H_n$ we write $A > 0$ ($A \geq 0$) if A is positive definite (nonnegative definite). The set of all unitary $n \times n$ matrices is denoted by $\mathcal{U}_n$.

For complex numbers $\lambda_1, \ldots, \lambda_n$ we denote by $\Lambda = [\lambda_1, \ldots, \lambda_n]$ the diagonal $n \times n$ matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

The following facts are well known:

• For every matrix $A \geq 0$, there exists a unique matrix $B \geq 0$ such that $A = BB$. We write $B = \sqrt{A}$; note that $A > 0$ is equivalent to $\sqrt{A} > 0$.

• Every matrix $A > 0$ may be represented as $A = TT^*$, where T is a uniquely determined lower triangular matrix with positive diagonal entries.

• Every $A \in H_n$ may be represented as $A = V\Lambda V^*$, where $V \in H_n$, $\Lambda = [\lambda_1, \ldots, \lambda_n]$ and $\lambda_1 \geq \ldots \geq \lambda_n$. Moreover, Λ is uniquely determined and $A > 0$ ($A \geq 0$) is equivalent to $\lambda_n > 0$ ($\lambda_n \geq 0$).

• Every $A \in M^*_{n,n}$ admits a representation $A = TU$, where $U \in \mathcal{U}_n$, $T \in M_{n,n}$ is a lower triangular matrix with positive diagonal entries, and both T and U are uniquely determined.

1.2. In [12, Theorem 2.1.2] it was established that for every $Z \in M_{m,n}$ the conditions $I^{(m)} - ZZ^* > 0$ (≥ 0) and $I^{(n)} - Z^*Z > 0$ (≥ 0) are equivalent and, furthermore,

$$\det(I^{(m)} - ZZ^*) = \det(I^{(n)} - Z^*Z). \tag{1.5}$$

This fact will often be used in what follows. For instance, we have (see (0.5))

$$\begin{aligned} R_{m,n} &= \{Z \in M_{m,n} : I^{(m)} - ZZ^* > 0\} \\ &= \{Z \in M_{m,n} : I^{(n)} - Z^*Z > 0\}. \end{aligned} \tag{1.6}$$

Also, (1.5) implies the identity

$$\det(I^{(m)} - \mathcal{Z}Z^*) \equiv \det(I^{(n)} - Z^*\mathcal{Z}), \qquad \mathcal{Z}, Z \in M_{m,n}. \tag{1.7}$$

Further, in [12, §2.2] two recursion relations were derived for integrals over $R_{m,n}$ relative to the Lebesgue measure $\mu_{m,n}$:

Formula I. Evidently, every $Z \in M_{m,n}$ can be written as

$$Z = (Z_1\ q), \qquad Z_1 \in M_{m,n-1},\ q \in M_{m,1} \cong \mathbb{C}^m. \tag{1.8}$$

Then one can show that

$$\begin{aligned} &R_{m,n} = \{Z = (Z_1\ q) \in M_{m,n} : Z_1 \in R_{m,n-1}, \\ &\qquad\qquad q = \sqrt{I^{(m)} - Z_1 Z_1^*}\,\omega,\ \omega \in R_{m,1} \cong \mathbb{B}_m\}, \\ &\det(I^{(m)} - ZZ^*) = \det(I^{(m)} - Z_1 Z_1^*)(1 - \omega^*\omega). \end{aligned} \tag{1.9}$$

Furthermore, for every nonnegative measurable function $f(Z), Z \in R_{m,n}$, the following integral formula holds:

$$(1.10)\qquad \int_{R_{m,n}} f(Z)\,d\mu_{m,n}(Z) =$$

$$\int_{R_{m,n-1}} \det(I^{(m)} - Z_1Z_1^*)d\mu_{m,n-1}(Z_1) \int_{R_{m,1}} f(Z_1\ \sqrt{I^{(m)} - Z_1Z_1^*}\,\omega)\,d\mu_{m,1}(\omega).$$

FORMULA II. Every $Z \in M_{m,n}$ can be written as

$$(1.11)\qquad Z = \begin{pmatrix} Z_1 \\ p \end{pmatrix}, \qquad Z_1 \in M_{m-1,n},\ p \in M_{1,n} = \mathbb{C}^n.$$

Then we have

$$(1.12)\qquad \begin{aligned} R_{m,n} = \Big\{ Z = \begin{pmatrix} Z_1 \\ p \end{pmatrix} \in M_{m,n} : Z_1 \in R_{m-1,n},& \\ p = \omega\sqrt{I^{(n)} - Z_1^*Z_1},\ \omega \in R_{1,n} = \mathbb{B}_n \Big\},& \\ \det(I^{(n)} - Z^*Z) = \det(I^{(n)} - Z_1^*Z_1)(1 - \omega\omega^*).& \end{aligned}$$

Furthermore, for every nonnegative measurable function $f(Z)$, $Z \in R_{m,n}$, the following integral formula holds:

$$(1.13)\qquad \int_{R_{m,n}} f(Z)\,d\mu_{m,n}(Z)$$

$$= \int_{R_{m-1,n}} \det(I^{(n)} - Z_1^*Z_1)\,d\mu_{m-1,n}(Z_1) \int_{R_{1,n}} f\begin{pmatrix} Z_1 \\ \omega\sqrt{I^{(n)} - Z_1^*Z_1} \end{pmatrix} d\mu_{1,n}(\omega).$$

1.3. We shall require some notations introduced in [12]. For $n \ge 1$ let $f_1 \ge \ldots \ge f_n \ge 0$ be integers. Then put

$$(1.14)\qquad M_{f_1,\ldots,f_n}(z_1,\ldots,z_n) := \det|z_j^{f_i+n-i}|_{i,j=1}^n, \qquad z = (z_1,\ldots,z_n) \in \mathbb{C}^n.$$

If $f_1 = \ldots = f_n = 0$, we get

$$(1.15)\qquad M_{0,\ldots,0}(z_1,\ldots,z_n) = \det|z_j^{n-i}|_{i,j=1}^n, \qquad z = (z_1,\ldots,z_n) \in \mathbb{C}^n.$$

In other words, $M_{0,\ldots,0}(z_1,\ldots,z_n)$ is the well-known Vandermonde determinant. We have

$$(1.16)\qquad \det|z_j^{n-i}|_{i,j=1}^n \equiv D(z_1,\ldots,z_n), \qquad z = (z_1,\ldots,z_n) \in \mathbb{C}^n,$$

where

$$(1.17)\qquad D(z_1,\ldots,z_n) := \prod_{1\le i<j\le n} (z_i - z_j), \qquad z = (z_1,\ldots,z_n) \in \mathbb{C}^n.$$

Next, for arbitrary integers $f_1 \geq \ldots \geq f_n \geq 0$ we set

$$N(f_1, \ldots, f_n) = \frac{D(f_1 + n - 1, f_2 + n - 2, \ldots, f_{n-1} + 1, f_n)}{D(n-1, n-2, \ldots, 1, 0)}. \tag{1.18}$$

Note that $N(f_1, \ldots, f_n)$ is a natural number.

1.4. Recall that $\mathcal{U}_n$ ($n \geq 1$) denotes the group of all unitary $n \times n$ matrices. Let Γ_n be the subgroup of all diagonal unitary matrices. We say that $U_1, U_2 \in \mathcal{U}_n$ are equivalent ($U_1 \sim U_2$) if $U_1^{-1} U_2 \in \Gamma_n$. The set of the corresponding equivalence classes is denoted by $[\mathcal{U}_n]$. Further, let dU and $d[U]$ be the volume elements in $\mathcal{U}_n$ and $[\mathcal{U}_n]$, respectively. In [17, Ch. VII, 4] and [12, §3.2] a relation between dU and $d[U]$ was established, but we do not dwell on this. Also, it was shown in [12, Theorems 3.1.1 and 3.2.1] that

$$\omega_n = \int\limits_{\mathcal{U}_n} dU = \frac{(2\pi)^{n(n+1)/2}}{D(n-1, n-2, \ldots, 1, 0)}, \tag{1.19}$$

$$\omega'_n = \int\limits_{[\mathcal{U}_n]} d[U] = \frac{(2\pi)^{n(n-1)/2}}{D(n-1, n-2, \ldots, 1, 0)}. \tag{1.20}$$

