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# On positive solutions of a class of second order nonlinear differential equations on the halfline

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**Abstract.** The differential equation of the form $(q(t)k(u)(u')^a)' = f(t)h(u)u'$, $a \in (0,\infty)$, is considered and solutions $u$ with $u(0) = 0$ and $(u(t))^2 + (u'(t))^2 > 0$ on $(0,\infty)$ are studied. Theorems about existence, uniqueness, boundedness and dependence of solutions on a parameter are given.

**1. Introduction.** In [9] the differential equation $(q(t)k(u)u')' = F(t,u)u'$ was considered and the author gave sufficient conditions for the existence and uniqueness of solutions $u$ such that $u(0) = 0$ and $(u(t))^2 + (u'(t))^2 > 0$ for $t \in (0,\infty)$. This problem is connected with the description of the mathematical model of infiltration of water. For more details see e.g. [3], [4] and [6]. Naturally, a question arises of what are the properties of solutions of the differential equation $(q(t)k(u)(u')^a)' = F(t,u)u'$, where $a$ is a positive constant. For the sake of simplicity of our assumptions, results and proofs we will consider the differential equations of the type

$$(q(t)k(u)(u')^a)' = f(t)h(u)u', \qquad a \in (0,\infty). \tag{1}$$

We also study the qualitative dependence of solutions of (1) on the parameter $a$. As special cases we obtain results of [9] (with $F(t,u) = f(t)h(u)$ and $a = 1$), of [8] (where $a = 1$, $f \in C^1(\mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$) and of [7] (where $a = 1$, $q(t) \equiv 1$, $h(u) \equiv 1$). We observe that special cases of (1) (with $a = 1$) were also considered in [1], [2], [4] and [6].

**2. Notations and lemmas.** We consider equation (1) in which the functions $q, k, f$ and $h$ satisfy the following assumptions:

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(H$_1$) $q \in C^0(\mathbb{R}_+)$, $q(t) > 0$ for $t > 0$ and $\int_0 (1/q(s))^{1/a}\,ds < \infty$;

(H$_2$) $k \in C^0(\mathbb{R}_+)$, $k(0) = 0$, $k(u) > 0$ for $u > 0$ and $\int_0 (k(s))^{1/a}\,ds < \infty$, $\int^\infty (k(s))^{1/a}\,ds = \infty$;

(H$_3$) $f \in C^0(\mathbb{R}_+)$, $f(t) > 0$ for $t \in \mathbb{R}_+$ and $f$ is decreasing on $\mathbb{R}_+$;

(H$_4$) $h \in C^0(\mathbb{R}_+)$, $h(u) \ge 0$ for $u \in \mathbb{R}_+$ and $H(u) = \int_0^u h(s)\,ds$ is strictly increasing on $\mathbb{R}_+$;

(H$_5$) $\int_0 (k(s)/H(s))^{1/a}\,ds < \infty$, $\int^\infty (k(s)/H(s))^{1/a}\,ds = \infty$.

We say that $u$ is a *solution* of (1) if $u \in C^0(\mathbb{R}_+) \cap C^1((0,\infty))$, $u(0) = 0$, $u(t) \ge 0$ on $\mathbb{R}_+$, $(u(t))^2 + (u'(t))^2 > 0$ for $t \in (0,\infty)$, $q(t)k(u(t))(u'(t))^a$ is continuously differentiable on $(0,\infty)$, $\lim_{t\to 0_+} q(t)k(u(t))(u'(t))^a = 0$ and (1) is satisfied on $(0,\infty)$.

Let $p \in C^0(\mathbb{R})$, $p(0) = 0$. We say that $u$ is a *solution* of the differential equation

$$(q(t)k(u)p(u'))' = f(t)h(u)u' \tag{2}$$

if $u \in C^0(\mathbb{R}_+) \cap C^1((0,\infty))$, $u(0) = 0$, $u(t) \ge 0$ on $\mathbb{R}_+$, $(u(t))^2 + (u'(t))^2 > 0$ for $t \in (0,\infty)$, $q(t)k(u(t))p(u'(t))$ is continuously differentiable on $(0,\infty)$, $\lim_{t\to 0_+} q(t)k(u(t))p(u'(t)) = 0$ and (2) is satisfied on $(0,\infty)$.

Lemma 1. *Let $u(t)$ be a solution of* (2). *Then $u'(t) > 0$ for $t \in (0,\infty)$.*

Proof. We see that

$$q(t)k(u(t))p(u'(t)) = \int_0^t f(s)h(u(s))u'(s)\,ds \quad \text{for } t > 0. \tag{3}$$

Suppose that there exist $0 < t_1 < t_2$ such that $u'(t_1) = u'(t_2) = 0$ and $u'(t) > 0$ (resp. $u'(t) < 0$) on $(t_1, t_2)$. Then $u(t) > 0$ for $t \in [t_1, t_2]$ and (3) implies

$$0 = q(t_2)k(u(t_2))p(u'(t_2)) - q(t_1)k(u(t_1))p(u'(t_1)) = \int_{t_1}^{t_2} f(s)h(u(s))u'(s)\,ds,$$

which contradicts

$$\int_{t_1}^{t_2} f(s)h(u(s))u'(s)\,ds \ge f(t_2)\int_{u(t_1)}^{u(t_2)} h(s)\,ds > 0$$

$$\Big(\text{resp. } \int_{t_1}^{t_2} f(s)h(u(s))u'(s)\,ds \le f(t_2)\int_{u(t_1)}^{u(t_2)} h(s)\,ds < 0\Big).$$

Assume $u'(\tau) = 0$ for a $\tau \in (0,\infty)$ and $u'(t) \ne 0$ on $(0,\tau)$. Then necessarily

$u'(t) > 0$ on $(0,\tau)$ since $u(t) \ge 0$ for $t \in \mathbb{R}_+$, and (cf. (3))

$$0 = q(\tau)k(u(\tau))p(u'(\tau)) = \int_0^\tau f(s)h(u(s))u'(s)\,ds,$$

which contradicts

$$\int_0^\tau f(s)h(u(s))u'(s)\,ds \ge f(\tau)\int_0^{u(\tau)} h(s)\,ds > 0.$$

Therefore by virtue of $(u(t))^2 + (u'(t))^2 > 0$ on $(0,\infty)$ we conclude $u'(t) > 0$ for $t \in (0,\infty)$. ∎

COROLLARY 1. *Let $u(t)$ be a solution of* (1). *Then $u'(t) > 0$ for $t \in (0,\infty)$.*

Proof. If $a = m/n$, where $m, n \in \mathbb{N}$ and $n$ is odd, then the function $v^a$ is defined for all $v \in \mathbb{R}$ and Corollary 1 follows from Lemma 1. Assume $a = m/n$, where $m, n \in \mathbb{N}$ and $n$ is even or $a$ is an irrational number. Then the function $v^a$ is defined for all $v \in \mathbb{R}_+$, and for every $p_1 \in C^0((-\infty, 0])$ with $p_1(0) = 0$, the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(v) = v^a$ for $v \in \mathbb{R}_+$ and $p(v) = p_1(v)$ for $v \in (-\infty, 0)$ is continuous on $\mathbb{R}$, $p(0) = 0$ and, moreover, $u(t)$ is a solution of (2). Hence $u'(t) > 0$ on $(0,\infty)$ by Lemma 1. ∎

Remark 1. It follows from Corollary 1 that $u \in \mathcal{A}$ for any solution $u$ of (1), where

$$\mathcal{A} = \{u \in C^0(\mathbb{R}_+) : u(0) = 0,\ u \text{ is strictly increasing on } \mathbb{R}_+\}.$$

Set

$$k_1(u) = (k(u))^{1/a}, \quad K_1(u) = \int_0^u k_1(s)\,ds, \quad P(u) = \int_0^u \left(\frac{k(s)}{H(s)}\right)^{1/a} ds$$

for $u \in \mathbb{R}_+$. Obviously, $k_1 \in C^0(\mathbb{R}_+)$, $K_1 \in C^1(\mathbb{R}_+)$, $P \in C^0(\mathbb{R}_+) \cap C^1((0,\infty))$, $K_1$ and $P$ are strictly increasing on $\mathbb{R}_+$, $\lim_{u\to\infty} K_1(u) = \infty$ by $(H_2)$ and $\lim_{u\to\infty} P(u) = \infty$ by $(H_5)$.

