On positive solutions of a class of second order nonlinear differential equations on the halfline

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Abstract. The differential equation of the form $(q(t)k(u)u')' = f(t)h(u)u'$, $a \in (0, \infty)$, is considered and solutions $u$ with $u(0) = 0$ and $(u(t))^2 + (u'(t))^2 > 0$ on $(0, \infty)$ are studied. Theorems about existence, uniqueness, boundedness and dependence of solutions on a parameter are given.

1. Introduction. In [9] the differential equation $(q(t)k(u)u')' = F(t, u)u'$ was considered and the author gave sufficient conditions for the existence and uniqueness of solutions $u$ such that $u(0) = 0$ and $(u(t))^2 + (u'(t))^2 > 0$ for $t \in (0, \infty)$. This problem is connected with the description of the mathematical model of infiltration of water. For more details see e.g. [3], [4] and [6]. Naturally, a question arises of what are the properties of solutions of the differential equation $(q(t)k(u)(u')^a)' = F(t, u)u'$, where $a$ is a positive constant. For the sake of simplicity of our assumptions, results and proofs we will consider the differential equations of the type

$(1) \quad (q(t)k(u)(u')^a)' = f(t)h(u)u'$, $a \in (0, \infty)$.

We also study the qualitative dependence of solutions of (1) on the parameter $a$. As special cases we obtain results of [9] (with $F(t, u) = f(t)h(u)$ and $a = 1$), of [8] (where $a = 1, f \in C^1([0, \infty)), h(u) \equiv 1$) and of [7] (where $a = 1, q(t) \equiv 1, h(u) \equiv 1$). We observe that special cases of (1) (with $a = 1$) were also considered in [1], [2], [4] and [6].

2. Notations and lemmas. We consider equation (1) in which the functions $q, k, f$ and $h$ satisfy the following assumptions:

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(H₁) $q \in C^0(\mathbb{R}_+), q(t) > 0$ for $t > 0$ and $\int_0^1 (1/q(s))^{1/a} \, ds < \infty$;

(H₂) $k \in C^0(\mathbb{R}_+), k(0) = 0, k(u) > 0$ for $u > 0$ and $\int_0^1 (k(s))^{1/a} \, ds < \infty$;

(H₃) $f \in C^0(\mathbb{R}_+), f(t) > 0$ for $t \in \mathbb{R}_+$ and $f$ is decreasing on $\mathbb{R}_+$;

(H₄) $h \in C^0(\mathbb{R}_+), h(u) \geq 0$ for $u \in \mathbb{R}_+$ and $H(u) = \int_0^u h(s) \, ds$ is strictly increasing on $\mathbb{R}_+$;

(H₅) $\int_0^1 (k(s)/H(s))^{1/a} \, ds < \infty$, $\int_0^1 (k(s)/H(s))^{1/a} \, ds = \infty$.

We say that $u$ is a solution of (1) if $u \in C^0(\mathbb{R}_+) \cap C^1((0, \infty)), u(0) = 0$, $u(t) \geq 0$ on $\mathbb{R}_+$, $(u(t))^2 + (u'(t))^2 > 0$ for $t \in (0, \infty)$, $q(t)k(u(t))(u'(t))^a$ is continuously differentiable on $(0, \infty)$, $\lim_{t \to 0^+} q(t)k(u(t))(u'(t))^a = 0$ and (1) is satisfied on $(0, \infty)$.

Let $p \in C^0(\mathbb{R}), p(0) = 0$. We say that $u$ is a solution of the differential equation

$$ (q(t)k(u)p(u'))' = f(t)h(u)u' $$

if $u \in C^0(\mathbb{R}_+) \cap C^1((0, \infty)), u(0) = 0, u(t) \geq 0$ on $\mathbb{R}_+$, $(u(t))^2 + (u'(t))^2 > 0$ for $t \in (0, \infty)$, $q(t)k(u(t))p(u'(t))$ is continuously differentiable on $(0, \infty)$, $\lim_{t \to 0^+} q(t)k(u(t))(u'(t)) = 0$ and (2) is satisfied on $(0, \infty)$.

**Lemma 1.** Let $u(t)$ be a solution of (2). Then $u'(t) > 0$ for $t \in (0, \infty)$.

**Proof.** We see that

$$ q(t)k(u(t))p(u'(t)) = \int_0^t f(s)h(u(s))u'(s) \, ds \quad \text{for } t > 0. $$

Suppose that there exist $0 < t_1 < t_2$ such that $u'(t_1) = u'(t_2) = 0$ and $u'(t) > 0$ (resp. $u'(t) < 0$) on $(t_1, t_2)$. Then $u(t) > 0$ for $t \in [t_1, t_2]$ and (3) implies

$$ 0 = q(t_2)k(u(t_2))p(u'(t_2)) - q(t_1)k(u(t_1))p(u'(t_1)) = \int_{t_1}^{t_2} f(s)h(u(s))u'(s) \, ds, $$

which contradicts

$$ \int_{t_1}^{t_2} f(s)h(u(s))u'(s) \, ds \geq f(t_2) \int_{u(t_1)}^{u(t_2)} h(s) \, ds > 0 $$

( resp. $\int_{t_1}^{t_2} f(s)h(u(s))u'(s) \, ds \leq f(t_2) \int_{u(t_1)}^{u(t_2)} h(s) \, ds < 0$).

Assume $u'(\tau) = 0$ for a $\tau \in (0, \infty)$ and $u'(t) \neq 0$ on $(0, \tau)$. Then necessarily
Let \( u(t) > 0 \) on \((0, \tau)\) since \( u(t) \geq 0 \) for \( t \in \mathbb{R}_+ \), and (cf. (3))

\[
0 = q(\tau)k(u(\tau))p(u'(\tau)) = \int_0^\tau f(s)h(u(s))u'(s) \, ds,
\]
which contradicts

\[
\int_0^\tau f(s)h(u(s))u'(s) \, ds \geq f(\tau) \int_0^\tau h(s) \, ds > 0.
\]

Therefore by virtue of \((u(t))' + (u'(t))^2 > 0\) on \((0, \infty)\) we conclude \( u'(t) > 0 \) for \( t \in (0, \infty) \).