Now let us introduce polar coordinates in $M_{n,n}$ (see [12, §3.4]). If $Z \in M^*_{n,n}$, then $Z = TU$, where $U \in \mathcal{U}_n$ and $T \in M_{n,n}$ is a lower triangular matrix with positive diagonal entries. Next, since $ZZ^* = TT^* > 0$ we have a representation $ZZ^* = V\Lambda V^*$, where $V \in \mathcal{U}_n$, $\Lambda = [\lambda_1, \ldots, \lambda_n]$, $\lambda_1 \geq \ldots \geq \lambda_n > 0$ and the matrix Λ is uniquely determined. If we assume in addition that $\lambda_1 > \ldots > \lambda_n > 0$, then $V \in \mathcal{U}_n$ in the above representation is in a sense also uniquely determined. To be more precise, $ZZ^* = V_1 \Lambda V_1^* = V_2 \Lambda V_2^*$ implies that V_1 and V_2 belong to the same equivalence class $[V] \in [\mathcal{U}_n]$. Thus, every matrix $Z \in M^*_{n,n}$ such that all eigenvalues of ZZ^* are distinct (the other matrices Z form in $M_{n,n}$ a variety of dimension less than $n^2 = \dim M_{n,n}$) uniquely defines a triple $\{\Lambda, U, [V]\}$, where $\Lambda = [\lambda_1, \ldots, \lambda_n]$, $\lambda_1 > \ldots > \lambda_n > 0$, $U \in \mathcal{U}_n$, $[V] \in [\mathcal{U}_n]$. This triple is called the *polar coordinates* of the matrix Z. Notice that Z may be recovered from its polar coordinates as follows: put $A = V\Lambda V^*$, where $V \in [V]$ (A does not depend on the choice of $V \in [V]$); then $A > 0$, so $A = TT^*$, where T is lower triangular with positive diagonal entries; finally, set $Z = TU$.

In conclusion, note that the Lebesgue measure $\mu_{n,n}$ on $M_{n,n}$ can be written in polar coordinates as follows:

$$d\mu_{n,n}(Z) = 2^{-n^2} D^2(\lambda_1, \ldots, \lambda_n)\, d\lambda_1 \ldots d\lambda_n\, dU\, d[V]. \tag{1.21}$$

1.5. Assume that $n \geq 1$ and $f_1 \geq \ldots \geq f_n \geq 0$ are arbitrary integers. In H. Weyl's monograph [17, Ch. IV], starting from rather complicated algebraic considerations, a certain mapping

$$A \to X_{f_1 \ldots f_n}(A) \tag{1.22}$$

from $M_{n,n}$ into $M_{N,N}$ was constructed, where $N = N(f_1, \ldots, f_n)$ (see (1.18)). This mapping has the following important properties:

(a) $X_{f_1 \ldots f_n}(AB) = X_{f_1 \ldots f_n}(A) X_{f_1 \ldots f_n}(B)$, $\forall A, B \in M_{n,n}$;
(b) if $A \in M^*_{n,n}$, then $X_{f_1 \ldots f_n}(A) \in M^*_{N,N}$;
(c) if $U \in \mathcal{U}_n$, then $X_{f_1 \ldots f_n}(U) \in \mathcal{U}_N$;
(d) $X_{f_1 \ldots f_n}(A^*) = (X_{f_1 \ldots f_n}(A))^*$, $\forall A \in M_{n,n}$;
(e) the entries of the matrix $X_{f_1 \ldots f_n}(A)$, where $A = (a_{ij})_{i,j=1}^n \in M_{n,n}$, are homogeneous polynomials of degree $f = f_1 + \ldots + f_n$ in a_{ij}, $1 \le i, j \le n$.

Algebraically, the properties (a)–(c) can be stated as follows:

- the correspondence $A \to X_{f_1 \ldots f_n}(A)$, $A \in M^*_{n,n}$, is an $N(f_1, \ldots, f_n)$-dimensional linear representation of the group $M^*_{n,n}$;
- the correspondence $U \to X_{f_1 \ldots f_n}(U)$, $U \in \mathcal{U}_n$, is a unitary $N(f_1, \ldots, f_n)$-dimensional linear representation of the group $\mathcal{U}_n$.

In [17, Ch. IV] it was also established that both these representations are irreducible.

Next, set

$$\chi_{f_1 \ldots f_n}(A) := \operatorname{sp}(X_{f_1 \ldots f_n}(A)), \qquad A \in M_{n,n}. \tag{1.23}$$

Combining (1.4) with (a), (b), (d), we get

$$\begin{aligned} &\chi_{f_1 \ldots f_n}(AB) = \chi_{f_1 \ldots f_n}(BA), && A, B \in M_{n,n}; \\ &\chi_{f_1 \ldots f_n}(BAB^{-1}) = \chi_{f_1 \ldots f_n}(A), && A \in M_{n,n},\ B \in M^*_{n,n}; \\ &\chi_{f_1 \ldots f_n}(A^*) = \overline{\chi_{f_1 \ldots f_n}(A)}, && A \in M_{n,n}. \end{aligned} \tag{1.24}$$

Moreover, if $\Lambda = [\lambda_1, \ldots, \lambda_n]$ and $\lambda_i \ne \lambda_j$ for $i \ne j$, then (see [17, Ch. VII])

$$\chi_{f_1 \ldots f_n}(\Lambda) = \frac{M_{f_1, \ldots, f_n}(\lambda_1, \ldots, \lambda_n)}{D(\lambda_1, \ldots, \lambda_n)}. \tag{1.25}$$

For $A \in M_{n,n}$ we denote by $\psi^{(i)}_{f_1 \ldots f_n}(A)$, $i = 1, \ldots, q(f_1, \ldots, f_n)$, the entries of the matrix $X_{f_1 \ldots f_n}(A)$ numbered in a definite way. To be more precise, we set (see the notation (1.2))

$$\{\psi^{(i)}_{f_1 \ldots f_n}(A)\}_{i=1}^{q(f_1, \ldots, f_n)} = (X_{f_1 \ldots f_n}(A))^\wedge. \tag{1.26}$$

It is easy to see that $q(f_1, \ldots, f_n) = N^2(f_1, \ldots, f_n)$. Also, one can easily check the following relations:

$$(1.27)\qquad \chi_{f_1\ldots f_n}(\mathcal{Z}Z^*)=\sum_{i=1}^{q(f_1,\ldots,f_n)}\psi^{(i)}_{f_1\ldots f_n}(\mathcal{Z})\overline{\psi^{(i)}_{f_1\ldots f_n}(Z)},\qquad \forall \mathcal{Z},Z\in M_{n,n},$$

$$(1.28)\qquad \chi_{f_1\ldots f_n}(ZZ^*)=\sum_{i=1}^{q(f_1,\ldots,f_n)}|\psi^{(i)}_{f_1\ldots f_n}(Z)|^2,\qquad \forall Z\in M_{n,n}.$$

1.6. Now we establish some important auxiliary facts.

LEMMA 1.1. *Let $f_1\ge\ldots\ge f_n\ge 0$ and $g_1\ge\ldots\ge g_n\ge 0$ be arbitrary integers. Also, let $1\le i\le q(f_1,\ldots,f_n)$, $1\le j\le q(g_1,\ldots,g_n)$ and $\alpha>-1$. Then*

$$(1.29)\qquad \int\limits_{R_{n,n}}\psi^{(i)}_{f_1\ldots f_n}(Z)\overline{\psi^{(j)}_{g_1\ldots g_n}(Z)}[\det(I^{(n)}-ZZ^*)]^{\alpha}\,d\mu_{n,n}(Z) =\begin{cases}0, & (f_1,\ldots,f_n)\ne(g_1,\ldots,g_n),\\ \delta_{ij}\varrho^{(\alpha)}_{f_1\ldots f_n}, & (f_1,\ldots,f_n)=(g_1,\ldots,g_n),\end{cases}$$

where δ_{ij} is the Kronecker symbol and $\varrho^{(\alpha)}_{f_1\ldots f_n}>0$ does not depend on i.

In [12, §5.1] this fact was established for $\alpha=0$. However, the proof given in [12] and based on the well-known Schur lemma (see, for example, [14, Ch. II, §3]) remains valid in the more general case of $\alpha>-1$. So we omit the proof of Lemma 1.1.