LEMMA 2. *If $u(t)$ is a solution of* (1), *then*

$$u(t) = K_1^{-1}\bigg(\int_0^t \bigg(\frac{1}{q(s)}\int_0^{u(s)} f(u^{-1}(\tau))h(\tau)\,d\tau\bigg)^{1/a} ds\bigg), \qquad t \in \mathbb{R}_+, \tag{4}$$

*where $K_1^{-1}$ and $u^{-1}$ denote the inverse functions to $K_1$ and $u$, respectively. Conversely, if $u \in \mathcal{A}$ is a solution of* (4), *then $u(t)$ is a solution of* (1).

Proof. Let $u$ be a solution of (1). Then $u \in \mathcal{A}$ (cf. Remark 1) and

$$(k_1(u(t))u'(t))^a = \frac{1}{q(t)}\int_0^t f(s)h(u(s))u'(s)\,ds, \qquad t > 0.$$

Hence

$$(K_1(u(t)))' = \left(\frac{1}{q(t)}\int_0^{u(t)} f(u^{-1}(s))h(s)\,ds\right)^{1/a}, \qquad t > 0, \tag{5}$$

and integrating (5) from 0 to $t$, we obtain

$$K_1(u(t)) = \int_0^t \left(\frac{1}{q(s)}\int_0^{u(s)} f(u^{-1}(\tau))h(\tau)\,d\tau\right)^{1/a} ds, \qquad t \in \mathbb{R}_+,$$

and consequently, equality (4) is satisfied.

Conversely, let $u \in \mathcal{A}$ be a solution of (4). Then $u \in C^1((0,\infty))$,

$$\lim_{t\to 0_+} q(t)k(u(t))(u'(t))^a = \lim_{t\to 0_+} \int_0^{u(t)} f(u^{-1}(s))h(s)\,ds = 0$$

and $(q(t)k(u(t))(u'(t))^a)' = f(t)h(u(t))u'(t)$ for $t \in (0,\infty)$. Hence $u$ is a solution of (1). ■

Define $\underline{\varphi}, \overline{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\underline{\varphi}(t) = P^{-1}\left(\int_0^t \left(\frac{f(s)}{q(s)}\right)^{1/a} ds\right), \qquad \overline{\varphi}(t) = P^{-1}\left(\int_0^t \left(\frac{f(0)}{q(s)}\right)^{1/a} ds\right),$$

where $P^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the inverse function to $P$. Obviously, $\underline{\varphi}(t) \le \overline{\varphi}(t)$ on $\mathbb{R}_+$ by ($H_3$).

Lemma 3. *Let $u(t)$ be a solution of* (1). *Then*

$$\underline{\varphi}(t) \le u(t) \le \overline{\varphi}(t) \quad \textit{for } t \in \mathbb{R}_+. \tag{6}$$

Proof. Since

$$f(t)H(u(t)) = f(t)\int_0^{u(t)} h(s)\,ds \le \int_0^t f(s)h(u(s))u'(s)\,ds \le f(0)\int_0^{u(t)} h(s)\,ds = f(0)H(u(t)),$$

we have

$$f(t)H(u(t)) \le q(t)(k_1(u(t))u'(t))^a \le f(0)H(u(t)), \qquad t \in (0,\infty).$$

Thus

$$\left(\frac{f(t)}{q(t)}H(u(t))\right)^{1/a} \le k_1(u(t))u'(t) \le \left(\frac{f(0)}{q(t)}H(u(t))\right)^{1/a}$$

and

$$(7)\qquad \left(\frac{f(t)}{q(t)}\right)^{1/a} \le \left(\frac{k(u(t))}{H(u(t))}\right)^{1/a} u'(t)\ (=(P(u(t)))') \le \left(\frac{f(0)}{q(t)}\right)^{1/a}, \quad t \in (0,\infty).$$

Integrating (7) from 0 to $t$ we obtain

$$\int_0^t \left(\frac{f(s)}{q(s)}\right)^{1/a} ds \le P(u(t)) \le \int_0^t \left(\frac{f(0)}{q(s)}\right)^{1/a} ds, \qquad t \in \mathbb{R}_+,$$

and (6) holds. ∎

Set

$$\mathcal{K} = \{u \in \mathcal{A} : \underline{\varphi}(t) \le u(t) \le \overline{\varphi}(t) \text{ for } t \in \mathbb{R}_+ \text{ and}$$
$$u(t_2) - u(t_1) \ge (f(t_2)H(\underline{\varphi}(t_1)))^{1/a} \int_{t_1}^{t_2} (1/q(s))^{1/a}\, ds$$
$$\times [\max\{k_1(u) : \underline{\varphi}(t_1) \le u \le \overline{\varphi}(t_2)\}]^{-1} \text{ for } 0<t_1<t_2\}.$$

Remark 2. We now verify that $\underline{\varphi} \in \mathcal{K}$ and thus $\mathcal{K}$ is a nonempty subset of $\mathcal{A}$. Fix $0 < t_1 < t_2$. Then

$$P(\underline{\varphi}(t_2)) - P(\underline{\varphi}(t_1)) = \int_{t_1}^{t_2} \left(\frac{f(s)}{q(s)}\right)^{1/a} ds$$

and, by the Taylor formula, there exists $\xi \in (\underline{\varphi}(t_1), \underline{\varphi}(t_2)) \subset (\underline{\varphi}(t_1), \overline{\varphi}(t_2))$ such that

$$P'(\xi)(\underline{\varphi}(t_2) - \underline{\varphi}(t_1)) \ge (f(t_2))^{1/a} \int_{t_1}^{t_2} \left(\frac{1}{q(s)}\right)^{1/a} ds.$$

Since

$$P'(\xi) = \frac{k_1(\xi)}{(H(\xi))^{1/a}} \le \max\{k_1(u) : \underline{\varphi}(t_1) \le u \le \overline{\varphi}(t_2)\}\left(\frac{1}{H(\underline{\varphi}(t_1))}\right)^{1/a},$$

we get

$$\underline{\varphi}(t_2) - \underline{\varphi}(t_1) \ge \frac{1}{P'(\xi)}(f(t_2))^{1/a} \int_{t_1}^{t_2} \left(\frac{1}{q(s)}\right)^{1/a} ds$$
$$\ge (f(t_2)H(\underline{\varphi}(t_1)))^{1/a} \int_{t_1}^{t_2} \left(\frac{1}{q(s)}\right)^{1/a} ds$$
$$\times [\max\{k_1(u) : \underline{\varphi}(t_1) \le u \le \overline{\varphi}(t_2)\}]^{-1}$$

and therefore $\underline{\varphi} \in \mathcal{K}$. Analogously we can show that $\overline{\varphi} \in \mathcal{K}$ as well.