**Corollary 1.** Let \( u(t) \) be a solution of \((1)\). Then \( u'(t) > 0 \) for \( t \in (0, \infty) \).

**Proof.** If \( a = m/n \), where \( m, n \in \mathbb{N} \) and \( n \) is odd, then the function \( v^a \) is defined for all \( v \in \mathbb{R} \) and Corollary 1 follows from Lemma 1. Assume \( a = m/n \), where \( m, n \in \mathbb{N} \) and \( n \) is even or \( a \) is an irrational number. Then the function \( v^a \) is defined for all \( v \in \mathbb{R}_+ \), and for every \( p_1 \in C^0((-\infty, 0]) \) with \( p_1(0) = 0 \), the function \( p : \mathbb{R} \to \mathbb{R} \) defined by \( p(v) = v^a \) for \( v \in \mathbb{R}_+ \) and \( p(v) = p_1(v) \) for \( v \in (-\infty, 0) \) is continuous on \( \mathbb{R} \), \( p(0) = 0 \) and, moreover, \( u(t) \) is a solution of \((2)\). Hence \( u'(t) > 0 \) on \((0, \infty)\) by Lemma 1.

**Remark 1.** It follows from Corollary 1 that \( u \in \mathcal{A} \) for any solution \( u \) of \((1)\), where

\[
\mathcal{A} = \{ u \in C^0(\mathbb{R}_+) : u(0) = 0, \ u \text{ is strictly increasing on } \mathbb{R}_+ \}.
\]

Set

\[
k_1(u) = (k(u))^{1/a}, \quad K_1(u) = \int_0^u k_1(s) \, ds, \quad P(u) = \int_0^u \left( \frac{k(s)}{H(s)} \right)^{1/a} \, ds
\]

for \( u \in \mathbb{R}_+ \). Obviously, \( k_1 \in C^0(\mathbb{R}_+) \), \( K_1 \in C^1(\mathbb{R}_+) \), \( P \in C^0(\mathbb{R}_+) \cap C^1((0, \infty)) \), \( K_1 \) and \( P \) are strictly increasing on \( \mathbb{R}_+ \), \( \lim_{u \to \infty} K_1(u) = \infty \) by (H2) and \( \lim_{u \to \infty} P(u) = \infty \) by (H5).

**Lemma 2.** If \( u(t) \) is a solution of \((1)\), then

\[
(4) \quad u(t) = K_1^{-1}'\left( \int_0^t \left( \frac{1}{q(s)} \int_0^{u(s)} f(u^{-1}(\tau))h(\tau) \, d\tau \right)^{1/a} \, ds \right), \quad t \in \mathbb{R}_+,
\]

where \( K_1^{-1} \) and \( u^{-1} \) denote the inverse functions to \( K_1 \) and \( u \), respectively. Conversely, if \( u \in \mathcal{A} \) is a solution of \((4)\), then \( u(t) \) is a solution of \((1)\).

**Proof.** Let \( u \) be a solution of \((1)\). Then \( u \in \mathcal{A} \) (cf. Remark 1) and

\[
(k_1(u(t)))' = \frac{1}{q(t)} \int_0^u f(s)h(u(s))u'(s) \, ds, \quad t > 0.
\]
Hence

\[(5) \quad (K_1(u(t)))' = \left( \frac{1}{q(t)} \int_0^{u(t)} f(u^{-1}(s)) h(s) \, ds \right)^{1/a}, \quad t > 0,\]

and integrating (5) from 0 to \( t \), we obtain

\[K_1(u(t)) = \int_0^t \left( \frac{1}{q(s)} \int_0^{u(s)} f(u^{-1}(\tau)) h(\tau) \, d\tau \right)^{1/a} \, ds, \quad t \in \mathbb{R}_+,\]

and consequently, equality (4) is satisfied.

Conversely, let \( u \in \mathcal{A} \) be a solution of (4). Then \( u \in C^1((0, \infty)) \),

\[\lim_{t \to 0^+} q(t)k(u(t))(u'(t))^a = \lim_{t \to 0^+} \int_0^{u(t)} f(u^{-1}(s)) h(s) \, ds = 0\]

and \((q(t)k(u(t))(u'(t))^a)' = f(t)h(u(t))u'(t)\) for \( t \in (0, \infty) \). Hence \( u \) is a solution of (1).

Define \( \varphi, \overline{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[\varphi(t) = P^{-1} \left( \int_0^t \left( \frac{f(s)}{q(s)} \right)^{1/a} \, ds \right), \quad \overline{\varphi}(t) = P^{-1} \left( \int_0^t \left( \frac{f(0)}{q(s)} \right)^{1/a} \, ds \right),\]

where \( P^{-1} : \mathbb{R}_+ \to \mathbb{R}_+ \) denotes the inverse function to \( P \). Obviously, \( \varphi(t) \leq \overline{\varphi}(t) \) on \( \mathbb{R}_+ \) by \( (H_3) \).