LEMMA 1.2. *Let $n\ge 1$ and $\alpha>-1$.*

(i) *For arbitrary integers $f_1\ge\ldots\ge f_n\ge 0$,*

$$(1.30)\qquad q(f_1,\ldots,f_n)\varrho^{(\alpha)}_{f_1\ldots f_n} =\int\limits_{R_{n,n}}\chi_{f_1\ldots f_n}(ZZ^*)[\det(I^{(n)}-ZZ^*)]^{\alpha}\,d\mu_{n,n}(Z).$$

(ii) *For arbitrary integers $f_1\ge\ldots\ge f_n\ge 0$, $g_1\ge\ldots\ge g_n\ge 0$ and for $\mathcal{Z}\in M_{n,n}$ we have*

$$(1.31)\qquad \int\limits_{R_{n,n}}\chi_{f_1\ldots f_n}(\mathcal{Z}Z^*)\chi_{g_1\ldots g_n}(Z\mathcal{Z}^*)[\det(I^{(n)}-ZZ^*)]^{\alpha}\,d\mu_{n,n}(Z) =\begin{cases}0, & (f_1,\ldots,f_n)\ne(g_1,\ldots,g_n),\\ \varrho^{(\alpha)}_{f_1\ldots f_n}\chi_{f_1\ldots f_n}(\mathcal{Z}\mathcal{Z}^*), & (f_1,\ldots,f_n)=(g_1,\ldots,g_n).\end{cases}$$

Proof. Lemma 1.1 gives, for $1\le i\le q(f_1,\ldots,f_n)$,

$$(1.32)\qquad \int\limits_{R_{n,n}}|\psi^{(i)}_{f_1\ldots f_n}(Z)|^2[\det(I^{(n)}-ZZ^*)]^{\alpha}\,d\mu_{n,n}(Z)=\varrho^{(\alpha)}_{f_1\ldots f_n}.$$

This together with (1.28) establishes (1.30), and (1.31) follows immediately from (1.27)–(1.29).

We now turn to the computation of the explicit value of the constant $\varrho^{(\alpha)}_{f_1\dots f_n}$. For $\alpha = 0$ it was computed in [12, §5.2]. The general case of $\alpha > -1$ turns out to be much more complicated. The computation is essentially based on the following non-trivial fact established in [13]:

THEOREM 1.1. *For* $\operatorname{Re} l_k > -1$ $(1 \le k \le n)$ *and* $\operatorname{Re}\alpha > -1$,

$$\det|B(l_i + j, \alpha + 1)|_{i,j=1}^n \equiv \prod_{k=1}^n \frac{\Gamma(l_k+1)\Gamma(\alpha+1)}{\Gamma(l_k+n+1+\alpha)} D(l_1, \dots, l_n)\mathcal{P}_n(\alpha), \tag{1.33}$$

where $\mathcal{P}_n(\alpha)$, $\alpha \in \mathbb{C}$, *is a polynomial of degree* $\le n(n-1)/2$.

Remark 1.1. Here B and Γ denote the well-known Euler functions. In [13] the polynomial $\mathcal{P}_n$ is written in an explicit form. For $\alpha = 0$ we obtain $\det|(l_i + j)^{-1}|_{i,j=1}^n$ on the left hand side of (1.33), which is a special case of the Cauchy determinant.

We need the following

LEMMA 1.3. *Let* α, a *and* $\{l_k\}_{k=1}^n$, $\{m_k\}_{k=1}^n$ *be arbitrary complex numbers which satisfy*

$$\operatorname{Re}\alpha > -1, \quad \operatorname{Re}(l_i + m_j + a) > -1, \quad 1 \le i,\, j \le n. \tag{1.34}$$

Then

$$\begin{aligned} I := \int_0^1 \dots \int_0^1 & \det|\lambda_j^{l_i}|_{i,j=1}^n \cdot \det|\lambda_j^{m_i}|_{i,j=1}^n \\ & \times \prod_{k=1}^n \lambda_k^a (1-\lambda_k)^\alpha \, d\lambda_1 \dots d\lambda_n \\ = {} & n! \det|B(l_i + m_j + a + 1, \alpha + 1)|_{i,j=1}^n. \end{aligned} \tag{1.35}$$

Proof. In view of (1.1),

$$\begin{aligned} \det|\lambda_j^{l_i}|_{i,j=1}^n \cdot \det|\lambda_j^{m_i}|_{i,j=1}^n & = \sum_j \delta_{j_1\dots j_n} \lambda_{j_1}^{l_1} \dots \lambda_{j_n}^{l_n} \sum_s \delta_{s_1\dots s_n} \lambda_1^{m_{s_1}} \dots \lambda_n^{m_{s_n}} \\ & = \sum_j \lambda_{j_1}^{l_1} \dots \lambda_{j_n}^{l_n} \sum_s \delta_{s_{j_1}\dots s_{j_n}} \lambda_{j_1}^{m_{s_{j_1}}} \dots \lambda_{j_n}^{m_{s_{j_n}}} \\ & = \sum_j \lambda_{j_1}^{l_1} \dots \lambda_{j_n}^{l_n} \sum_k \delta_{k_1\dots k_n} \lambda_{j_1}^{m_{k_1}} \dots \lambda_{j_n}^{m_{k_n}} \\ & = \sum_j \sum_k \delta_{k_1\dots k_n} \lambda_{j_1}^{l_1+m_{k_1}} \dots \lambda_{j_n}^{l_n+m_{k_n}}. \end{aligned} \tag{1.36}$$

Inserting (1.36) into the integral I, we get

$$(1.37)\qquad I = \sum_j \sum_k \delta_{k_1\ldots k_n} \int_0^1 \ldots \int_0^1 \lambda_{j_1}^{l_1+m_{k_1}+a} \ldots \lambda_{j_n}^{l_n+m_{k_n}+a} \times (1-\lambda_{j_1})^\alpha \ldots (1-\lambda_{j_n})^\alpha \, d\lambda_1 \ldots d\lambda_n$$
$$= \sum_j \sum_k \delta_{k_1\ldots k_n} B(l_1+m_{k_1}+a+1, \alpha+1) \times \ldots \times B(l_n+m_{k_n}+a+1,\alpha+1)$$
$$= n! \det |B(l_i+m_j+a+1,\alpha+1)|_{i,j=1}^n.$$

R e m a r k 1.2. In fact, we have repeated the proof of Theorem 5.2.1 of [12], where (1.35) was established for $\alpha = 0$.

Setting in (1.35), $a = 0$, $m_k = n-k$ $(1 \le k \le n)$, we get, in view of (1.16), (1.17) and (1.33), the following assertion.

LEMMA 1.4. *If* $\operatorname{Re}\alpha > -1$ *and* $\operatorname{Re} l_k > -1$ $(1 \le k \le n)$, *then*

$$(1.38)\qquad \int_0^1 \ldots \int_0^1 \det |\lambda_j^{l_i}|_{i,j=1}^n \cdot D(\lambda_1,\ldots,\lambda_n) \prod_{k=1}^n (1-\lambda_k)^\alpha \, d\lambda_1 \ldots d\lambda_n$$
$$= n!(-1)^{n(n-1)/2} \prod_{k=1}^n \frac{\Gamma(l_k+1)\Gamma(\alpha+1)}{\Gamma(l_k+n+1+\alpha)} D(l_1,\ldots,l_n)\mathcal{P}_n(\alpha).$$

The final result of this section is

LEMMA 1.5. *Suppose that* $\alpha > -1$, $f_1 \ge \ldots \ge f_n \ge 0$ *are arbitrary integers and set* $l_i = f_i + n - i$ $(1 \le i \le n)$. *Then*

$$(1.39)\qquad q(f_1,\ldots,f_n)\varrho^{(\alpha)}_{f_1\ldots f_n} = 2^{-n^2}\omega_n\omega_n'(-1)^{n(n-1)/2} \prod_{i=1}^n \frac{\Gamma(l_i+1)\Gamma(\alpha+1)}{\Gamma(l_i+n+1+\alpha)} \times D(l_1,\ldots,l_n)\mathcal{P}_n(\alpha).$$

P r o o f. Introducing the polar coordinates in the right hand side of (1.30), we get, in view of (1.21) and (1.25),

$$(1.40)\qquad q(f_1,\ldots,f_n)\varrho^{(\alpha)}_{f_1\ldots f_n} = \omega_n\omega_n' \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \ldots \int_0^{\lambda_{n-1}} d\lambda_n \, \chi_{f_1\ldots f_n}([\lambda_1,\ldots,\lambda_n]) \times \prod_{k=1}^n (1-\lambda_k)^\alpha 2^{-n^2} D^2(\lambda_1,\ldots,\lambda_n)$$

$$= 2^{-n^2}\omega_n\omega_n' \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \ldots \int_0^{\lambda_{n-1}} d\lambda_n \prod_{k=1}^n (1-\lambda_k)^\alpha$$
$$\times M_{f_1,\ldots,f_n}(\lambda_1,\ldots,\lambda_n) D(\lambda_1,\ldots,\lambda_n)$$
$$= \frac{2^{-n^2}\omega_n\omega_n'}{n!} \int_0^1 \ldots \int_0^1 \det|\lambda_j^{f_i+n-i}|_{i,j=1}^n D(\lambda_1,\ldots,\lambda_n)$$
$$\times \prod_{k=1}^n (1-\lambda_k)^\alpha \, d\lambda_1 \ldots d\lambda_n.$$

Combining (1.40) with (1.38) and taking into account the definition of l_i, we obtain (1.39).

