A counterexample to
a conjecture of Drużkowski and Rusek

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Abstract. Let \( F = X + H \) be a cubic homogeneous polynomial automorphism from \( \mathbb{C}^n \) to \( \mathbb{C}^n \). Let \( p \) be the nilpotence index of the Jacobian matrix \( JH \). It was conjectured by Drużkowski and Rusek in \([4]\) that \( \deg F^{-1} \leq 3^{p-1} \). We show that the conjecture is true if \( n \leq 4 \) and false if \( n \geq 5 \).

1. Introduction. In \([1]\) and \([7]\) it was shown that it suffices to prove the Jacobian Conjecture for cubic homogeneous polynomial maps from \( \mathbb{C}^n \) to \( \mathbb{C}^n \), i.e. maps of the form \( F = (F_1, \ldots, F_n) \) with \( F_i = X_i + H_i \), where each \( H_i \) is either zero or a homogeneous polynomial of degree 3. In \([2]\) it was shown that it even suffices to consider cubic linear polynomial maps, i.e. maps such that each \( H_i \) is of the form \( H_i = l_i^3 \), where \( l_i \) is a linear form.

A crucial result (cf. \([1]\) and \([6]\)) asserts that the degree of the inverse of a polynomial automorphism \( F \) is bounded by \( (\deg F)^{n-1} \) (where \( \deg F = \max \deg F_i \)). In \([4]\) Drużkowski and Rusek proved that for cubic homogeneous (resp. cubic linear) automorphisms this degree estimate could be improved in some special cases; more precisely, if \( \text{ind } JH \) denotes the index of nilpotency of \( JH \) then they showed that \( \deg F^{-1} \leq 3^{\text{ind } JH - 1} \) if \( \text{ind } JH \leq 2 \) and also if \( H \) is cubic linear and \( \text{ind } JH \leq 3 \). This led them to the following conjecture:

Conjecture 1.1 (D–R) \([4]\), 1985. If \( F = X + H \) is a cubic homogeneous polynomial automorphism, then \( \deg F^{-1} \leq 3^{p-1} \), where \( p = \text{ind } JH \).

Recently, in \([3]\), Drużkowski showed that Conjecture D–R is true in case all coefficients of \( H \) are real numbers \( \leq 0 \) (in which case the map \( F \) is stably tame, a result obtained by Yu in \([8]\)).

In the present paper we show that the conjecture is true if \( n \leq 4 \) and false if \( n \geq 5 \).

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2. The counterexample for \( n \geq 5 \). Let \( n \geq 5 \) and consider the polynomial ring \( \mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n] \).

**Theorem 2.1.** For each \( n \geq 5 \) the polynomial automorphism
\[
F = (X_1 + 3X_2^4X_2 - 2X_4X_5X_3, X_2 + X_4^2X_5, X_3 + X_4^2X_5, X_4 + X_5^3, X_5, \ldots, X_n)
\]
is a counterexample to Conjecture D–R.

**Proof.** Put \( H = F - X \). Then one easily verifies that \((JH)^3 = 0\) and \((JH)^2 \neq 0\). Thus \( \text{ind } JH = 3 \). So if Conjecture D–R is true, then \( \deg F^{-1} \leq 9 \). However, the inverse \( G = (G_1, \ldots, G_n) \) of \( F \) is given by the following formulas:
\[
G_1 = X_1 - 3(X_4 - X_5^3)^2(X_2 - (X_4 - X_5^3)^2X_5) + 2(X_4 - X_5^3)X_5(X_3 - (X_4 - X_5^3)^3),
\]
\[
G_2 = X_2 - (X_4 - X_5^3)^2X_5,
\]
\[
G_3 = X_3 - (X_4 - X_5^3)^3,
\]
\[
G_4 = X_4 - X_5^3,
\]
\[
G_i = X_i \quad \text{for all } 5 \leq i \leq n.
\]

So looking at the highest power of \( X_5 \) appearing in \( G_1 \), one easily verifies that \( \deg G_1 = 13 > 9 \). \( \blacksquare \)

3. The case \( n \leq 4 \). The main result of this section is

**Proposition 3.1.** Conjecture D–R is true if \( n \leq 4 \).

To prove this result we need the following theorem (cf. [5]):