**Lemma 3.** Let \( u(t) \) be a solution of (1). Then

\[(6) \quad \varphi(t) \leq u(t) \leq \overline{\varphi}(t) \quad \text{for} \ t \in \mathbb{R}_+.\]

**Proof.** Since

\[f(t)H(u(t)) = f(t) \int_0^{u(t)} h(s) \, ds \leq \int_0^t f(s)h(u(s))u'(s) \, ds \leq f(0) \int_0^{u(t)} h(s) \, ds = f(0)H(u(t)),\]

we have

\[f(t)H(u(t)) \leq q(t)(k_1(u(t))u'(t))^a \leq f(0)H(u(t)), \quad t \in (0, \infty).\]

Thus

\[\left( \frac{f(t)}{q(t)}H(u(t)) \right)^{1/a} \leq k_1(u(t))u'(t) \leq \left( \frac{f(0)}{q(t)}H(u(t)) \right)^{1/a}\]

and
\[ (f(t)/q(t))^{1/a} \leq \left( \frac{k(u(t))}{H(u(t))} \right)^{1/a} u'(t) = (P(u(t)))' \leq \left( \frac{f(0)}{q(t)} \right)^{1/a}, \quad t \in (0, \infty). \]

Integrating (7) from 0 to \( t \) we obtain
\[ \int_0^t \left( \frac{f(s)}{q(s)} \right)^{1/a} ds \leq P(u(t)) \leq \int_0^t \left( \frac{f(0)}{q(s)} \right)^{1/a} ds, \quad t \in \mathbb{R}_+, \]
and (6) holds. \( \blacksquare \)

Set
\[ \mathcal{K} = \{ u \in \mathcal{A} : \varphi(t) \leq u(t) \leq \overline{\varphi}(t) \text{ for } t \in \mathbb{R}_+ \text{ and } \}
\[ u(t_2) - u(t_1) \geq (f(t_2)H(\varphi(t_1)))^{1/a} \int_{t_1}^{t_2} (1/q(s))^{1/a} ds \]
\[ \times [\max\{k_1(u) : \varphi(t_1) \leq u \leq \overline{\varphi}(t_2)\}]^{-1} \text{ for } 0 < t_1 < t_2 \} \]

Remark 2. We now verify that \( \varphi \in \mathcal{K} \) and thus \( \mathcal{K} \) is a nonempty subset of \( \mathcal{A} \). Fix \( 0 < t_1 < t_2 \). Then
\[ P(\varphi(t_2)) - P(\varphi(t_1)) = \int_{t_1}^{t_2} \left( \frac{f(s)}{q(s)} \right)^{1/a} ds \]
and, by the Taylor formula, there exists \( \xi \in (\varphi(t_1), \varphi(t_2)) \subset (\varphi(t_1), \overline{\varphi}(t_2)) \) such that
\[ P'(\xi)(\varphi(t_2) - \varphi(t_1)) \geq (f(t_2))^{1/a} \int_{t_1}^{t_2} \left( \frac{1}{q(s)} \right)^{1/a} ds. \]
Since
\[ P'(\xi) = \frac{k_1(\xi)}{(H(\xi))^{1/a}} \leq \max\{k_1(u) : \varphi(t_1) \leq u \leq \overline{\varphi}(t_2)\} \left( \frac{1}{H(\varphi(t_1))} \right)^{1/a}, \]
we get
\[ \varphi(t_2) - \varphi(t_1) \geq \frac{1}{P'(\xi)} (f(t_2))^{1/a} \int_{t_1}^{t_2} \left( \frac{1}{q(s)} \right)^{1/a} ds \]
\[ \geq (f(t_2)H(\varphi(t_1)))^{1/a} \int_{t_1}^{t_2} \left( \frac{1}{q(s)} \right)^{1/a} ds \]
\[ \times [\max\{k_1(u) : \varphi(t_1) \leq u \leq \overline{\varphi}(t_2)\}]^{-1} \]
and therefore \( \varphi \in \mathcal{K} \). Analogously we can show that \( \overline{\varphi} \in \mathcal{K} \) as well.
Define the operator $T : \mathcal{K} \rightarrow C^0(\mathbb{R}_+)$ by

$$(Tu)(t) = K_1^{-1}\left(\int_0^t \left(\frac{1}{q(s)} \int_0^u f(u^{-1}(\tau)) h(\tau) \, d\tau\right)^{1/a} \, ds\right), \quad t \in \mathbb{R}_+.$$  

**Lemma 4.** $T : \mathcal{K} \rightarrow \mathcal{K}$. 

**Proof.** Let $u \in \mathcal{K}$. Set

$$\gamma(t) = \int_0^t \left(\frac{1}{q(s)} \int_0^u f(u^{-1}(\tau)) h(\tau) \, d\tau\right)^{1/a} \, ds,$$

$$\alpha(t) = \gamma(t) - K_1(\varphi(t)), \quad \beta(t) = \gamma(t) - K_1(\varphi(t))$$

for $t \in \mathbb{R}_+$. Then

$$\alpha'(t) = \left(\frac{1}{q(t)} \int_0^u f(u^{-1}(s)) h(s) \, ds\right)^{1/a} - \frac{k_1(\varphi(t))}{P'(\varphi(t))} \left(\frac{f(t)}{q(t)}\right)^{1/a}$$

$$\geq \left(\frac{f(t)}{q(t)} H(u(t))\right)^{1/a} - \frac{k_1(\varphi(t))}{k_1(\varphi(t))} \left(\frac{f(t)}{q(t)} H(\varphi(t))\right)^{1/a} \geq 0,$$

$$\beta'(t) = \left(\frac{1}{q(t)} \int_0^u f(u^{-1}(s)) h(s) \, ds\right)^{1/a} - \frac{k_1(\varphi(t))}{P'(\varphi(t))} \left(\frac{f(0)}{q(t)}\right)^{1/a}$$

$$\leq \left(\frac{f(0)}{q(t)} H(u(t))\right)^{1/a} - \frac{k_1(\varphi(t))}{k_1(\varphi(t))} \left(\frac{f(0)}{q(t)} H(\varphi(t))\right)^{1/a} \leq 0$$

for $t \in (0, \infty)$. Since $\alpha(0) = \beta(0) = 0$ and $\alpha'(t) \geq 0$, $\beta'(t) \leq 0$ on $(0, \infty)$, we see that $\alpha(t) \geq 0$, $\beta(t) \leq 0$ for $t \in \mathbb{R}_+$, and consequently,

$$(8) \quad \varphi(t) \leq K_1^{-1}(\gamma(t)) = (Tu)(t) \leq \varphi(t) \quad \text{for } t \in \mathbb{R}_+.$$