2. Computation of the main integral

2.1. We begin with some new notations and auxiliary facts.

Let a and b be positive. We write $a \approx b$ if the ratio a/b is bounded from above as well as from below by fixed positive numbers. For example, the Euler Γ function admits the following well-known asymptotic estimate: if $\mu = \mu_1 + i\mu_2 \in \mathbb{C}$, then

$$|\Gamma(\mu + R)| \approx R^{\mu_1 - 1/2 + R} e^{-R} \tag{2.1}$$

as $R \to \infty$ (i.e. for $R_0 \le R < \infty$).

Further, for $k \ge 1$ we denote by G_k the set of all matrices $A \in M_{k,k}$ with eigenvalues less than 1 in modulus. It is not difficult to verify that G_k is a complete circular domain in $M_{k,k}$. This means that if $A \in G_k$ and $\alpha \in \mathbb{C}$, $|\alpha| \le 1$, then $\alpha A \in G_k$. In particular, G_k is starlike with respect to the null-matrix $0 \in M_{k,k}$; consequently, G_k is simply connected. Furthermore, we have:

- if $A \in M_{k,k}$, then $A \in G_k \Leftrightarrow A^* \in G_k$;
- if $A \in M_{k,k}$ and $X \in M_{k,k}^*$, then $A \in G_k \Leftrightarrow XAX^{-1} \in G_k$.

Also, $R_{k,k} \subset G_k$ for $k \ge 1$. If $m, n \ge 1$, then

$$\mathcal{Z}Z^* \in R_{m,m} \subset G_m \quad \text{and} \quad Z\mathcal{Z}^* \in R_{m,m} \subset G_m \tag{2.2}$$

for $\mathcal{Z} \in R_{m,n}$, $Z \in \overline{R_{m,n}}$ (closure in $M_{m,n}$).

Next, it is easy to see that $\det(I^{(n)} - A) \ne 0$ for $A \in G_n$.

Since $G_n \subset M_{n,n}$ is simply connected, there exists a unique holomorphic function $\varphi : G_n \to \mathbb{C}$ which satisfies

$$\exp\{\varphi(A)\} = \det(I^{(n)} - A), \quad A \in G_n, \quad \varphi(0) = 0. \tag{2.3}$$

We write $\varphi(A) = \ln\det(I^{(n)} - A)$, $A \in G_n$. Then for every $\beta \in \mathbb{C}$ we define

$$[\det(I^{(n)} - A)]^\beta := \exp\{\beta \ln\det(I^{(n)} - A)\}, \quad A \in G_n. \tag{2.4}$$

One can easily verify the following assertions:

• if $A = [\lambda_1, \ldots, \lambda_n]$, then $A \in G_n \Leftrightarrow |\lambda_i| < 1$ $(1 \le i \le n)$; moreover,

$$\ln\det(I^{(n)} - A) = \sum_{i=1}^{n} \ln(1-\lambda_i), \tag{2.5}$$

$$[\det(I^{(n)} - A)]^{\beta} = \prod_{i=1}^{n} (1-\lambda_i)^{\beta}, \qquad \forall \beta \in \mathbb{C}; \tag{2.6}$$

• if $A \in G_n$, then

$$\ln\det(I^{(n)} - A^*) = \overline{\ln\det(I^{(n)} - A)}, \tag{2.7}$$

$$[\det(I^{(n)} - A^*)]^{\beta} = \overline{[\det(I^{(n)} - A)]^{\beta}}, \qquad \forall \beta \in \mathbb{R}, \tag{2.8}$$

$$\mathrm{Re}[\ln\det(I^{(n)} - A)] = \ln|\det(I^{(n)} - A)|, \tag{2.9}$$

$$|[\det(I^{(n)} - A)]^{\beta}| = |\det(I^{(n)} - A)|^{\beta}, \qquad \forall \beta \in \mathbb{R}. \tag{2.10}$$

Finally, we shall require the following important fact established in [12, Theorem 1.2.5 and §5.3]. Let $n \ge 1$, $\mathrm{Re}\,\varrho > 0$ and set

$$a_l^{\varrho} = \Gamma(\varrho + l)/(\Gamma(\varrho)\Gamma(l+1)), \qquad l = 0, 1, 2, \ldots \tag{2.11}$$

Then

$$[\det(I^{(n)} - A)]^{-(\varrho+n-1)} = C_{\varrho} \sum_{l_1 > \ldots > l_n \ge 0} a_{l_1}^{\varrho} \ldots a_{l_n}^{\varrho} N(f_1, \ldots, f_n) \chi_{f_1 \ldots f_n}(A), \qquad A \in G_n, \tag{2.12}$$

where $C_{\varrho} = (a_0^{\varrho} a_1^{\varrho} \ldots a_{n-1}^{\varrho})^{-1}$ and $l_i = f_i + n - i$ $(1 \le i \le n)$.

2.2. For $m, n \ge 1$ and $t > -1$, $c \in \mathbb{R}$ we consider the integral

$$J_{m,n,c}^{t}(\mathcal{Z}) \equiv \int_{R_{m,n}} \frac{[\det(I^{(m)} - ZZ^*)]^{t}}{|\det(I^{(m)} - \mathcal{Z}Z^*)|^{m+n+t+c}}\, d\mu_{m,n}(Z), \qquad \mathcal{Z} \in R_{m,n}. \tag{2.13}$$

The behaviour of this integral is described by

THEOREM 2.1. *For $m, n \ge 1$, $t > -1$ and $c > \min\{m, n\} - 1$,*

$$J_{m,n,c}^{t}(\mathcal{Z}) \approx [\det(I^{(m)} - \mathcal{Z}\mathcal{Z}^*)]^{-c}, \qquad \mathcal{Z} \in R_{m,n}. \tag{2.14}$$

Proof. We break up the proof into three steps.

Step 1. First we establish (2.14) in the case $m = n$, when $t > -1$, $c > n - 1$ and

$$J_{n,n,c}^{t}(\mathcal{Z}) = \int_{R_{n,n}} \frac{[\det(I^{(n)} - ZZ^*)]^{t}}{|\det(I^{(n)} - \mathcal{Z}Z^*)|^{2n+t+c}}\, d\mu_{n,n}(Z). \tag{2.15}$$

Notice that

(2.16)
$$|\det(I^{(n)} - \mathcal{Z}Z^*)|^{-(2n+t+c)} = [\det(I^{(n)} - \mathcal{Z}Z^*)]^{-(n+(t+c)/2)} \times [\det(I^{(n)} - Z\mathcal{Z}^*)]^{-(n+(t+c)/2)}, \quad \mathcal{Z}, Z \in R_{n,n}.$$

Using (2.12) and (2.2) we obtain the expansions

(2.17)
$$[\det(I^{(n)} - \mathcal{Z}Z^*)]^{-(n+(t+c)/2)} = C_\varrho \sum_{l_1>\ldots>l_n\ge 0} a^\varrho_{l_1}\ldots a^\varrho_{l_n} N(f_1,\ldots,f_n)\chi_{f_1\ldots f_n}(\mathcal{Z}Z^*), \quad \mathcal{Z}, Z \in R_{n,n},$$

(2.18)
$$[\det(I^{(n)} - Z\mathcal{Z}^*)]^{-(n+(t+c)/2)} = C_\varrho \sum_{l_1>\ldots>l_n\ge 0} a^\varrho_{l_1}\ldots a^\varrho_{l_n} N(f_1,\ldots,f_n)\chi_{f_1\ldots f_n}(Z\mathcal{Z}^*), \quad \mathcal{Z}, Z \in R_{n,n}.$$

Note that in both (2.17) and (2.18), $\varrho = 1+(t+c)/2$ and $l_i = f_i+n-i$ $(1 \le i \le n)$. Combining (2.15)–(2.18) with (1.31), we see that

(2.19)
$$J^t_{n,n,c}(\mathcal{Z}) = C^2_\varrho \sum_{l_1>\ldots>l_n\ge 0} [a^\varrho_{l_1}\ldots a^\varrho_{l_n}]^2 N^2(f_1,\ldots,f_n)\varrho^{(t)}_{f_1\ldots f_n}\chi_{f_1\ldots f_n}(\mathcal{Z}\mathcal{Z}^*), \quad \mathcal{Z} \in R_{n,n}.$$

Further, by (1.39) and (1.18) (together with the asymptotic formula (2.1)) we have

$$N(f_1,\ldots,f_n)\varrho^{(t)}_{f_1\ldots f_n} \approx \prod_{i=1}^n \frac{1}{(l_i+1)^{n+t}}. \tag{2.20}$$

Furthermore, from (2.11) it follows that

$$a^\varrho_{l_i} \approx (l_i+1)^{\varrho-1} = (l_i+1)^{(t+c)/2} \quad (1 \le i \le n). \tag{2.21}$$

Using all these formulas, we obtain

(2.22)
$$J^t_{n,n,c}(\mathcal{Z}) \approx \sum_{l_1>\ldots>l_n\ge 0} N(f_1,\ldots,f_n)\prod_{i=1}^n \frac{1}{(l_i+1)^{n-c}}\chi_{f_1\ldots f_n}(\mathcal{Z}\mathcal{Z}^*)$$
$$\approx \sum_{l_1>\ldots>l_n\ge 0} N(f_1,\ldots,f_n)\prod_{i=1}^n \frac{\Gamma(l_i+c-n+1)}{\Gamma(l_i+1)\Gamma(c-n+1)}\chi_{f_1\ldots f_n}(\mathcal{Z}\mathcal{Z}^*), \quad \mathcal{Z} \in R_{n,n}.$$

It remains to note that (2.12) and (2.22) yield (2.14) for $m = n$.