Define the operator $\mathrm{T}:\mathcal{K}\to C^0(\mathbb{R}_+)$ by

$$(\mathrm{T}u)(t)=K_1^{-1}\biggl(\int_0^t\biggl(\frac{1}{q(s)}\int_0^{u(s)}f(u^{-1}(\tau))h(\tau)\,d\tau\biggr)^{1/a}ds\biggr),\qquad t\in\mathbb{R}_+.$$

Lemma 4. $\mathrm{T}:\mathcal{K}\to\mathcal{K}$.

Proof. Let $u\in\mathcal{K}$. Set

$$\gamma(t)=\int_0^t\biggl(\frac{1}{q(s)}\int_0^{u(s)}f(u^{-1}(\tau))h(\tau)\,d\tau\biggr)^{1/a}ds,$$
$$\alpha(t)=\gamma(t)-K_1(\underline{\varphi}(t)),\qquad \beta(t)=\gamma(t)-K_1(\overline{\varphi}(t))$$

for $t\in\mathbb{R}_+$. Then

$$\begin{aligned}\alpha'(t)&=\biggl(\frac{1}{q(t)}\int_0^{u(t)}f(u^{-1}(s))h(s)\,ds\biggr)^{1/a}-\frac{k_1(\underline{\varphi}(t))}{P'(\underline{\varphi}(t))}\biggl(\frac{f(t)}{q(t)}\biggr)^{1/a}\\&\ge\biggl(\frac{f(t)}{q(t)}H(u(t))\biggr)^{1/a}-\frac{k_1(\underline{\varphi}(t))}{k_1(\underline{\varphi}(t))}\biggl(\frac{f(t)}{q(t)}H(\underline{\varphi}(t))\biggr)^{1/a}\ge 0,\\ \beta'(t)&=\biggl(\frac{1}{q(t)}\int_0^{u(t)}f(u^{-1}(s))h(s)\,ds\biggr)^{1/a}-\frac{k_1(\overline{\varphi}(t))}{P'(\overline{\varphi}(t))}\biggl(\frac{f(0)}{q(t)}\biggr)^{1/a}\\&\le\biggl(\frac{f(0)}{q(t)}H(u(t))\biggr)^{1/a}-\frac{k_1(\overline{\varphi}(t))}{k_1(\overline{\varphi}(t))}\biggl(\frac{f(0)}{q(t)}H(\overline{\varphi}(t))\biggr)^{1/a}\le 0\end{aligned}$$

for $t\in(0,\infty)$. Since $\alpha(0)=\beta(0)=0$ and $\alpha'(t)\ge 0$, $\beta'(t)\le 0$ on $(0,\infty)$, we see that $\alpha(t)\ge 0$, $\beta(t)\le 0$ for $t\in\mathbb{R}_+$, and consequently,

$$\underline{\varphi}(t)\le K_1^{-1}(\gamma(t))=(\mathrm{T}u)(t)\le\overline{\varphi}(t)\qquad\text{for } t\in\mathbb{R}_+. \tag{8}$$

Let $0<t_1<t_2$. Then

$$\begin{aligned}K_1((\mathrm{T}u)(t_2))-K_1((\mathrm{T}u)(t_1))&=\int_{t_1}^{t_2}\biggl(\frac{1}{q(s)}\int_0^{u(s)}f(u^{-1}(\tau))h(\tau)\,d\tau\biggr)^{1/a}ds\\&\ge\int_{t_1}^{t_2}\biggl(\frac{f(s)}{q(s)}H(u(s))\biggr)^{1/a}ds\\&\ge(H(\underline{\varphi}(t_1))f(t_2))^{1/a}\int_{t_1}^{t_2}\biggl(\frac{1}{q(s)}\biggr)^{1/a}ds\end{aligned}$$

and

$$\begin{aligned}K_1((\mathrm{T}u)(t_2)) - K_1((\mathrm{T}u)(t_1)) \\ &= k_1(\xi)[(\mathrm{T}u)(t_2) - (\mathrm{T}u)(t_1)] \\ &\le \max\{k_1(u) : \underline{\varphi}(t_1) \le u \le \overline{\varphi}(t_2)\}[(\mathrm{T}u)(t_2) - (\mathrm{T}u)(t_1)]\end{aligned}$$

by the Taylor formula (here $\xi \in ((\mathrm{T}u)(t_1), (\mathrm{T}u)(t_2)) \subset (\underline{\varphi}(t_1), \overline{\varphi}(t_2)))$. Hence (with $A = [\max\{k_1(u) : \underline{\varphi}(t_1) \le u \le \overline{\varphi}(t_2)\}]^{-1}$)

$$(9)\qquad \begin{aligned}(\mathrm{T}u)(t_2) - (\mathrm{T}u)(t_1) &\ge A[K_1((\mathrm{T}u)(t_2)) - K_1((\mathrm{T}u)(t_1))] \\ &\ge A(H(\underline{\varphi}(t_1))f(t_2))^{1/a} \int_{t_1}^{t_2} \left(\frac{1}{q(s)}\right)^{1/a} ds.\end{aligned}$$

From (8) and (9) it follows that $\mathrm{T}u \in \mathcal{K}$ for each $u \in \mathcal{K}$, and consequently, $\mathrm{T} : \mathcal{K} \to \mathcal{K}$. ∎

## 3. Existence theorem

THEOREM 1. *Let assumptions* ($\mathrm{H}_1$)–($\mathrm{H}_5$) *be satisfied. Then there exists a solution of* (1).

Proof. By Lemma 2 and Corollary 1, $u \in \mathcal{A}$ is a solution of (1) if and only if $u$ is a solution of (4). Therefore in order to prove Theorem 1 it is enough to show that the operator T has a fixed point.

Let $\mathbf{X}$ be the Fréchet space of $C^0$-functions on $\mathbb{R}_+$ with the topology of uniform convergence on compact subintervals of $\mathbb{R}_+$. Then $\mathcal{K}$ is a bounded closed convex subset of $\mathbf{X}$ and $\mathrm{T} : \mathcal{K} \to \mathcal{K}$ (by Lemma 4). Let $\{u_n\} \subset \mathcal{K}$ be a convergent sequence, $\lim_{n\to\infty} u_n = u$ ($\in \mathcal{K}$). Then $\lim_{n\to\infty} u_n^{-1} = u^{-1}$ ($u_n^{-1}$ and $u^{-1}$ denote the inverse functions to $u_n$ and $u$, respectively) and consequently, $\lim_{n\to\infty} \mathrm{T}u_n = \mathrm{T}u$. This proves that T is a continuous operator.