Let $0 < t_1 < t_2$. Then

$$K_1((Tu)(t_2)) - K_1((Tu)(t_1)) = \int_{t_1}^{t_2} \left(\frac{1}{q(s)} \int_0^u f(u^{-1}(\tau)) h(\tau) \, d\tau\right)^{1/a} \, ds$$

$$\geq \int_{t_1}^{t_2} \left(\frac{f(s)}{q(s)} H(u(s))\right)^{1/a} \, ds$$

$$\geq (H(\varphi(t_1))) f(t_2)^{1/a} \int_{t_1}^{t_2} \left(\frac{1}{q(s)}\right)^{1/a} \, ds$$

and
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\[ K_1((T(u))(t_2)) - K_1((T(u))(t_1)) \]
\[ = k_1(\xi)\left[(T(u))(t_2) - (T(u))(t_1)\right] \]
\[ \leq \max\{k_1(u) : \varphi(t_1) \leq u \leq \varphi(t_2)\}[\xi] \]

by the Taylor formula (here \( \xi \in (\varphi(t_1), (T(u))(t_2)) \subset (\varphi(t_1), \varphi(t_2))\)). Hence (with \( A = \max\{k_1(u) : \varphi(t_1) \leq u \leq \varphi(t_2)\}^{-1}\))

\[ (T(u))(t_2) - (T(u))(t_1) \geq A[K_1((T(u))(t_2)) - K_1((T(u))(t_1))] \]
\[ \geq A[H(\varphi(t_1))f(t_2)]^{1/a} \int_{t_1}^{t_2} \left( \frac{1}{q(s)} \right)^{1/a} ds. \]

From (8) and (9) it follows that \( Tu \in K \) for each \( u \in K \), and consequently, \( T : K \to K \).

3. Existence theorem

**Theorem 1.** Let assumptions \((H_1)\)–\((H_5)\) be satisfied. Then there exists a solution of (1).

**Proof.** By Lemma 2 and Corollary 1, \( u \in A \) is a solution of (1) if and only if \( u \) is a solution of (4). Therefore in order to prove Theorem 1 it is enough to show that the operator \( T \) has a fixed point.

Let \( X \) be the Fréchet space of \( C^0 \)-functions on \( \mathbb{R}_+ \) with the topology of uniform convergence on compact subintervals of \( \mathbb{R}_+ \). Then \( K \) is a bounded closed convex subset of \( X \) and \( T : K \to K \) (by Lemma 4). Let \( \{u_n\} \subset K \) be a convergent sequence, \( \lim_{n \to \infty} u_n = u \in K \). Then \( \lim_{n \to \infty} u_n^{-1} = u^{-1} \) (\( u_n^{-1} \) and \( u^{-1} \) denote the inverse functions to \( u_n \) and \( u \), respectively) and consequently, \( \lim_{n \to \infty} Tu_n = Tu \). This proves that \( T \) is a continuous operator.

It follows from the inequalities \((0 \leq t_1 < t_2 \leq t_3, u \in K)\)

\[ (0 \leq K_1((T(u))(t_2)) - K_1((T(u))(t_1)) \]
\[ = \int_{t_1}^{t_2} \left( \frac{1}{q(s)} \int_0^{u(s)} f(u^{-1}(\tau)) h(\tau) d\tau \right)^{1/a} ds \]
\[ \leq \int_{t_1}^{t_2} \left( \frac{f(0)}{q(s)} H(u(s)) \right)^{1/a} ds \]
\[ \leq (f(0)H(\varphi(t_3)))^{1/a} \int_{t_1}^{t_2} \left( \frac{1}{q(s)} \right)^{1/a} ds \]

and from the Arzelà–Ascoli theorem that \( T(K) \) is a relatively compact subset of \( K \). By the Tikhonov–Schauder fixed point theorem, there exists a fixed point of \( T \). Hence Theorem 1 is proved. □
THEOREM 2. Let assumptions (H₁)–(H₅) be satisfied. If there exist two different solutions \( u(t) \) and \( v(t) \) of (1) then
\[
\text{for } t \in (0, \infty).
\]

Proof. Assume \( u, v \) are different solutions of (1). Assume there exists a \( t_1 > 0 \) such that \( u(t_1) = v(t_1) \) and \( u(t) \neq v(t) \) on \((0, t_1)\), say \( u(t) < v(t) \) for \( t \in (0, t_1) \). Then
\[
0 = v(t_1) - u(t_1) = K_1((Tv)(t_1)) - K_1((Tu)(t_1))
\]
\[
= \int_0^{t_1} \left( \frac{1}{q(s)} \int_0^s f(v^{-1}(\tau))h(\tau) \, d\tau \right)^{1/a} \, ds
\]
\[
- \int_0^{t_1} \left( \frac{1}{q(s)} \int_0^s f(u^{-1}(\tau))h(\tau) \, d\tau \right)^{1/a} \, ds,
\]
which contradicts
\[
\int_0^{t_1} \left( \frac{1}{q(s)} \int_0^s f(v^{-1}(\tau))h(\tau) \, d\tau \right)^{1/a} \, ds > \int_0^{t_1} \left( \frac{1}{q(s)} \int_0^s f(u^{-1}(\tau))h(\tau) \, d\tau \right)^{1/a} \, ds.
\]
Let \( 0 < t_1 < t_2 \) be such that \( u(t_1) = v(t_1) \), \( u(t_2) = v(t_2) \), \( u(t) \neq v(t) \) on \((t_1, t_2)\), say \( u(t) > v(t) \) for \( t \in (t_1, t_2) \). Then \( u'(t_1) \geq v'(t_1) \), \( u'(t_2) \leq v'(t_2) \) and
\[
0 \leq q(t_1)k(u(t_1))(u'(t_1))^a - (v'(t_1))^a
\]
\[
- q(t_2)k(u(t_2))(u'(t_2))^a - (v'(t_2))^a
\]
\[
= \int_{t_1}^{t_2} f(s)h(u(s))u'(s) \, ds - \int_{t_2}^{t_1} f(s)h(v(s))v'(s) \, ds
\]
\[
= \int_{u(t_1)}^{u(t_2)} [f(u^{-1}(s)) - f(v^{-1}(s))]h(s) \, ds.
\]
On the other hand, since \( u(t_2) > u(t_1) \) and \( f(u^{-1}(t)) - f(v^{-1}(t)) \geq 0 \) on \([u(t_1), u(t_2)]\),
\[
\int_{u(t_1)}^{u(t_2)} [f(u^{-1}(s)) - f(v^{-1}(s))]h(s) \, ds \leq 0.
\]
Thus by (10), \( u'(t_1) = v'(t_1), u'(t_2) = v'(t_2) \) and \( f(u^{-1}(t)) = f(v^{-1}(t)) \) for \( t \in [u(t_1), u(t_2)] \). Since
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\[ q(t)(K_1(u(t)))' - q(t_1)k(u(t_1))(u'(t_1)) = \int_{u(t_1)}^{u(t)} f(u^{-1}(s))h(s) ds, \]