Step 2. Assume that $m > n \geq 1$; then $t > -1$ and $c > n-1$. First note that for all $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$,

$$J^t_{m,n,c}(U\mathcal{Z}V) = J^t_{m,n,c}(\mathcal{Z}), \qquad \mathcal{Z} \in R_{m,n}. \tag{2.23}$$

Further, for every $\mathcal{Z} \in R_{m,n}$ there exists $U \in \mathcal{U}_m$ such that

$$W := U\mathcal{Z} \in R_{m,n} \tag{2.24}$$

has the form

$$W = \begin{pmatrix} W_1 \\ 0 \end{pmatrix}, \qquad W_1 \in R_{m-1,n},\ 0 \in \mathbb{C}^n, \tag{2.25}$$

and, moreover,

$$\det(I^{(m)} - \mathcal{Z}\mathcal{Z}^*) = \det(I^{(m-1)} - W_1W_1^*). \tag{2.26}$$

Consequently, by (1.13) we have

$$\begin{aligned} J^t_{m,n,c}(\mathcal{Z}) &= J^t_{m,n,c}(W) \\ &= \int\limits_{R_{m,n}} \frac{[\det(I^{(m)} - ZZ^*)]^t}{|\det(I^{(m-1)} - W_1Z_1^*)|^{m+n+t+c}}\, d\mu_{m,n}(Z) \\ &= \int\limits_{R_{m-1,n}} \frac{[\det(I^{(m-1)} - Z_1Z_1^*)]^{t+1}}{|\det(I^{(m-1)} - W_1Z_1^*)|^{m+n+t+c}}\, d\mu_{m-1,n}(Z_1) \\ &\quad \times \int\limits_{\mathbb{B}_n} (1-\omega\omega^*)^t\, d\mu_{1,n}(\omega) \\ &= J^{t+1}_{m-1,n,c}(W_1)\, \frac{\Gamma(t+1)\pi^n}{\Gamma(t+n+1)}\,. \end{aligned} \tag{2.27}$$

Thus, we have established the following fact: if $m > n \geq 1$, $t > -1$ and $c > n-1$, then for every $\mathcal{Z} \in R_{m,n}$ there exists $W_1 \in R_{m-1,n}$ such that

$$\det(I^{(m)} - \mathcal{Z}\mathcal{Z}^*) = \det(I^{(m-1)} - W_1W_1^*), \tag{2.28}$$

$$J^t_{m,n,c}(\mathcal{Z}) = J^{t+1}_{m-1,n,c}(W_1)\, \frac{\Gamma(t+1)}{\Gamma(t+n+1)}\, \pi^n. \tag{2.29}$$

It follows from (2.28) and (2.29) that one can reduce the parameter m step by step and thus reduce the problem to the case $m = n \geq 1$ examined above.

Step 3. The case $n > m \geq 1$ is considered in a similar way, except that we now use the integral formula (1.10) instead of (1.13).

Thus, the theorem is proved.

Remark 2.1. For $m = 1$ the estimate (2.14) was originally obtained in [11], where the case of arbitrary $c \in \mathbb{R}$ was considered.

R e m a r k 2.2. The results of [16] give

$$J^t_{m,n,c}(\mathcal{Z}) \equiv \mathrm{const}[\det(I^{(m)} - \mathcal{Z}\mathcal{Z}^*)]^{-c}, \quad \mathcal{Z} \in R_{m,n}, \tag{2.30}$$

where $m, n \geq 1$, $t \geq 0$ and $c = t + m + n$. Of course, (2.30) is more explicit than (2.14), but we consider the conditions $t \geq 0$ and $c = t + m + n$ to be rather restrictive.

3. Bounded projections in $L^p_\alpha(R_{m,n})$

3.1. Recall (see (0.14)) that for $m, n \geq 1$ and $\operatorname{Re}\beta > -1$ we have defined the integral operator $T^\beta_{m,n}$ acting on functions $f(Z)$, $Z \in R_{m,n}$. The assertion of Theorem D can be reformulated as follows: if $m, n \geq 1$, $1 \leq p < \infty$, $\alpha > -1$ and the complex number β satisfies $\operatorname{Re}\beta > (\alpha+1)/p - 1$ for $1 < p < \infty$ and $\operatorname{Re}\beta \geq \alpha$ for $p = 1$, then $T^\beta_{m,n}$ is a reproducing operator for the class $H^p_\alpha(R_{m,n})$. As an important addition to Theorem D we have

THEOREM 3.1. *Suppose that $m, n \geq 1$, $1 \leq p < \infty$, $\alpha > (p-1)\min\{m,n\} - p$ and β is a complex number satisfying*

$$\operatorname{Re}\beta > \frac{\alpha + \min\{m,n\}}{p} - 1. \tag{3.1}$$

Then $T^\beta_{m,n}$ is a bounded projection of $L^p_\alpha(R_{m,n})$ onto $H^p_\alpha(R_{m,n})$.

P r o o f. Since the assumptions of Theorem 3.1 imply those of Theorem D, it suffices to show that $T^\beta_{m,n}$ is bounded from $L^p_\alpha(R_{m,n})$ into $H^p_\alpha(R_{m,n})$. Furthermore, in view of [8, Corollary 3.1 to Lemma 3.1], $T^\beta_{m,n}(f)(\mathcal{Z})$ is holomorphic in $\mathcal{Z} \in R_{m,n}$, for every $f \in L^p_\alpha(R_{m,n})$. Consequently, to prove Theorem 3.1 we need to establish an estimate of the form

$$\|T^\beta_{m,n}(f)\|_{p,\alpha} \leq \mathrm{const}\, \|f\|_{p,\alpha}, \quad \forall f \in L^p_\alpha(R_{m,n}), \tag{3.2}$$

where the constant may depend on m, n and p, α, β, but not on $f \in L^p_\alpha(R_{m,n})$.

Note first that in view of Lemma 1.2 of [10],

$$|T^\beta_{m,n}(f)(\mathcal{Z})| \leq A^\beta_{m,n} \int\limits_{R_{m,n}} \frac{|f(Z)|[\det(I^{(m)} - ZZ^*)]^{\operatorname{Re}\beta}}{|\det(I^{(m)} - \mathcal{Z}Z^*)|^{m+n+\operatorname{Re}\beta}}\, d\mu_{m,n}(Z) \qquad \mathcal{Z} \in R_{m,n}, \tag{3.3}$$

where

$$A^\beta_{m,n} = |c_{m,n}(\beta)| \exp\{\pi m |\operatorname{Im}\beta|\}. \tag{3.4}$$

First we assume $p = 1$. Then

$$\begin{aligned}
(3.5)\quad &\|T^{\beta}_{m,n}(f)\|_{1,\alpha} \\
&= \int\limits_{R_{m,n}} |T^{\beta}_{m,n}(f)(\mathcal{Z})|[\det(I^{(m)} - \mathcal{Z}\mathcal{Z}^*)]^{\alpha}\, d\mu_{m,n}(\mathcal{Z}) \\
&\le A^{\beta}_{m,n} \int\limits_{R_{m,n}} [\det(I^{(m)} - \mathcal{Z}\mathcal{Z}^*)]^{\alpha} \\
&\quad \times \int\limits_{R_{m,n}} \frac{|f(Z)|[\det(I^{(m)} - ZZ^*)]^{\operatorname{Re}\beta}}{|\det(I^{(m)} - \mathcal{Z}Z^*)|^{m+n+\operatorname{Re}\beta}}\, d\mu_{m,n}(Z)\, d\mu_{m,n}(\mathcal{Z}) \\
&\le A^{\beta}_{m,n} \int\limits_{R_{m,n}} |f(Z)|[\det(I^{(m)} - ZZ^*)]^{\operatorname{Re}\beta} J^{\alpha}_{m,n,\operatorname{Re}\beta-\alpha}(Z)\, d\mu_{m,n}(Z).
\end{aligned}$$

Further, for $p = 1$ the assumptions of the theorem can be written as

$$(3.6)\qquad \alpha > -1, \qquad \operatorname{Re}\beta > \alpha + \min\{m,n\} - 1.$$

In view of Theorem 2.1, (3.5) gives

$$\begin{aligned}
(3.7)\quad \|T^{\beta}_{m,n}(f)\|_{1,\alpha} &\le \text{const} \int\limits_{R_{m,n}} |f(Z)|[\det(I^{(m)} - ZZ^*)]^{\operatorname{Re}\beta} \\
&\quad \times [\det(I^{(m)} - ZZ^*)]^{-(\operatorname{Re}\beta-\alpha)} d\mu_{m,n}(Z) = \text{const}\, \|f\|_{1,\alpha}.
\end{aligned}$$

So Theorem 3.1 is established for $p = 1$.