It follows from the inequalities ($0 \le t_1 < t_2 \le t_3$, $u \in \mathcal{K}$)

$$\begin{aligned}(0 \le)\ K_1((\mathrm{T}u)(t_2)) - K_1((\mathrm{T}u)(t_1)) \\ &= \int_{t_1}^{t_2} \left(\frac{1}{q(s)} \int_0^{u(s)} f(u^{-1}(\tau))h(\tau)\, d\tau\right)^{1/a} ds \\ &\le \int_{t_1}^{t_2} \left(\frac{f(0)}{q(s)} H(u(s))\right)^{1/a} ds \\ &\le (f(0)H(\overline{\varphi}(t_3)))^{1/a} \int_{t_1}^{t_2} \left(\frac{1}{q(s)}\right)^{1/a} ds\end{aligned}$$

and from the Arzelà–Ascoli theorem that $\mathrm{T}(\mathcal{K})$ is a relatively compact subset of $\mathcal{K}$. By the Tikhonov–Schauder fixed point theorem, there exists a fixed point of T. Hence Theorem 1 is proved. ∎

Theorem 2. *Let assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_5)$ *be satisfied. If there exist two different solutions* $u(t)$ *and* $v(t)$ *of* (1) *then*

$$u(t) \neq v(t) \quad \text{for } t \in (0, \infty).$$

Proof. Assume $u, v$ are different solutions of (1). Assume there exists a $t_1 > 0$ such that $u(t_1) = v(t_1)$ and $u(t) \neq v(t)$ on $(0, t_1)$, say $u(t) < v(t)$ for $t \in (0, t_1)$. Then

$$\begin{aligned} 0 = v(t_1) - u(t_1) &= K_1((\mathrm{T}v)(t_1)) - K_1((\mathrm{T}u)(t_1)) \\ &= \int_0^{t_1} \left( \frac{1}{q(s)} \int_0^{v(s)} f(v^{-1}(\tau))h(\tau)\,d\tau \right)^{1/a} ds \\ &\quad - \int_0^{t_1} \left( \frac{1}{q(s)} \int_0^{u(s)} f(u^{-1}(\tau))h(\tau)\,d\tau \right)^{1/a} ds, \end{aligned}$$

which contradicts

$$\int_0^{t_1} \left( \frac{1}{q(s)} \int_0^{v(s)} f(v^{-1}(\tau))h(\tau)\,d\tau \right)^{1/a} ds > \int_0^{t_1} \left( \frac{1}{q(s)} \int_0^{u(s)} f(u^{-1}(\tau))h(\tau)\,d\tau \right)^{1/a} ds.$$

Let $0 < t_1 < t_2$ be such that $u(t_1) = v(t_1)$, $u(t_2) = v(t_2)$, $u(t) \neq v(t)$ on $(t_1, t_2)$, say $u(t) > v(t)$ for $t \in (t_1, t_2)$. Then $u'(t_1) \geq v'(t_1)$, $u'(t_2) \leq v'(t_2)$ and

$$\begin{aligned} 0 &\leq q(t_1)k(u(t_1))((u'(t_1))^a - (v'(t_1))^a) \\ &\quad - q(t_2)k(u(t_2))((u'(t_2))^a - (v'(t_2))^a) \\ &= \int_{t_2}^{t_1} f(s)h(u(s))u'(s)\,ds - \int_{t_2}^{t_1} f(s)h(v(s))v'(s)\,ds \\ &= \int_{u(t_2)}^{u(t_1)} [f(u^{-1}(s)) - f(v^{-1}(s))]h(s)\,ds. \end{aligned} \tag{10}$$

On the other hand, since $u(t_2) > u(t_1)$ and $f(u^{-1}(t)) - f(v^{-1}(t)) \geq 0$ on $[u(t_1), u(t_2)]$,

$$\int_{u(t_2)}^{u(t_1)} [f(u^{-1}(s)) - f(v^{-1}(s))]h(s)\,ds \leq 0.$$

Thus by (10), $u'(t_1) = v'(t_1)$, $u'(t_2) = v'(t_2)$ and $f(u^{-1}(t)) = f(v^{-1}(t))$ for $t \in [u(t_1), u(t_2)]$. Since

$$q(t)((K_1(u(t)))')^a - q(t_1)k(u(t_1))(u'(t_1))^a = \int\limits_{u(t_1)}^{u(t)} f(u^{-1}(s))h(s)\,ds,$$

$$q(t)((K_1(v(t)))')^a - q(t_1)k(v(t_1))(v'(t_1))^a = \int\limits_{u(t_1)}^{v(t)} f(v^{-1}(s))h(s)\,ds$$

on $(0,\infty)$, $q(t_1)k(u(t_1))(u'(t_1))^a = q(t_1)k(v(t_1))(v'(t_1))^a$, $0 < f(u^{-1}(s)) = f(v^{-1}(s))$ for $s \in [u(t_1), u(t_2)]$ and $u(t) > v(t)$ on $(t_1, t_2)$, we obtain

$$((K_1(u(t)))')^a - ((K_1(v(t)))')^a = \frac{1}{q(t)} \int\limits_{v(t)}^{u(t)} f(u^{-1}(s))h(s)\,ds > 0, \quad t \in (t_1, t_2).$$

Thus

(11) $$(K_1(u(t)))' > (K_1(v(t)))' \quad \text{for } t \in (t_1, t_2),$$

and consequently, $K_1(u(t_2)) - K_1(u(t_1)) > K_1(v(t_2)) - K_1(v(t_1))$, which contradicts $u(t_1) = v(t_1)$, $u(t_2) = v(t_2)$. So either $u(t) \neq v(t)$ on $(0,\infty)$ or there exists a $t_0 \in (0,\infty)$ such that $u(t) = v(t)$ for $t \in [0, t_0]$ and $u(t) \neq v(t)$ on $(t_0,\infty)$, say for example $u(t) > v(t)$ for $t \in (t_0,\infty)$. Assume that the second case occurs. Then, by the Bonnet mean value theorem, there exists a $\xi \in [t_0, t]$ such that

(12) $$\begin{aligned} ((K_1(u(t)))')^a &- ((K_1(v(t)))')^a \\ &= \frac{1}{q(t)} \int\limits_{t_0}^{t} f(s)[h(u(s))u'(s) - h(v(s))v'(s)]\,ds \\ &= \frac{1}{q(t)} \Big[ f(t_0) \int\limits_{t_0}^{\xi} (h(u(s))u'(s) - h(v(s))v'(s))\,ds \\ &\quad + f(t) \int\limits_{\xi}^{t} (h(u(s))u'(s) - h(v(s))v'(s))\,ds \Big] \\ &= \frac{1}{q(t)} [(f(t_0) - f(t))(H(u(\xi)) - H(v(\xi))) \\ &\quad + f(t)(H(u(t)) - H(v(t)))], \quad t \ge t_0. \end{aligned}$$

Set

$$\begin{aligned} M &= a \min\{q(t) : t_0 \le t \le t_0+1\} \cdot \min\{(k_1(z))^{a-1} : u(t_0) \le z \le u(t_0+1)\} \\ &\quad \times \min\{\min\{(u'(t))^{a-1}, (v'(t))^{a-1}\} : t_0 \le t \le t_0+1\} \ (>0), \end{aligned}$$

$$M_1 = \min\{k_1(z) : u(t_0) \le z \le u(t_0+1)\} \ (>0),$$

$$L = \max\{h(z) : u(t_0) \le z \le u(t_0+1)\} \ (>0),$$
$$V(t) = \max\{u(s) - v(s) : t_0 \le s \le t\} \quad \text{for } t \in [t_0, t_0+1].$$

Obviously, $V(t_0) = 0$ and $V$ is continuous nondecreasing on $[t_0, t_0+1]$.