\[ q(t)(K_1(v(t)))' - q(t_1)k(v(t_1))(v'(t_1)) = \int_{v(t_1)}^{v(t)} f(v^{-1}(s))h(s) ds \]

on \((0, \infty), q(t_1)k(u(t_1))(u'(t_1)) = q(t_1)k(v(t_1))(v'(t_1)), 0 < f(u^{-1}(s)) = f(v^{-1}(s))\) for \(s \in [u(t_1), u(t_2)]\) and \(u(t) > v(t)\) on \((t_1, t_2)\), we obtain

\[ ((K_1(u(t)))') - ((K_1(v(t)))') = \frac{1}{q(t)} \int_{v(t)}^{u(t)} f(u^{-1}(s))h(s) ds > 0, \quad t \in (t_1, t_2). \]

Thus

\[ (K_1(u(t)))' > (K_1(v(t)))' \quad \text{for} \quad t \in (t_1, t_2), \]

and consequently, 

\[ K_1(u(t_2)) - K_1(u(t_1)) > K_1(v(t_2)) - K_1(v(t_1)), \]

which contradicts \(u(t_1) = v(t_1), u(t_2) = v(t_2)\). So either \(u(t) = v(t)\) on \((0, \infty)\) or there exists a \(t_0 \in (0, \infty)\) such that \(u(t) = v(t)\) for \(t \in [0, t_0]\) and \(u(t) = v(t)\) on \((t_0, \infty), \) say for example \(u(t) > v(t)\) for \(t \in (t_0, \infty)\). Assume that the second case occurs. Then, by the Bonnet mean value theorem, there exists a \(\xi \in [t_0, t]\) such that

\[ ((K_1(u(t)))') - ((K_1(v(t)))') = \frac{1}{q(t)} \int_{v(t)}^{u(t)} f(s)[h(u(s))u'(s) - h(v(s))v'(s)] ds \]

\[ = \frac{1}{q(t)} \int_{t_0}^{t} f(t_0) \int_{t_0}^{\xi} (h(u(s))u'(s) - h(v(s))v'(s)) ds \]

\[ + f(t) \int_{t_0}^{\xi} (h(u(s))u'(s) - h(v(s))v'(s)) ds \]

\[ = \frac{1}{q(t)} [(f(t_0) - f(t))(H(u(\xi)) - H(v(\xi))) + f(t)(H(u(t)) - H(v(t)))], \quad t \geq t_0. \]

Set

\[ M = a \min\{q(t) : t_0 \leq t \leq t_0 + 1\} \cdot \min\{(k_1(z))^{\alpha - 1} : u(t_0) \leq z \leq u(t_0 + 1)\} \]

\[ \times \min\{\min\{(u'(t))^{\alpha - 1}, (v'(t))^{\alpha - 1}\} : t_0 \leq t \leq t_0 + 1\} (> 0), \]

\[ M_1 = \min\{k_1(z) : u(t_0) \leq z \leq u(t_0 + 1)\} (> 0), \]
\[ L = \max \{ h(z) : u(t_0) \leq z \leq u(t_0 + 1) \} > 0, \]
\[ V(t) = \max \{ u(s) - v(s) : t_0 \leq s \leq t \} \text{ for } t \in [t_0, t_0 + 1]. \]

Obviously, \( V(t_0) = 0 \) and \( V \) is continuous nondecreasing on \([t_0, t_0 + 1]\).

By the Taylor formula, there exists a \( B (= B(t)) \) in the interval with end points \((K_1(u(t)))'\) and \((K_1(v(t)))'\) such that
\[
((K_1(u(t)))')^a - ((K_1(v(t)))')^a = aB^{a-1}(K_1(u(t)) - K_1(v(t)))',
\]
\[ t \in [t_0, t_0 + 1], \]
and therefore (cf. (12))
\[
(K_1(u(t)) - K_1(v(t)))' \leq \frac{1}{M} [(f(t_0) - f(t))(H(u(\xi)) - H(v(\xi)))
+ f(t)(H(u(t)) - H(v(t)))]
\]
\[
\leq \frac{f(t_0)}{M} [(H(u(\xi)) - H(v(\xi)) + (H(u(t)) - H(v(t)))]
\]
\[
\leq \frac{2}{M} Lf(t_0)V(t), \quad t \in [t_0, t_0 + 1].
\]

Then
\[
K_1(u(t)) - K_1(v(t)) \leq \frac{2}{M} Lf(t_0) \int_{t_0}^{t} V(s) \, ds,
\]
and consequently,
\[
u(t) - v(t) \leq \frac{2Lf(t_0)}{MK_1(\varepsilon)} \int_{t_0}^{t} V(s) \, ds \leq \frac{2Lf(t_0)}{MM_1} \int_{t_0}^{t} V(s) \, ds, \quad t \in [t_0, t_0 + 1],
\]
where \( \varepsilon \in [v(t), u(t)] \) by the Taylor formula. Hence
\[
V(t) \leq \frac{2Lf(t_0)}{MM_1} \int_{t_0}^{t} V(s) \, ds \leq \frac{2Lf(t_0)}{MM_1} V(t) \int_{t_0}^{t} ds
\]
\[
= \frac{2Lf(t_0)}{MM_1} V(t)(t - t_0), \quad t \in [t_0, t_0 + 1].
\]