Suppose now that $1 < p < \infty$ and put $q = p/(p-1) \in (1,\infty)$. Set

$$(3.8)\qquad d\nu(Z) := [\det(I^{(m)} - ZZ^*)]^{\alpha} d\mu_{m,n}(Z), \qquad Z \in R_{m,n},$$

$$(3.9)\qquad Q(\mathcal{Z}, Z) := \frac{[\det(I^{(m)} - ZZ^*)]^{\operatorname{Re}\beta-\alpha}}{|\det(I^{(m)} - \mathcal{Z}Z^*)|^{m+n+\operatorname{Re}\beta}}, \qquad \mathcal{Z}, Z \in R_{m,n}.$$

Now, (3.3) can be written as

$$(3.10)\qquad |T^{\beta}_{m,n}(f)(\mathcal{Z})| \le A^{\beta}_{m,n} \int\limits_{R_{m,n}} |f(Z)| Q(\mathcal{Z}, Z)\, d\nu(Z), \qquad \mathcal{Z} \in R_{m,n}.$$

Hence, to prove (3.2) we have to show the boundedness of the integral operator

$$(3.11)\qquad \psi(Z) \to \int\limits_{R_{m,n}} \psi(Z) Q(\mathcal{Z}, Z)\, d\nu(Z), \qquad \mathcal{Z} \in R_{m,n},$$

in the space $L^p(R_{m,n}; d\nu) = L^p_{\alpha}(R_{m,n})$. For this we invoke the Forelli–Rudin lemma [11]. It asserts that the operator (3.11) is bounded provided that there exists a positive measurable function $g(\mathcal{Z})$, $\mathcal{Z} \in R_{m,n}$, such that

$$(3.12)\qquad \int\limits_{R_{m,n}} Q(\mathcal{Z}, Z)[g(Z)]^q d\nu(Z) \le \text{const}[g(\mathcal{Z})]^q, \qquad \mathcal{Z} \in R_{m,n},$$

$$\int\limits_{R_{m,n}} Q(\mathcal{Z},Z)[g(\mathcal{Z})]^p d\nu(\mathcal{Z}) \le \text{const}\,[g(Z)]^p, \qquad Z\in R_{m,n}. \tag{3.13}$$

In view of (3.8) and (3.9), these inequalities can be written as

$$\int\limits_{R_{m,n}} \frac{[g(Z)]^q[\det(I^{(m)}-ZZ^*)]^{\operatorname{Re}\beta}}{|\det(I^{(m)}-\mathcal{Z}Z^*)|^{m+n+\operatorname{Re}\beta}}\, d\mu_{m,n}(Z) \le \text{const}\,[g(\mathcal{Z})]^q, \qquad \mathcal{Z}\in R_{m,n}, \tag{3.14}$$

$$\int\limits_{R_{m,n}} \frac{[g(\mathcal{Z})]^p[\det(I^{(m)}-\mathcal{Z}\mathcal{Z}^*)]^{\alpha}}{|\det(I^{(m)}-\mathcal{Z}Z^*)|^{m+n+\operatorname{Re}\beta}} [\det(I^{(m)}-ZZ^*)]^{\operatorname{Re}\beta-\alpha}\, d\mu_{m,n}(\mathcal{Z}) \le \text{const}\,[g(Z)]^p, \qquad Z\in R_{m,n}. \tag{3.15}$$

We set

$$g(\mathcal{Z}) = [\det(I^{(m)}-\mathcal{Z}\mathcal{Z}^*)]^{-(\delta+(\min\{m,n\}-1)/q)}, \qquad \mathcal{Z}\in R_{m,n}, \tag{3.16}$$

where $\delta\in(0,\infty)$. By Theorem 2.1, the two inequalities hold under the following conditions:

$$\begin{aligned} &\operatorname{Re}\beta-(q\delta+\min\{m,n\}-1) > -1,\\ &\alpha - p\Big(\delta+\frac{\min\{m,n\}-1}{q}\Big) > -1,\\ &\operatorname{Re}\beta-\alpha+p\Big(\delta+\frac{\min\{m,n\}-1}{q}\Big) > \min\{m,n\}-1. \end{aligned} \tag{3.17}$$

It is easy to verify that in view of our assumptions such a choice of $\delta\in(0,\infty)$ is possible, so the case $1<p<\infty$ is also settled. Thus, Theorem 3.1 is established.

R e m a r k 3.1. For $m=1$, this theorem coincides with the assertion (ii) of Theorem C. Furthermore, for $m,n\ge 1$ and for the particular values $p=1$, $\alpha=0$, $\beta=m+n$, Theorem 3.1 follows from the results of [16] on bounded projections in L^1-spaces on arbitrary bounded symmetric domains.

3.2. For $p=2$, Theorem 3.1 has an important supplement. But first we need one more notation. If $m,n\ge 1$ and $\alpha>-1$, then for all $f,g\in L^2_\alpha(R_{m,n})$ we define

$$\{f,g\}_\alpha := \int\limits_{R_{m,n}} f(Z)\overline{g(Z)}\,[\det(I^{(m)}-ZZ^*)]^{\alpha}\, d\mu_{m,n}(Z). \tag{3.18}$$

Clearly, $\{\cdot,\cdot\}_\alpha$ is an inner product in $L^2_\alpha(R_{m,n})$. Moreover, with this inner product $L^2_\alpha(R_{m,n})$ is a Hilbert space and $H^2_\alpha(R_{m,n})$ is its closed subspace. Notice also that $\{f,f\}_\alpha=\|f\|^2_{2,\alpha}, \forall f\in L^2_\alpha(R_{m,n})$. For $f,g\in L^2_\alpha(R_{m,n})$ we write $f\perp g$ if $\{f,g\}_\alpha=0$.

THEOREM 3.2. *If $m,n \geq 1$ and $\alpha > -1$, then $T^{\alpha}_{m,n}$ acts in $L^2_{\alpha}(R_{m,n})$ as the orthogonal projection onto $H^2_{\alpha}(R_{m,n})$.*

Proof. Fix $f \in L^2_{\alpha}(R_{m,n})$. Then we have the representation

$$f = f_1 + f_2, \tag{3.19}$$

where $f_1 \in H^2_{\alpha}(R_{m,n})$ and $f_2 \perp H^2_{\alpha}(R_{m,n})$, i.e.

$$f_2 \perp \varphi, \quad \forall \varphi \in H^2_{\alpha}(R_{m,n}). \tag{3.20}$$

Further, in view of Theorem D (for $p = 2$, $\alpha > -1$, $\beta = \alpha$) we get

$$T^{\alpha}_{m,n}(f) = T^{\alpha}_{m,n}(f_1) + T^{\alpha}_{m,n}(f_2) = f_1 + T^{\alpha}_{m,n}(f_2). \tag{3.21}$$

Consequently, it suffices to show that

$$T^{\alpha}_{m,n}(f_2)(\mathcal{Z}) \equiv 0, \quad \mathcal{Z} \in R_{m,n}. \tag{3.22}$$

Note that

$$T^{\alpha}_{m,n}(f_2)(\mathcal{Z}) = \{f_2, \varphi_{\mathcal{Z}}\}_{\alpha}, \quad \mathcal{Z} \in R_{m,n}, \tag{3.23}$$

where

$$\varphi_{\mathcal{Z}}(Z) := c_{m,n}(\alpha)[\det(I^{(m)} - Z\mathcal{Z}^*)]^{-(m+n+\alpha)}, \quad Z \in \overline{R_{m,n}}. \tag{3.24}$$

In view of Proposition 2.2(c) of [8], for fixed $\mathcal{Z} \in R_{m,n}$ the function $\varphi_{\mathcal{Z}}$ is continuous on $\overline{R_{m,n}}$ and holomorphic in $R_{m,n}$. Hence, $\varphi_{\mathcal{Z}} \in H^2_{\alpha}(R_{m,n})$. It remains to note that (3.22) follows from (3.23) and (3.20).