**Theorem 3.2.** Let \( K \) be a field of characteristic zero and \( F = X - H \) a cubic homogeneous polynomial map in dimension four such that \( \text{Det}(JF) = 1 \). Then there exists some \( T \in GL_4(K) \) such that \( T^{-1}FT \) is of one of the following forms:

\[
(1) \begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 - a_4x_1^3 - b_4x_1^2x_2 - c_4x_1x_2^2 - d_4x_1x_2x_3
\end{pmatrix},
\]

\[
(2) \begin{pmatrix}
    x_1 \\
    x_2 - \frac{1}{5}x_1^3 - h_2x_1x_2^2 - q_2x_3^3 \\
    x_3 \\
    x_4 - x_1^2x_3 - h_4x_1x_3^2 - q_4x_3^3
\end{pmatrix},
\]
Theorem 3.1. As remarked in the introduction, the case \( \text{ind} \, JH = n \) was proved in [1] and [6]. The case \( \text{ind} \, JH = 2 \) was done in [4]. So we may assume that \( 2 < \text{ind} \, JH < n \). Therefore only the case \( n = 4 \) and \( \text{ind} \, JH = 3 \) remains. By the classification theorem of Hubbers ([5, Theorem 2.7]) we know that there exists \( T \in GL_4(\mathbb{C}) \) such that \( T^{-1}FT \) has one of the eight forms described above. One easily verifies that in each of the eight

\[
\begin{align*}
(3) & \\
& \begin{pmatrix}
  x_1 \\
  x_2 - \frac{1}{3} x_1^3 + c_1 x_1 x_2 x_3 - \frac{16q_4 c_2 - q_1 c_2}{48 c_1^2} x_1^2 x_3 \\
  x_3 - \frac{q_4}{4} x_1 x_2 x_3 + x_4 x_2 x_3^2 - \frac{q_4}{12 c_1} x_3^3 - \frac{r}{16 c_1} x_1^2 x_3 \\
  x_4 - x_1 x_2^3 + \frac{r}{4} x_1 x_2 x_3^2 - 3 c_1 x_1 x_2 x_3 + 9 c_1 x_1 x_2 x_3^2 \\
  - q_4 x_2^3 - \frac{r}{4} x_2 x_3^2 x_4
\end{pmatrix}, \\
(4) & \\
& \begin{pmatrix}
  x_1 \\
  x_2 - \frac{1}{4} x_1^3 \\
  x_3 - x_1 x_2^2 - e_3 x_1 x_2^2 - k_3 x_2^3 \\
  x_4 - e_4 x_1 x_2^2 - k_4 x_2^3
\end{pmatrix}, \\
(5) & \\
& \begin{pmatrix}
  x_1 \\
  x_2 - \frac{1}{3} x_1^3 + i_3 x_1 x_2 x_3 - j_2 x_1 x_2 x_3^2 - t_2 x_4 \\
  x_3 - x_1 x_2^3 - \frac{2 m_4}{n_4} x_1 x_2 x_3 - i_3 x_1 x_2 x_3 - k_3 x_2^3 \\
  - s_3 x_1 x_2^3 - t_3 x_4 \\
  x_4
\end{pmatrix}, \\
(6) & \\
& \begin{pmatrix}
  x_1 \\
  x_2 - \frac{1}{4} x_1^3 - j_2 x_1 x_2^2 - t_2 x_4 \\
  x_3 - x_1 x_2^3 - e_3 x_1 x_2^2 - g_3 x_1 x_2 x_3 - j_3 x_1 x_2 x_3^2 - k_3 x_2^3 \\
  - m_3 x_1 x_2 x_3 - p_3 x_2 x_3^2 - t_4 x_4 \\
  x_4
\end{pmatrix}, \\
(7) & \\
& \begin{pmatrix}
  x_1 \\
  x_2 - \frac{1}{5} x_1^3 \\
  x_3 - x_1 x_2^2 - e_3 x_1 x_2^2 - k_3 x_2^3 \\
  x_4 - x_1 x_2^3 - e_4 x_1 x_2^2 - f_4 x_1 x_2 x_3 - h_4 x_1 x_2^3 - k_4 x_2^3 \\
  - l_4 x_2 x_3 - n_4 x_2 x_3^2 - q_4 x_4
\end{pmatrix}, \\
(8) & \\
& \begin{pmatrix}
  x_1 \\
  x_2 - \frac{1}{5} x_1^3 \\
  x_3 - x_1 x_2^3 - e_3 x_1 x_2^2 + g_4 x_1 x_2 x_3 - k_3 x_2^3 + m_4 x_2^3 x_3 + g_4 x_2^3 x_4 \\
  - x_1 x_2 x_3 - e_4 x_1 x_2^2 - \frac{2 m_4}{n_4} x_1 x_2 x_3 - g_4 x_1 x_2 x_4 - k_4 x_2^3
\end{pmatrix}.
\end{align*}
\]

Proof. See [5, Theorem 2.7].
cases in which the nilpotency index of $JH$ equals 3, $\deg(T^{-1}FT)^{-1} \leq 9$, so $\deg F^{-1} \leq 9$. ■

References