Since \( V(t) > 0 \) for \( t \in (t_0, t_0 + 1) \), we obtain (cf. (13))
\[
1 \leq \frac{2Lf(t_0)}{MM_1} (t - t_0) \quad \text{for } t \in (t_0, t_0 + 1],
\]
a contradiction. \( \blacksquare \)
Theorem 3. Let assumptions (H1)–(H5) be satisfied. Then there exist solutions \( \varphi(t) \) and \( \bar{\pi}(t) \) of (1) such that

\[ \varphi(t) \leq u(t) \leq \bar{\pi}(t) \leq \bar{\Phi}(t), \quad t \in \mathbb{R}_+, \]

for any solution \( u(t) \) of (1).

Proof. Denote by \( \mathcal{B} \) the set of all solutions of (1). By Theorem 1, \( \mathcal{B} \) is a nonempty set. If \( \mathcal{B} \) is a finite set, then Theorem 3 follows from Theorem 2. Assume \( \mathcal{B} \) is an infinite set. Set

\[ u(t) = \inf \{ u(t) : u \in \mathcal{B} \}, \quad \bar{\pi}(t) = \sup \{ u(t) : u \in \mathcal{B} \} \quad \text{for } t \in \mathbb{R}_+. \]

Then \( \varphi(t) \leq u(t) \leq \bar{\pi}(t) \leq \bar{\Phi}(t) \) on \( \mathbb{R}_+ \) and to prove Theorem 3 it is enough to show that \( \varphi \) and \( \bar{\pi} \) are solutions of (1). By Theorem 2, there exists a sequence \( \{ u_n \} \subset \mathcal{B}, u_1(t) < \ldots < u_n(t) < \ldots < \bar{\pi}(t), t \in (0, \infty), \) such that \( \bar{\pi}(t) = \lim_{n \to \infty} u_n(t) \) for \( t \in \mathbb{R}_+ \). Now we prove that \( \lim_{n \to \infty} u_n'(t) = b(t) \) for all \( t \in (0, \infty) \) and \( b = \bar{\pi}' \). Evidently,

\[
(K_1(u_{n+1}(t)))' - (K_1(u_n(t)))' = \left( \frac{1}{q(t)} \int_0^{u_{n+1}(t)} f(u_{n+1}(s))h(s)ds \right)^{1/a} - \left( \frac{1}{q(t)} \int_0^{u_n(t)} f(u_n^{-1}(s))h(s)ds \right)^{1/a} > \left( \frac{1}{q(t)} \int_0^{u_n(t)} f(u_n^{-1}(s))h(s)ds \right)^{1/a} - \left( \frac{1}{q(t)} \int_0^{u_n(t)} f(u_n^{-1}(s))h(s)ds \right)^{1/a} = 0
\]

for \( t \in (0, \infty) \) and \( n \in \mathbb{N} \). Therefore the sequence \( \{ k_1(u_n(t))u_n'(t) \} \) is strictly increasing for each \( t \in (0, \infty) \). Setting \( \alpha(t) = \lim_{n \to \infty} k_1(u_n(t))u_n'(t), t \in (0, \infty), \) we see that

\[
\lim_{n \to \infty} u_n'(t) = \lim_{n \to \infty} \frac{k_1(u_n(t))u_n'(t)}{k_1(u_n(t))} = \frac{\alpha(t)}{k_1(\bar{\pi}(t))} =: \beta(t), \quad t \in (0, \infty),
\]

and using the Lebesgue dominated convergence theorem in the equalities

\[
u_n(t) = \int_0^t u_n'(s)ds, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N},
\]

we get \( \bar{\pi}(t) = \int_0^t \beta(s)ds \) on \( \mathbb{R}_+ \); hence \( \beta(t) = \bar{\pi}'(t) \) for \( t \in (0, \infty) \). Applying again the Lebesgue theorem to the equalities

\[
k_1(u_n(t))u_n'(t) = \left( \frac{1}{q(t)} \int_0^t f(s)h(u_n(s))u_n'(s)ds \right)^{1/a}, \quad t \in (0, \infty), \quad n \in \mathbb{N},
\]

we obtain

\[
k_1(\bar{\pi}(t))\bar{\pi}'(t) = \left( \frac{1}{q(t)} \int_0^t f(s)h(\bar{\pi}(s))\bar{\pi}'(s)ds \right)^{1/a}, \quad t \in (0, \infty),
\]

Positive solutions of nonlinear equations
and consequently, \( \pi \) is a solution of (1). Analogously we can prove that \( u \) is a solution of (1). ■

4. Bounded and unbounded solutions

Theorem 4. Let assumptions (H1)–(H5) be satisfied. Then

(i) some (and then any) solution of (1) is bounded if and only if

\[ \int_0^\infty \left( \frac{1}{q(t)} \right)^{1/a} dt < \infty, \]

(ii) some (and then any) solution of (1) is unbounded if and only if

\[ \int_0^\infty \left( \frac{1}{q(t)} \right)^{1/a} dt = \infty. \]

Proof. First note that either \( \int_0^\infty (1/q(t))^{1/a} dt < \infty \) or \( \int_0^\infty (1/q(t))^{1/a} dt = \infty \). In the first case, by Lemma 3, any solution \( u \) of (1) is bounded. Now assume \( \int_0^\infty (1/q(t))^{1/a} dt = \infty \) and \( u \) is a solution of (1). Then

\[
\lim_{t \to \infty} \frac{\int_0^t \left( \frac{1}{q(s)} \int_0^s f(u^{-1}(\tau))h(\tau) d\tau \right)^{1/a} ds}{\int_0^t \left( \frac{1}{q(s)} \right)^{1/a} ds}
\]

\[= \lim_{t \to \infty} \left( \int_0^t f(u^{-1}(s))h(s) ds \right)^{1/a}, \]

\[= \lim_{t \to \infty} \left( \int_0^t f(s)h(u(s))u'(s) ds \right)^{1/a} > 0, \]

and consequently,

\[ \lim_{t \to \infty} K_1(u(t)) = \lim_{t \to \infty} \frac{\int_0^t \left( \frac{1}{q(s)} \int_0^s f(u^{-1}(\tau))h(\tau) d\tau \right)^{1/a} ds}{\int_0^t \left( \frac{1}{q(s)} \right)^{1/a} ds} = \infty. \]

Hence \( \lim_{t \to \infty} u(t) = \infty \) and \( u \) is unbounded.