By the Taylor formula, there exists a $B$ $(= B(t))$ in the interval with end points $(K_1(u(t)))'$ and $(K_1(v(t)))'$ such that

$$((K_1(u(t)))')^a - ((K_1(v(t)))')^a = aB^{a-1}(K_1(u(t)) - K_1(v(t)))', \quad t \in [t_0, t_0+1],$$

and therefore (cf. (12))

$$\begin{aligned}(K_1(u(t)) - K_1(v(t)))' &\le \frac{1}{M}[(f(t_0) - f(t))(H(u(\xi)) - H(v(\xi))) \\ &\quad + f(t)(H(u(t)) - H(v(t)))] \\ &\le \frac{f(t_0)}{M}[(H(u(\xi)) - H(v(\xi))) + (H(u(t)) - H(v(t)))] \\ &\le \frac{2}{M} L f(t_0) V(t), \quad t \in [t_0, t_0+1].\end{aligned}$$

Then

$$K_1(u(t)) - K_1(v(t)) \le \frac{2}{M} L f(t_0) \int_{t_0}^{t} V(s)\,ds,$$

and consequently,

$$u(t) - v(t) \le \frac{2Lf(t_0)}{Mk_1(\varepsilon)} \int_{t_0}^{t} V(s)\,ds \le \frac{2Lf(t_0)}{MM_1} \int_{t_0}^{t} V(s)\,ds, \quad t \in [t_0, t_0+1],$$

where $\varepsilon \in [v(t), u(t)]$ by the Taylor formula. Hence

$$\begin{aligned}V(t) &\le \frac{2Lf(t_0)}{MM_1} \int_{t_0}^{t} V(s)\,ds \le \frac{2Lf(t_0)}{MM_1} V(t) \int_{t_0}^{t} ds \\ &= \frac{2Lf(t_0)}{MM_1} V(t)(t - t_0), \quad t \in [t_0, t_0+1].\end{aligned} \tag{13}$$

Since $V(t) > 0$ for $t \in (t_0, t_0+1]$, we obtain (cf. (13))

$$1 \le \frac{2Lf(t_0)}{MM_1}(t - t_0) \quad \text{for } t \in (t_0, t_0+1],$$

a contradiction. ∎

THEOREM 3. *Let assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_5)$ *be satisfied. Then there exist solutions* $\underline{u}(t)$ *and* $\overline{u}(t)$ *of* (1) *such that*

$$\underline{\varphi}(t) \le \underline{u}(t) \le u(t) \le \overline{u}(t) \le \overline{\varphi}(t), \qquad t \in \mathbb{R}_+,$$

*for any solution* $u(t)$ *of* (1).

Proof. Denote by $\mathcal{B}$ the set of all solutions of (1). By Theorem 1, $\mathcal{B}$ is a nonempty set. If $\mathcal{B}$ is a finite set, then Theorem 3 follows from Theorem 2. Assume $\mathcal{B}$ is an infinite set. Set

$$\underline{u}(t) = \inf\{u(t) : u \in \mathcal{B}\}, \quad \overline{u}(t) = \sup\{u(t) : u \in \mathcal{B}\} \quad \text{for } t \in \mathbb{R}_+.$$

Then $\underline{\varphi}(t) \le \underline{u}(t) \le \overline{u}(t) \le \overline{\varphi}(t)$ on $\mathbb{R}_+$ and to prove Theorem 3 it is enough to show that $\underline{u}$ and $\overline{u}$ are solutions of (1). By Theorem 2, there exists a sequence $\{u_n\} \subset \mathcal{B}$, $u_1(t) < \ldots < u_n(t) < \ldots < \overline{u}(t)$, $t \in (0,\infty)$, such that $\overline{u}(t) = \lim_{n\to\infty} u_n(t)$ for $t \in \mathbb{R}_+$. Now we prove that $\lim_{n\to\infty} u_n'(t) =: b(t)$ exists for all $t \in (0,\infty)$ and $b = \overline{u}'$. Evidently,

$$\begin{aligned}
&(K_1(u_{n+1}(t)))' - (K_1(u_n(t)))' \\
&= \left(\frac{1}{q(t)} \int_0^{u_{n+1}(t)} f(u_{n+1}^{-1}(s))h(s)\,ds\right)^{1/a} - \left(\frac{1}{q(t)} \int_0^{u_n(t)} f(u_n^{-1}(s))h(s)\,ds\right)^{1/a} \\
&> \left(\frac{1}{q(t)} \int_0^{u_n(t)} f(u_n^{-1}(s))h(s)\,ds\right)^{1/a} - \left(\frac{1}{q(t)} \int_0^{u_n(t)} f(u_n^{-1}(s))h(s)\,ds\right)^{1/a} = 0
\end{aligned}$$

for $t \in (0,\infty)$ and $n \in \mathbb{N}$. Therefore the sequence $\{k_1(u_n(t))u_n'(t)\}$ is strictly increasing for each $t \in (0,\infty)$. Setting $\alpha(t) = \lim_{n\to\infty} k_1(u_n(t))u_n'(t)$, $t \in (0,\infty)$, we see that

$$\lim_{n\to\infty} u_n'(t) = \lim_{n\to\infty} \frac{k_1(u_n(t))u_n'(t)}{k_1(u_n(t))} = \frac{\alpha(t)}{k_1(\overline{u}(t))} =: \beta(t), \qquad t \in (0,\infty),$$

and using the Lebesgue dominated convergence theorem in the equalities

$$u_n(t) = \int_0^t u_n'(s)\,ds, \qquad t \in \mathbb{R}_+,\ n \in \mathbb{N},$$

we get $\overline{u}(t) = \int_0^t \beta(s)\,ds$ on $\mathbb{R}_+$; hence $\beta(t) = \overline{u}'(t)$ for $t \in (0,\infty)$. Applying again the Lebesgue theorem to the equalities

$$k_1(u_n(t))u_n'(t) = \left(\frac{1}{q(t)} \int_0^t f(s)h(u_n(s))u_n'(s)\,ds\right)^{1/a}, \qquad t \in (0,\infty),\ n \in \mathbb{N},$$

we obtain

$$k_1(\overline{u}(t))\overline{u}'(t) = \left(\frac{1}{q(t)} \int_0^t f(s)h(\overline{u}(s))\overline{u}'(s)\,ds\right)^{1/a}, \qquad t \in (0,\infty),$$

and consequently, $\overline{u}$ is a solution of (1). Analogously we can prove that $\underline{u}$ is a solution of (1). ■

## 4. Bounded and unbounded solutions

THEOREM 4. *Let assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_5)$ *be satisfied. Then*

(i) *some* (*and then any*) *solution of* (1) *is bounded if and only if*

$$\int_0^\infty \left(\frac{1}{q(t)}\right)^{1/a} dt < \infty,$$

(ii) *some* (*and then any*) *solution of* (1) *is unbounded if and only if*

$$\int_0^\infty \left(\frac{1}{q(t)}\right)^{1/a} dt = \infty.$$

Proof. First note that either $\int_0^\infty (1/q(t))^{1/a}\, dt < \infty$ or $\int_0^\infty (1/q(t))^{1/a}\, dt = \infty$. In the first case, by Lemma 3, any solution $u$ of (1) is bounded. Now assume $\int_0^\infty (1/q(t))^{1/a}\, dt = \infty$ and $u$ is a solution of (1). Then

$$\lim_{t\to\infty} \frac{\int_0^t \left(\frac{1}{q(s)} \int_0^{u(s)} f(u^{-1}(\tau))h(\tau)\, d\tau\right)^{1/a} ds}{\int_0^t \left(\frac{1}{q(s)}\right)^{1/a} ds}$$
$$= \lim_{t\to\infty} \left(\int_0^{u(t)} f(u^{-1}(s))h(s)\, ds\right)^{1/a}$$
$$= \lim_{t\to\infty} \left(\int_0^{t} f(s)h(u(s))u'(s)\, ds\right)^{1/a} > 0,$$

and consequently,

$$\lim_{t\to\infty} K_1(u(t)) = \lim_{t\to\infty} \int_0^t \left(\frac{1}{q(s)} \int_0^{u(s)} f(u^{-1}(\tau))h(\tau)\, d\tau\right)^{1/a} ds = \infty.$$

Hence $\lim_{t\to\infty} u(t) = \infty$ and $u$ is unbounded.