Remark 3.2. For $\alpha = 0$ this result coincides with the assertion (ii) of Theorem B. Note also that Theorem 3.2 is a corollary of Theorem 3.1 only for $\alpha > \min\{m,n\} - 2$.

4. Integral representations and inequalities for pluriharmonic functions

4.1. Let Ω be an arbitrary open set in $\mathbb{C}^k$ $(k \geq 1)$. We denote by $H(\Omega)$ the space of all holomorphic functions in Ω. A function $g(\omega)$, $\omega \in \Omega$, is called *antiholomorphic* if the function $f(\omega) := \overline{g(\omega)}$ is holomorphic. The space of all antiholomorphic functions in Ω will be denoted by $\overline{H}(\Omega)$. Further, a complex function $f \in C^2(\Omega)$ is said to be *pluriharmonic* provided that its restriction to an arbitrary complex line is an ordinary harmonic function of one complex variable. It is well known that this condition can also be written as

$$\frac{\partial^2 f}{\partial \omega_j \partial \overline{\omega}_i} \equiv 0, \quad \omega = (\omega_1, \ldots, \omega_k) \in \Omega \ (1 \leq j, i \leq k). \tag{4.1}$$

The space of all pluriharmonic functions in Ω will be denoted by $h(\Omega)$. Note the inclusion

$$H(\Omega) + \overline{H}(\Omega) \subset h(\Omega). \tag{4.2}$$

Moreover, if $f \in h(\Omega)$, then $\bar{f} \in h(\Omega)$, $\operatorname{Re} f \in h(\Omega)$ and $\operatorname{Im} f \in h(\Omega)$. In particular, the real part of any holomorphic function in Ω is a real pluriharmonic function. The natural question arises: is every real pluriharmonic function the real part of some holomorphic function? In general, this is not so for every open set $\Omega \subset \mathbb{C}^k$. However, for convex domains the answer is affirmative. In other words, for every convex domain $\Omega \subset \mathbb{C}^k$ real pluriharmonic functions coincide with real parts of holomorphic functions. Hence, for such domains we have (compare with (4.2))

$$H(\Omega) + \overline{H}(\Omega) = h(\Omega). \tag{4.3}$$

Finally, observe that $R_{m,n} \subset M_{m,n} \cong \mathbb{C}^{mn}$ is convex.

4.2. Let $m, n \geq 1$ and $1 \leq p < \infty$, $\alpha > -1$. Then together with the space $H^p_\alpha(R_{m,n}) = H(R_{m,n}) \cap L^p_\alpha(R_{m,n})$ we also consider the spaces

$$\begin{aligned} \overline{H}^p_\alpha(R_{m,n}) &= \overline{H}(R_{m,n}) \cap L^p_\alpha(R_{m,n}), \\ h^p_\alpha(R_{m,n}) &= h(R_{m,n}) \cap L^p_\alpha(R_{m,n}). \end{aligned} \tag{4.4}$$

It is easy to see that

$$H^p_\alpha(R_{m,n}) + \overline{H}^p_\alpha(R_{m,n}) \subset h^p_\alpha(R_{m,n}). \tag{4.5}$$

Further, let $\operatorname{Re} \beta > -1$. Then apart from the operator

$$\begin{aligned} &T^\beta_{m,n}(f)(\mathcal{Z}) \\ &\quad = c_{m,n}(\beta) \int\limits_{R_{m,n}} \frac{f(Z)[\det(I^{(m)} - ZZ^*)]^\beta}{[\det(I^{(m)} - \mathcal{Z}Z^*)]^{m+n+\beta}} d\mu_{m,n}(Z), \qquad \mathcal{Z} \in R_{m,n}, \end{aligned} \tag{4.6}$$

which was already considered, we introduce the following integral operator:

$$\begin{aligned} &\mathcal{P}^\beta_{m,n}(f)(\mathcal{Z}) \\ &\quad = c_{m,n}(\beta) \int\limits_{R_{m,n}} f(Z)[\det(I^{(m)} - ZZ^*)]^\beta \\ &\qquad \times \bigg\{ \frac{1}{[\det(I^{(m)} - \mathcal{Z}Z^*)]^{m+n+\beta}} \\ &\qquad + \frac{1}{[\det(I^{(m)} - Z\mathcal{Z}^*)]^{m+n+\beta}} - 1 \bigg\} d\mu_{m,n}(Z), \qquad \mathcal{Z} \in R_{m,n}. \end{aligned} \tag{4.7}$$

The operators (4.6) and (4.7) are connected by the following simple (but useful) relation:

$$\mathcal{P}^\beta_{m,n}(f)(\mathcal{Z}) \equiv T^\beta_{m,n}(f)(\mathcal{Z}) + \overline{T^{\bar{\beta}}_{m,n}(\bar{f})(\mathcal{Z})} - T^\beta_{m,n}(f)(0), \quad \mathcal{Z} \in R_{m,n}. \tag{4.8}$$

LEMMA 4.1. *Let* $m, n \geq 1$, $1 \leq p < \infty$, $\alpha > -1$ *and* $f \in L^p_\alpha(R_{m,n})$. *Then*

(i) *For fixed* $\mathcal{Z} \in R_{m,n}$, *both* $T^\beta_{m,n}(f)(\mathcal{Z})$ *and* $\mathcal{P}^\beta_{m,n}(f)(\mathcal{Z})$ (*as functions of* β) *are holomorphic in the domain* $\{\operatorname{Re}\beta > (\alpha+1)/p-1\}$ *if* $1 < p < \infty$, *and are holomorphic in* $\{\operatorname{Re}\beta > \alpha\}$ *and continuous in* $\{\operatorname{Re}\beta \geq \alpha\}$ *if* $p = 1$.

(ii) *If* $\operatorname{Re}\beta > (\alpha+1)/p - 1$ (*for* $1 < p < \infty$) *and* $\operatorname{Re}\beta \geq \alpha$ (*for* $p = 1$), *then* $T^\beta_{m,n}(f)(\mathcal{Z})$ *is holomorphic* (*in* $\mathcal{Z}$) *in* $R_{m,n}$, *and* $\mathcal{P}^\beta_{m,n}(f)(\mathcal{Z})$ *is pluriharmonic* (*in* $\mathcal{Z}$) *in* $R_{m,n}$.

Proof. For $T^\beta_{m,n}$ the assertions of the lemma were established in [8, Corollaries 3.1 and 3.2 of Lemma 3.1]. The case of $\mathcal{P}^\beta_{m,n}$ is similar.

The following main theorem holds:

THEOREM 4.1. *Let* $m, n \geq 1$. *Then*

(i) *If* $1 \leq p < \infty$, $\alpha > -1$ *and* $\operatorname{Re}\beta > (\alpha+1)/p-1$ *for* $1 < p < \infty$, *and* $\operatorname{Re}\beta \geq \alpha$ *for* $p = 1$, *then for each* $u \in h^p_\alpha(R_{m,n})$ *we have a representation*

$$u(\mathcal{Z}) = \mathcal{P}^\beta_{m,n}(u)(\mathcal{Z}), \qquad \mathcal{Z} \in R_{m,n}. \tag{4.9}$$

(ii) *If* $1 \leq p < \infty$, $\alpha > (p-1)\min\{m,n\} - p$ *and*

$$\operatorname{Re}\beta > \frac{\alpha + \min\{m,n\}}{p} - 1, \tag{4.10}$$

then $\mathcal{P}^\beta_{m,n}$ *is a bounded projection of* $L^p_\alpha(R_{m,n})$ *onto* $h^p_\alpha(R_{m,n})$.

(iii) *If* $\alpha > -1$, *then* $\mathcal{P}^\alpha_{m,n}$ *is the orthogonal projection of* $L^2_\alpha(R_{m,n})$ *onto* $h^2_\alpha(R_{m,n})$.