Let \( u \) be a solution of (1). If \( u \) is bounded, then \( \int_0^\infty (1/q(t))^{1/a} dt < \infty \) since in the opposite case \( u \) is unbounded by the first part of the proof. Analogously, \( u \) unbounded implies \( \int_0^\infty (1/q(t))^{1/a} dt = \infty \). ■
5. Uniqueness theorem

**Theorem 5.** Let assumptions \((H_1)-(H_5)\) be satisfied. Moreover, assume that

\[(H_6)\] There exist positive numbers \(\varepsilon\) and \(L\) such that

(i) \(|f(t_1) - f(t_2)| \leq L|t_1 - t_2|\) for all \(t_1, t_2 \in [0, \varepsilon]\),

(ii) the modulus of continuity \(\gamma(t) = \sup\{(q(t_1))^{1/\alpha} - (q(t_2))^{1/\alpha} : t_1, t_2 \in [0, \varepsilon], |t_1 - t_2| \leq t\}\) of \((q(t))^{1/\alpha}\) on \([0, \varepsilon]\) satisfies

\[
\limsup_{t \to 0^+} \frac{\gamma(t)}{t} < \infty.
\]

Then (1) admits a unique solution.

**Proof.** By Theorem 1, there exists at least one solution of (1). Let \(u_1, u_2\) be different solutions of (1), say \(u_1(t) < u_2(t)\) on \([0, \infty)\) (see Theorem 2). According to the last part of the proof of Theorem 2 it is enough to show that \(u_1(t) = u_2(t)\) on \([0, t_0]\) for a positive number \(t_0\). Setting \(A_i = \lim_{t \to \infty} u_i(t)\) and \(w_i = u_i^{-1} (i = 1, 2)\), we see that \(0 < A_1 \leq A_2 \leq \infty\), \(w_i : [0, A_i) \to \mathbb{R}_+\) are continuous strictly increasing functions and

\[
w_i(t) = \int_0^t k_1(s) \left( \frac{1}{q(w_i(s))} \right)^{1/\alpha} \left( \int_0^s f(w_i(\tau)) h(\tau) d\tau \right)^{-1/\alpha} ds,
\]

\(t \in [0, A_i), i = 1, 2.\)

Then (for \(t \in [0, A_1]\))

\[
(14) \quad (0 \leq) w_1(t) - w_2(t)
\]

\[
= \int_0^t k_1(s)[(q(w_1(s)))^{1/\alpha} - (q(w_2(s)))^{1/\alpha}] \left( \int_0^s f(w_2(\tau)) h(\tau) d\tau \right)^{-1/\alpha} ds
\]

\[
+ \int_0^t \frac{k_1(s)(q(w_1(s)))^{1/\alpha}}{(\int_0^s f(w_1(\tau)) h(\tau) d\tau)^{1/\alpha} \int_0^s f(w_2(\tau)) h(\tau) d\tau)^{1/\alpha}}
\]

\[
\times \left[ \left( \int_0^s f(w_2(\tau)) h(\tau) d\tau \right)^{1/\alpha} - \left( \int_0^s f(w_1(\tau)) h(\tau) d\tau \right)^{1/\alpha} \right] ds.
\]

Let \(\varepsilon > 0\) be as in assumption \((H_6)\) and set \(b = \min\{u_1(\varepsilon), \varepsilon\}, A = \max\{(q(t))^{1/\alpha} : 0 \leq t \leq \varepsilon\}\) and \(X(t) = \max\{w_1(s) - w_2(s) : 0 \leq s \leq t\}\) for \(t \in (0, b]\). Then \(X\) is continuous nondecreasing, \(X(0) = 0\), \(X(t) > 0\) for \(t \in (0, b]\) and (cf. \((H_6)\))

\[
|(q(w_1(t)))^{1/\alpha} - (q(w_2(t)))^{1/\alpha}| \leq \gamma(X(t)) \quad \text{for} \ t \in [0, b].
\]
1. Let $a = 1$. Then (cf. (14))

$$w_1(t) - w_2(t) \leq \frac{1}{f(\varepsilon)} \int_0^t k(s)\gamma(X(s))(H(s))^{-1}ds$$

$$+ \frac{L}{(f(\varepsilon))^2} \int_0^t k(s)q(w_1(s)) \int_0^s h(\tau)(w_1(\tau) - w_2(\tau)) d\tau ds$$

$$\leq \frac{1}{f(\varepsilon)} \gamma(X(t))P(t) + \frac{LA}{(f(\varepsilon))^2} X(t)P(t), \quad t \in [0, b].$$

Hence

$$X(t) \leq \frac{1}{f(\varepsilon)} \gamma(X(t))P(t) + \frac{LA}{(f(\varepsilon))^2} X(t)P(t), \quad t \in [0, b],$$

and

$$1 \leq \frac{\gamma(X(t))}{f(\varepsilon)X(t)} P(t) + \frac{LA}{(f(\varepsilon))^2} P(t), \quad t \in (0, b].$$

Since

$$\limsup_{t \to 0^+} \frac{\gamma(X(t))}{X(t)} = \limsup_{t \to 0^+} \frac{\gamma(t)}{t} < \infty \quad \text{(by (H_6))}$$

and

$$\lim_{t \to 0^+} P(t) = 0,$$

we get

$$\lim_{t \to 0^+} \left[ \frac{\gamma(X(t))}{f(\varepsilon)X(t)} P(t) + \frac{LA}{(f(\varepsilon))^2} P(t) \right] = 0,$$

which contradicts (15).