Let $u$ be a solution of (1). If $u$ is bounded, then $\int_0^\infty (1/q(t))^{1/a}\, dt < \infty$ since in the opposite case $u$ is unbounded by the first part of the proof. Analogously, $u$ unbounded implies $\int_0^\infty (1/q(t))^{1/a}\, dt = \infty$. ■

## 5. Uniqueness theorem

THEOREM 5. *Let assumptions* ($H_1$)–($H_5$) *be satisfied. Moreover, assume that*

($H_6$) *There exist positive numbers* $\varepsilon$ *and* $L$ *such that*

(i) $|f(t_1) - f(t_2)| \le L|t_1 - t_2|$ *for all* $t_1, t_2 \in [0, \varepsilon]$,

(ii) *the modulus of continuity* $\gamma(t)$ $(= \sup\{|(q(t_1))^{1/a} - (q(t_2))^{1/a}| : t_1, t_2 \in [0,\varepsilon],\ |t_1 - t_2| \le t\})$ *of* $(q(t))^{1/a}$ *on* $[0,\varepsilon]$ *satisfies*

$$\limsup_{t\to 0_+} \frac{\gamma(t)}{t} < \infty.$$

*Then* (1) *admits a unique solution.*

P r o o f. By Theorem 1, there exists at least one solution of (1). Let $u_1$, $u_2$ be different solutions of (1), say $u_1(t) < u_2(t)$ on $(0,\infty)$ (see Theorem 2). According to the last part of the proof of Theorem 2 it is enough to show that $u_1(t) = u_2(t)$ on $[0, t_0]$ for a positive number $t_0$. Setting $A_i = \lim_{t\to\infty} u_i(t)$ and $w_i = u_i^{-1}$ $(i = 1, 2)$, we see that $0 < A_1 \le A_2 \le \infty$, $w_i : [0, A_i) \to \mathbb{R}_+$ are continuous strictly increasing functions and

$$w_i(t) = \int_0^t k_1(s)\Big(\frac{1}{q(w_i(s))}\int_0^s f(w_i(\tau))h(\tau)\,d\tau\Big)^{-1/a} ds, \qquad t \in [0, A_i),\ i = 1, 2.$$

Then (for $t \in [0, A_1)$)

$$\begin{aligned}(14)\qquad &(0 \le)\ w_1(t) - w_2(t)\\ &= \int_0^t k_1(s)[(q(w_1(s)))^{1/a} - (q(w_2(s)))^{1/a}]\Big(\int_0^s f(w_2(\tau))h(\tau)\,d\tau\Big)^{-1/a} ds\\ &\quad + \int_0^t \frac{k_1(s)(q(w_1(s)))^{1/a}}{(\int_0^s f(w_1(\tau))h(\tau)\,d\tau \int_0^s f(w_2(\tau))h(\tau)\,d\tau)^{1/a}}\\ &\quad \times \Big[\Big(\int_0^s f(w_2(\tau))h(\tau)\,d\tau\Big)^{1/a} - \Big(\int_0^s f(w_1(\tau))h(\tau)\,d\tau\Big)^{1/a}\Big]\, ds.\end{aligned}$$

Let $\varepsilon > 0$ be as in assumption ($H_6$) and set $b = \min\{u_1(\varepsilon), \varepsilon\}$, $A = \max\{(q(t))^{1/a} : 0 \le t \le \varepsilon\}$ and $X(t) = \max\{w_1(s) - w_2(s) : 0 \le s \le t\}$ for $t \in (0, b]$. Then $X$ is continuous nondecreasing, $X(0) = 0$, $X(t) > 0$ for $t \in (0, b]$ and (cf. ($H_6$))

$$|(q(w_1(t)))^{1/a} - (q(w_2(t)))^{1/a}| \le \gamma(X(t)) \quad \text{for } t \in [0, b].$$

1. Let $a = 1$. Then (cf. (14))

$$w_1(t) - w_2(t) \le \frac{1}{f(\varepsilon)} \int_0^t k(s)\gamma(X(s))(H(s))^{-1} ds + \frac{L}{(f(\varepsilon))^2} \int_0^t \frac{k(s)q(w_1(s))}{(H(s))^2} \int_0^s h(\tau)(w_1(\tau) - w_2(\tau))\, d\tau\, ds \le \frac{1}{f(\varepsilon)}\gamma(X(t))P(t) + \frac{LA}{(f(\varepsilon))^2} X(t)P(t), \quad t \in [0, b].$$

Hence

$$X(t) \le \frac{1}{f(\varepsilon)}\gamma(X(t))P(t) + \frac{LA}{(f(\varepsilon))^2} X(t)P(t), \quad t \in [0, b],$$

and

$$1 \le \frac{\gamma(X(t))}{f(\varepsilon)X(t)} P(t) + \frac{LA}{(f(\varepsilon))^2} P(t), \quad t \in (0, b]. \tag{15}$$

Since

$$\limsup_{t\to 0_+} \frac{\gamma(X(t))}{X(t)} = \limsup_{t\to 0_+} \frac{\gamma(t)}{t} < \infty \quad \text{(by (H}_6\text{))}$$

and $\lim_{t\to 0_+} P(t) = 0$, we get

$$\lim_{t\to 0_+} \left[ \frac{\gamma(X(t))}{f(\varepsilon)X(t)} P(t) + \frac{LA}{(f(\varepsilon))^2} P(t) \right] = 0,$$

which contradicts (15).

2. Let $a > 1$. Then there is a positive integer $n$ such that $(n + 1)/a > 1$ and

$$\Big(\int_0^t f(w_2(s))h(s)\, ds\Big)^{(n+1)/a} - \Big(\int_0^t f(w_1(s))h(s)\, ds\Big)^{(n+1)/a} = \Big[\Big(\int_0^t f(w_2(s))h(s)\, ds\Big)^{1/a} - \Big(\int_0^t f(w_1(s))h(s)\, ds\Big)^{1/a}\Big] \times \sum_{k=0}^{n} \Big(\int_0^t f(w_2(s))h(s)\, ds\Big)^{k/a} \Big(\int_0^t f(w_1(s))h(s)\, ds\Big)^{(n-k)/a}. \tag{16}$$

By the Taylor formula,

$$\Big(\int_0^t f(w_2(s))h(s)\, ds\Big)^{(n+1)/a} - \Big(\int_0^t f(w_1(s))h(s)\, ds\Big)^{(n+1)/a} = \frac{n+1}{a}\xi^{(n+1)/a-1} \int_0^t (f(w_2(s)) - f(w_1(s)))h(s)\, ds,$$

where $\xi = \xi(t)$ lies in the interval with end points $\int_0^t f(w_1(s))h(s)\,ds$, $\int_0^t f(w_2(s))h(s)\,ds$, and thus (cf. (14) and (16))