Proof. (i) Evidently, we can suppose that $u \in h^p_\alpha(R_{m,n})$ is real. Furthermore, in view of Lemma 4.1(i) and the uniqueness theorem (for analytic functions of one complex variable) we can additionally assume that $\beta > 0$. Since $R_{m,n}$ is convex, we have $u = \operatorname{Re} f$, where $f \in H(R_{m,n})$. Note that f need not be of class $H^p_\alpha(R_{m,n})$, in spite of the condition $u \in h^p_\alpha(R_{m,n})$. Nevertheless, for each $r \in (0,1)$ we have

$$f_r(Z) := f(rZ) \in H^p_\alpha(R_{m,n}). \tag{4.11}$$

Hence, Theorem D yields

$$f_r(\mathcal{Z}) \equiv T^\beta_{m,n}(f_r)(\mathcal{Z}), \qquad \mathcal{Z} \in R_{m,n} \ (0 < r < 1), \tag{4.12}$$

$$\overline{f_r(0)} \equiv T^\beta_{m,n}(\bar{f}_r)(\mathcal{Z}), \qquad \mathcal{Z} \in R_{m,n} \ (0 < r < 1). \tag{4.13}$$

Summing (4.12) and (4.13), we get

$$f_r(\mathcal{Z}) + \overline{f_r(0)} = 2T^\beta_{m,n}(u_r)(\mathcal{Z}), \qquad \mathcal{Z} \in R_{m,n} \ (0 < r < 1). \tag{4.14}$$

Then set $\mathcal{Z} = 0$ in (4.14):

$$u_r(0) = T^\beta_{m,n}(u_r)(0) \qquad (0 < r < 1). \tag{4.15}$$

Further, since β is real, (4.8) leads to

$$\mathcal{P}^{\beta}_{m,n}(u_r)(\mathcal{Z}) \equiv T^{\beta}_{m,n}(u_r)(\mathcal{Z}) + \overline{T^{\beta}_{m,n}(u_r)(\mathcal{Z})} - T^{\beta}_{m,n}(u_r)(0), \quad \mathcal{Z} \in R_{m,n} \ (0<r<1). \tag{4.16}$$

Taking real parts in (4.14), we obtain

$$u_r(\mathcal{Z}) + u_r(0) = 2\operatorname{Re} T^{\beta}_{m,n}(u_r)(\mathcal{Z}), \quad \mathcal{Z} \in R_{m,n} \ (0<r<1). \tag{4.17}$$

Using all these formulas, we get

$$u_r(\mathcal{Z}) = \mathcal{P}^{\beta}_{m,n}(u_r)(\mathcal{Z}), \quad \mathcal{Z} \in R_{m,n} \ (0<r<1). \tag{4.18}$$

Now note (see (4.7)) that (4.18) can be written as follows:

$$\begin{aligned} u(r\mathcal{Z}) = c_{m,n}(\beta) r^{-2m(n+\beta)} \int\limits_{rR_{m,n}} & u(Z)[\det(r^2 I^{(m)} - ZZ^*)]^{\beta} \\ & \times \bigg\{ \frac{1}{[\det(I^{(m)} - \mathcal{Z}(Z^*/r))]^{m+n+\beta}} \\ & + \frac{1}{[\det(I^{(m)} - (Z/r)\mathcal{Z}^*)]^{m+n+\beta}} - 1 \bigg\} \, d\mu_{m,n}(Z), \\ & \mathcal{Z} \in R_{m,n} \ (0<r<1), \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} rR_{m,n} &= \{rZ : Z \in R_{m,n}\} \\ &= \{Z \in M_{m,n} : r^2 I^{(m)} - ZZ^* > 0\} \quad (0<r<1). \end{aligned} \tag{4.20}$$

Letting r to tend to 1 in (4.19), we get (4.9) in view of the Lebesgue dominated convergence theorem.

Further, Theorem 3.1 together with Lemma 4.1(ii) and (4.8) give (ii). The proof of (iii) is merely a repetition of that of Theorem 3.2. Thus, Theorem 4.1 is proved.

Remark 4.1. The operator $\mathcal{P}^{\beta}_{1,n}$ was considered in [1]. There it was also established that for $\alpha > -1$, $\mathcal{P}^{\alpha}_{1,n}$ is the orthogonal projection of $L^2_{\alpha}(R_{1,n}) = L^2_{\alpha}(\mathbb{B}_n)$ onto $h^2_{\alpha}(R_{1,n}) = h^2_{\alpha}(\mathbb{B}_n)$.

4.3. We now give some applications of the main theorems established above.

Theorem 4.2. (a) *If* $1 \le p < \infty$ *and* $\alpha > (p-1)\min\{m,n\} - p$, *then*

$$h^p_{\alpha}(R_{m,n}) = H^p_{\alpha}(R_{m,n}) + \overline{H}^p_{\alpha}(R_{m,n}). \tag{4.21}$$

(b) *If* $\alpha > -1$, *then*

$$h^2_{\alpha}(R_{m,n}) = H^2_{\alpha}(R_{m,n}) + \overline{H}^2_{\alpha}(R_{m,n}). \tag{4.22}$$

P r o o f. We only prove (a) as (b) can be established in the same way. In view of (4.5), it suffices to show that

$$h^p_\alpha(R_{m,n}) \subset H^p_\alpha(R_{m,n}) + \overline{H}^p_\alpha(R_{m,n}). \tag{4.23}$$

Fix $\beta \in \mathbb{R}$ such that $\beta > (\alpha + \min\{m,n\})/p - 1$. By Theorem 4.1(i) and (4.8) we get

$$u(\mathcal{Z}) \equiv T^\beta_{m,n}(u)(\mathcal{Z}) + \overline{T^\beta_{m,n}(\overline{u})(\mathcal{Z})} - T^\beta_{m,n}(u)(0), \quad \mathcal{Z} \in R_{m,n}, \quad \forall u \in h^p_\alpha(R_{m,n}). \tag{4.24}$$

According to Theorem 3.1,

$$T^\beta_{m,n}(u) \in H^p_\alpha(R_{m,n}), \quad \overline{T^\beta_{m,n}(\overline{u})} \in \overline{H}^p_\alpha(R_{m,n}). \tag{4.25}$$

Combining (4.24) with (4.25), we see that $u \in H^p_\alpha(R_{m,n}) + \overline{H}^p_\alpha(R_{m,n})$, which completes the proof.

THEOREM 4.3. *Assume that either*

(a) $1 \le p < \infty$, $\alpha > (p-1)\min\{m,n\} - p$ *and* $\alpha \ge 0$, *or*
(b) $p = 2$, $\alpha \ge 0$.

Then

$$\|f\|_{p,\alpha} \le C\|u\|_{p,\alpha}, \quad C = C(p,\alpha) \in (0,\infty), \tag{4.26}$$

for all $f = u + iv \in H(R_{m,n})$ *with* $v(0) = 0$.

P r o o f. We first assume that $f = u + iv \in H^p_\alpha(R_{m,n})$ and $v(0) = 0$. Fix $\beta \in \mathbb{R}$ with

$$\beta > \frac{\alpha + \min\{m,n\}}{p} - 1 \quad \text{(in case (a))},$$
$$\beta = \alpha \quad \text{(in case (b))}.$$

In view of Theorem D we have

$$f(\mathcal{Z}) \equiv T^\beta_{m,n}(f)(\mathcal{Z}), \quad u(0) \equiv T^\beta_{m,n}(\bar{f})(\mathcal{Z}), \quad \mathcal{Z} \in R_{m,n}. \tag{4.27}$$

Consequently,

$$f(\mathcal{Z}) \equiv 2T^\beta_{m,n}(u)(\mathcal{Z}) - u(0), \quad \mathcal{Z} \in R_{m,n}, \tag{4.28}$$

or

$$f(\mathcal{Z}) \equiv 2T^\beta_{m,n}(u)(\mathcal{Z}) - T^\beta_{m,n}(u)(0), \quad \mathcal{Z} \in R_{m,n}. \tag{4.29}$$

From (4.29) and Theorems 3.1, 3.2 it follows that the estimate (4.26) is valid, but under the additional hypothesis $f \in H^p_\alpha(R_{m,n})$ (note that the assumption $\alpha \ge 0$ is not used yet). If we only have $f \in H(R_{m,n})$, then for $r \in (0,1)$, $f_r(Z) := f(rZ) \in H^p_\alpha(R_{m,n})$. Hence

$$\|f_r\|_{p,\alpha} \le C(p,\alpha)\|u_r\|_{p,\alpha}, \quad r \in (0,1). \tag{4.30}$$

This estimate can be written as follows:

$$(4.31)\qquad \int\limits_{rR_{m,n}} |f(Z)|^p[\det(r^2I^{(m)} - ZZ^*)]^\alpha d\mu_{m,n}(Z) \le \widetilde{C}(p,\alpha) \int\limits_{rR_{m,n}} |u(Z)|^p[\det(r^2I^{(m)} - ZZ^*)]^\alpha d\mu_{m,n}(Z).$$

The final step is to let r tend to 1 in (4.31). If we take into account the hypothesis $\alpha \ge 0$, then an application of the Lebesgue monotone convergence theorem makes it possible to derive the estimate (4.26) from (4.31). Thus, Theorem 4.3 is proved.

Remark 4.2. In [2] the estimates of type (4.26) were established for rather large classes of unbounded multidimensional domains. Moreover, there the conditions on the parameters p and α were not so restrictive as in Theorem 4.3.

Remark 4.3. For $p = 1$, $\alpha = 0$ and under the assumption $f(0) = 0$, Theorem 4.3 follows from [16], where, as mentioned earlier, the case of arbitrary bounded symmetric domains is considered.

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