$$\begin{aligned}
&w_1(t) - w_2(t)\\
&\quad\le \int_0^t k_1(s)\gamma(X(s))(f(\varepsilon)H(s))^{-1/a}\,ds\\
&\qquad+ \int_0^t \frac{k_1(s)(q(w_1(s)))^{1/a}}{(\int_0^s f(w_1(\tau))h(\tau)\,d\tau \int_0^s f(w_2(\tau))h(\tau)\,d\tau)^{1/a}}\\
&\qquad\times \frac{[(\int_0^s f(w_2(\tau))h(\tau)\,d\tau)^{(n+1)/a} - (\int_0^s f(w_1(\tau))h(\tau)\,d\tau)^{(n+1)/a}]}{\sum_{k=0}^n (\int_0^s f(w_2(\tau))h(\tau)\,d\tau)^{k/a}(\int_0^s f(w_1(\tau))h(\tau)\,d\tau)^{(n-k)/a}}\,ds\\
&\quad\le \gamma(X(t))P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a} + \frac{n+1}{a}A\left(\frac{1}{f(\varepsilon)}\right)^{(n+2)/a}\\
&\qquad\times \int_0^t \frac{k_1(s)\xi^{(n+1)/a-1}\int_0^s (f(w_2(\tau)) - f(w_1(\tau)))h(\tau)\,d\tau}{(n+1)(H(s))^{2/a}(H(s))^{n/a}}\,ds\\
&\quad\le \gamma(X(t))P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a} + \frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{(n+2)/a} L(f(0))^{(n+1)/a-1}\\
&\qquad\times \int_0^t \frac{k_1(s)(H(s))^{(n+1)/a}X(s)}{(H(s))^{(n+2)/a}}\,ds \le \gamma(X(t))P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}\\
&\qquad+ \frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{(n+2)/a}(f(0))^{(n+1)/a-1}LX(t)P(t)
\end{aligned}$$

for $t \in [0,b]$ since $|\xi(t)| \le f(0)H(t)$ on $[0,b]$. Then

$$\begin{aligned}
X(t) \le \gamma(X(t))P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}&\\
+ \frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{(n+2)/a}&(f(0))^{(n+1)/a-1}LX(t)P(t),
\end{aligned}$$

hence

$$\begin{aligned}
1 \le \frac{\gamma(X(t))}{X(t)}P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}&\\
+ \frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{(n+2)/a}&(f(0))^{(n+1)/a-1}LP(t)
\end{aligned} \tag{17}$$

for $t \in (0,b]$, and since $\limsup_{t\to 0_+} \gamma(X(t))/X(t) < \infty$ and $\lim_{t\to 0_+} P(t) = 0$, we get

$$\lim_{t\to 0_+}\left[\frac{\gamma(X(t))}{X(t)}P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}+\frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{(n+2)/a}(f(0))^{(n+1)/a-1}LP(t)\right]=0,$$

which contradicts (17).

3. Let $a<1$. By the Taylor formula,

$$\Big(\int_0^t f(w_2(s))h(s)\,ds\Big)^{1/a}-\Big(\int_0^t f(w_1(s))h(s)\,ds\Big)^{1/a}$$
$$=\frac{\nu^{1/a-1}}{a}\Big(\int_0^t f(w_2(s))h(s)\,ds-\int_0^t f(w_1(s))h(s)\,ds\Big),$$

where $\nu=\nu(t)$ lies in the interval with end points $\int_0^t f(w_2(s))h(s)\,ds$ and $\int_0^t f(w_1(s))h(s)\,ds$, and using (14) we obtain

$$\begin{aligned}
&w_1(t)-w_2(t)\\
&\quad=\gamma(X(t))P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}+\frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{2/a}(f(0))^{1/a-1}\\
&\qquad\times\int_0^t\frac{k_1(s)(H(s))^{1/a-1}}{(H(s))^{2/a}}\int_0^s(f(w_2(\tau))-f(w_1(\tau)))h(\tau)\,d\tau\,ds\\
&\quad\le\gamma(X(t))P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}+\frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{2/a}(f(0))^{1/a-1}L\int_0^t\frac{k_1(s)X(s)}{(H(s))^{1/a}}\,ds\\
&\quad\le\gamma(X(t))P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}+\frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{2/a}(f(0))^{1/a-1}LX(t)P(t)
\end{aligned}$$

for $t\in[0,b]$ since $|\nu(t)|\le f(0)H(t)$ on $[0,b]$. Then

$$X(t)\le\gamma(X(t))P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}+\frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{2/a}(f(0))^{1/a-1}LX(t)P(t),\qquad t\in[0,b],$$

and hence

$$1\le\frac{\gamma(X(t))}{X(t)}P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}+\frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{2/a}(f(0))^{1/a-1}LP(t),\qquad t\in(0,b],$$

which contradicts

$$\lim_{t\to 0_+}\left[\frac{\gamma(X(t))}{X(t)}P(t)\left(\frac{1}{f(\varepsilon)}\right)^{1/a}+\frac{A}{a}\left(\frac{1}{f(\varepsilon)}\right)^{2/a}(f(0))^{1/a-1}LP(t)\right]=0. \quad\blacksquare$$

**6. Dependence of solutions on a parameter.** Consider the differential equation

$$(q(t)k(u)(u')^a)' = \lambda f(t)h(u)u', \qquad \lambda > 0, \tag{$18_\lambda$}$$

depending on the positive parameter $\lambda$ with $q, k, f$ and $h$ satisfying assumptions $(H_1)$–$(H_5)$. Set

$$\underline{\varphi}(t,\lambda) = P^{-1}\Big(\int_0^t \Big(\lambda \frac{f(s)}{q(s)}\Big)^{1/a} ds\Big),$$

$$\overline{\varphi}(t,\lambda) = P^{-1}\Big(\int_0^t \Big(\lambda \frac{f(0)}{q(s)}\Big)^{1/a} ds\Big)$$

for $(t,\lambda) \in \mathbb{R}_+ \times (0,\infty)$. Denote by $u(t,\lambda)$ a solution of $(18_\lambda)$. By Theorem 3 (with $\lambda f$ instead of $f$), there exist solutions $\underline{u}(t,\lambda)$ and $\overline{u}(t,\lambda)$ of $(18_\lambda)$ such that

$$\underline{\varphi}(t,\lambda) \le \underline{u}(t,\lambda) \le u(t,\lambda) \le \overline{u}(t,\lambda) \le \overline{\varphi}(t,\lambda), \qquad (t,\lambda) \in \mathbb{R}_+ \times (0,\infty), \tag{19}$$

for any solution $u(t,\lambda)$ of $(18_\lambda)$.

THEOREM 6. *Let assumptions* $(H_1)$–$(H_5)$ *be satisfied. Then*

$$\overline{u}(t,\lambda_1) < \underline{u}(t,\lambda_2), \qquad t \in (0,\infty),$$

*for any* $0 < \lambda_1 < \lambda_2$.

Proof. Let $0 < \lambda_1 < \lambda_2$. Since

$$\lim_{t\to 0+} \frac{\int_0^t \Big(\lambda_2 \frac{f(s)}{q(s)}\Big)^{1/a} ds}{\int_0^t \Big(\lambda_1 \frac{f(0)}{q(s)}\Big)^{1/a} ds} = \lim_{t\to 0+} \frac{(\lambda_2 f(t))^{1/a}}{(\lambda_1 f(0))^{1/a}} = (\lambda_2/\lambda_1)^{1/a} > 1,$$

there exists an $\varepsilon > 0$ such that $\underline{\varphi}(t,\lambda_2) > \overline{\varphi}(t,\lambda_1)$ for $t \in (0,\varepsilon]$, and consequently,

$$\overline{u}(t,\lambda_1) < \underline{u}(t,\lambda_2) \quad \text{for } t \in (0,\varepsilon] \tag{20}$$

by (19). Assume $\overline{u}(t,\lambda_1) < \underline{u}(t,\lambda_2)$ on $(0,t_0)$ while $\overline{u}(t_0,\lambda_1) = \underline{u}(t_0,\lambda_2)$ for a $t_0 \in (\varepsilon,\infty)$. Then

$$0 = K_1(\underline{u}(t_0,\lambda_2)) - K_1(\overline{u}(t_0,\lambda_1))$$
$$= \int_0^{t_0} \left(\frac{\lambda_2}{q(t)} \int_0^{\underline{u}(t,\lambda_2)} f(\underline{u}^{-1}(s,\lambda_2))h(s)\,ds\right)^{1/a} dt$$
$$- \int_0^{t_0} \left(\frac{\lambda_1}{q(t)} \int_0^{\bar{u}(t,\lambda_1)} f(\overline{u}^{-1}(s,\lambda_1))h(s)\,ds\right)^{1/a} dt,$$

which contradicts

$$\left(\frac{\lambda_2}{q(t)} \int_0^{\underline{u}(t,\lambda_2)} f(\underline{u}^{-1}(s,\lambda_2))h(s)\,ds\right)^{1/a} - \left(\frac{\lambda_1}{q(t)} \int_0^{\bar{u}(t,\lambda_1)} f(\overline{u}^{-1}(s,\lambda_1))h(s)\,ds\right)^{1/a}$$
$$> \left(\frac{\lambda_2}{q(t)} \int_0^{\bar{u}(t,\lambda_1)} f(\overline{u}^{-1}(s,\lambda_1))h(s)\,ds\right)^{1/a}$$
$$- \left(\frac{\lambda_1}{q(t)} \int_0^{\bar{u}(t,\lambda_1)} f(\overline{u}^{-1}(s,\lambda_1))h(s)\,ds\right)^{1/a} > 0 \quad \text{for } 0 < t \le t_0. \ \blacksquare$$

Corollary 2. *Let assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_5)$ *be satisfied. Then there exists an at most countable set* $\mathcal{R} \subset (0,\infty)$ *such that equation* $(18_\lambda)$ *has a unique solution for every* $\lambda \in (0,\infty) - \mathcal{R}$.

Proof. Let $t_0 \in (0,\infty)$ and set $g(\lambda) = \underline{u}(t_0,\lambda)$ for $\lambda \in (0,\infty)$. Then $g$ is strictly increasing on $(0,\infty)$ by Theorem 6, and

$$\lim_{\lambda\to\infty} g(\lambda) = \lim_{\lambda\to\infty} \underline{u}(t_0,\lambda)$$
$$\ge \lim_{\lambda\to\infty} \underline{\varphi}(t_0,\lambda) = \lim_{\lambda\to\infty} P^{-1}\left(\int_0^{t_0} \left(\lambda\frac{f(s)}{g(s)}\right)^{1/a} ds\right) = \infty.$$

Evidently, if $g$ is continuous at a point $\lambda = \lambda_0$ then $(18_\lambda)$ has a unique solution for $\lambda = \lambda_0$. For each $n \in \mathbb{N}$ denote by $\mathcal{R}_n$ the set of points of discontinuity of $g$ on the interval $[1/n, n]$. By Theorem 1 of [5, p. 229], the set $\mathcal{R}_n$ is at most countable. Hence $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$ is the set of points of discontinuity of $g$ and since $\mathcal{R}$ is at most countable, the proof of Corollary 2 is finished. $\blacksquare$

Theorem 7. *Let assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_6)$ *be satisfied and, moreover,* $\int_0^\infty (1/q(t))^{1/a}\,dt < \infty$. *Then for each* $c \in (0,\infty)$ *there exists a unique* $\lambda_c \in (0,\infty)$ *such that equation* $(18_\lambda)$ *for* $\lambda = \lambda_c$ *has a* (*necessarily unique*) *solution* $u(t,\lambda_c)$ *with*

$$\lim_{t\to\infty} u(t,\lambda_c) = c.$$

P r o o f. By Theorem 5, equation $(18_\lambda)$ has a unique solution $u(t,\lambda)$ for each $\lambda \in (0,\infty)$. This solution is strictly increasing (by Corollary 1) and bounded on $\mathbb{R}_+$ (by Theorem 4). Define $g(\lambda) = \lim_{t\to\infty} u(t,\lambda)$ for all $\lambda > 0$. The function $g : (0,\infty) \to (0,\infty)$ is increasing by Theorem 6. To prove our theorem it is sufficient to show that $g$ is continuous, strictly increasing and maps $(0,\infty)$ onto itself. Assume $g(\lambda_1) = g(\lambda_2)$ for some $0 < \lambda_1 < \lambda_2$. Then $u(t,\lambda_1) < u(t,\lambda_2)$ on $(0,\infty)$ and thus

$$g(\lambda_1) = \int_0^\infty \left(\frac{\lambda_1}{q(t)} \int_0^{u(t,\lambda_1)} f(u^{-1}(s,\lambda_1))h(s)\,ds\right)^{1/a} dt$$
$$< \int_0^\infty \left(\frac{\lambda_2}{q(t)} \int_0^{u(t,\lambda_2)} f(u^{-1}(s,\lambda_2))h(s)\,ds\right)^{1/a} dt = g(\lambda_2),$$

a contradiction. Assume

$$\lim_{\lambda\to\lambda_{0+}} g(\lambda) - \lim_{\lambda\to\lambda_{0-}} g(\lambda) > 0 \quad \text{for a } \lambda_0 \in (0,\infty).$$

Set

$$\alpha(t) = \lim_{\lambda\to\lambda_{0+}} u(t,\lambda), \quad \beta(t) = \lim_{\lambda\to\lambda_{0-}} u(t,\lambda) \quad \text{for } t \in \mathbb{R}_+.$$

Then

$$\liminf_{t\to\infty}(\alpha(t) - \beta(t)) > 0. \tag{21}$$

Using the Lebesgue dominated convergence theorem as $\lambda \to \lambda_{0+}$ and $\lambda \to \lambda_{0-}$ in the equality

$$u(t,\lambda) = K_1^{-1}\left(\int_0^t \left(\frac{\lambda}{q(s)} \int_0^{u(s,\lambda)} f(u^{-1}(\tau,\lambda))h(\tau)\,d\tau\right)^{1/a} ds\right),$$
$$(t,\lambda) \in \mathbb{R}_+ \times (0,\infty),$$

we see (cf. Lemma 2) that $\alpha$ and $\beta$ are solutions of $(18_{\lambda_0})$. Consequently, $\alpha(t) = \beta(t) = u(t,\lambda_0)$ for $t \in \mathbb{R}_+$, which contradicts (21). Finally,

$$\lim_{\lambda\to\infty}\lim_{t\to\infty} \underline{\varphi}(t,\lambda) = \lim_{\lambda\to\infty} P^{-1}\left(\int_0^\infty \left(\frac{\lambda f(s)}{q(s)}\right)^{1/a} ds\right) = \infty,$$

$$\lim_{\lambda\to 0_+}\lim_{t\to\infty} \overline{\varphi}(t,\lambda) = \lim_{\lambda\to 0_+} P^{-1}\left(\int_0^\infty \left(\frac{\lambda f(0)}{q(s)}\right)^{1/a} ds\right) = 0,$$

since $\lim_{t\to\infty} P^{-1}(t) = \infty$, $\lim_{t\to 0_+} P^{-1}(t) = 0$,

$$0 < \int_0^\infty \left(\frac{f(s)}{q(s)}\right)^{1/a} ds < \int_0^\infty \left(\frac{f(0)}{q(s)}\right)^{1/a} ds < \infty$$

and therefore (cf. (19)) $\lim_{\lambda\to\infty} g(\lambda) = \infty$ and $\lim_{\lambda\to 0_+} g(\lambda) = 0$. ∎

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