2. Let $a > 1$. Then there is a positive integer $n$ such that $(n + 1)/a > 1$ and

$$\left( \int_0^t f(w_2(s))h(s) ds \right)^{(n+1)/a} - \left( \int_0^t f(w_1(s))h(s) ds \right)^{(n+1)/a}$$

$$= \left[ \left( \int_0^t f(w_2(s))h(s) ds \right)^{1/a} - \left( \int_0^t f(w_1(s))h(s) ds \right)^{1/a} \right]$$

$$\times \sum_{k=0}^{n} \left( \int_0^t f(w_2(s))h(s) ds \right)^{k/a} \left( \int_0^t f(w_1(s))h(s) ds \right)^{(n-k)/a}.$$

By the Taylor formula,

$$\left( \int_0^t f(w_2(s))h(s) ds \right)^{(n+1)/a} - \left( \int_0^t f(w_1(s))h(s) ds \right)^{(n+1)/a}$$

$$= \frac{n+1}{a} \varepsilon^{(n+1)/a-1} \int_0^t (f(w_2(s)) - f(w_1(s)))h(s) ds,$$
where \( \xi = \xi(t) \) lies in the interval with end points \( \int_0^1 f(w_1(s))h(s)\, ds \), \( \int_0^1 f(w_2(s))h(s)\, ds \), and thus (cf. (14) and (16))

\[
\begin{align*}
\int_0^t k_1(s) & \gamma(X(s))(f(\varepsilon)H(s))^{-1/a} \, ds \\
& + \int_0^t k_1(s)\left[q(w_1(s))\right]^{1/a} \\
& \times \left[ \int_0^1 f(w_2(\tau))h(\tau)\, d\tau \int_0^1 f(w_1(\tau))h(\tau)\, d\tau \right]^{1/a} \\
& \times \left[ \int_0^1 f(w_1(\tau))h(\tau)\, d\tau \right]^{(n+1)/a} - \left[ \int_0^1 f(w_1(\tau))h(\tau)\, d\tau \right]^{(n+1)/a} \\
& \times \sum_{k=0}^{\gamma} f\left[ \int_0^1 f(w_2(\tau))h(\tau)\, d\tau \right]^{k/a} f\left[ \int_0^1 f(w_1(\tau))h(\tau)\, d\tau \right]^{(n-k)/a} \\
& \leq \gamma(X(t))P(t) \left( \frac{1}{f(\varepsilon)} \right)^{1/a} + \frac{n+1}{a} A \left( \frac{1}{f(\varepsilon)} \right)^{(n+2)/a} \\
& \times \int_0^t k_1(s)\xi^{(n+1)/a-1} \int_0^1 f(w_2(\tau)) - f(w_1(\tau))h(\tau)\, d\tau \\
& \leq \gamma(X(t))P(t) \left( \frac{1}{f(\varepsilon)} \right)^{1/a} + \frac{A}{a} \left( \frac{1}{f(\varepsilon)} \right)^{(n+2)/a} L(f(0))^{(n+1)/a-1} \\
& \times \int_0^t k_1(s)\left( H(s) \right)^{(n+1)/a} X(s) \left( H(s)^{(n+2)/a} \right) \\
& \leq \gamma(X(t))P(t) \left( \frac{1}{f(\varepsilon)} \right)^{1/a} \\
& + \frac{A}{a} \left( \frac{1}{f(\varepsilon)} \right)^{(n+2)/a} (f(0))^{(n+1)/a-1} LX(t)P(t)
\end{align*}
\]

for \( t \in [0, b] \) since \( |\xi(t)| \leq f(0)H(t) \) on \( [0, b] \). Then

\[
X(t) \leq \gamma(X(t))P(t) \left( \frac{1}{f(\varepsilon)} \right)^{1/a} \\
+ \frac{A}{a} \left( \frac{1}{f(\varepsilon)} \right)^{(n+2)/a} (f(0))^{(n+1)/a-1} LX(t)P(t),
\]

hence

\[
1 \leq \frac{\gamma(X(t))P(t)}{X(t)} \left( \frac{1}{f(\varepsilon)} \right)^{1/a} \\
+ \frac{A}{a} \left( \frac{1}{f(\varepsilon)} \right)^{(n+2)/a} (f(0))^{(n+1)/a-1} LP(t)
\]

for \( t \in (0, b] \), and since \( \limsup_{t \to 0} \gamma(X(t))/X(t) < \infty \) and \( \lim_{t \to 0} P(t) = 0 \),
we get
\[
\lim_{t \to 0+} \left[ \gamma(X(t)) \frac{P(t)}{X(t)} \left( \frac{1}{f(x)} \right)^{1/a} + \frac{A}{a} \left( \frac{1}{f(x)} \right)^{(n+2)/a} (f(0))^{(n+1)/a-1} LP(t) \right] = 0,
\]
which contradicts (17).

3. Let \( a < 1 \). By the Taylor formula,
\[
\left( \int_0^t f(w_2(s))h(s) ds \right)^{1/a} - \left( \int_0^t f(w_1(s))h(s) ds \right)^{1/a} = \frac{\nu^{1/a-1}}{a} \left( \int_0^t f(w_2(s))h(s) ds - \int_0^t f(w_1(s))h(s) ds \right),
\]
where \( \nu = \nu(t) \) lies in the interval with end points \( \int_0^t f(w_2(s))h(s) ds \) and \( \int_0^t f(w_1(s))h(s) ds \), and using (14) we obtain
\[
w_1(t) - w_2(t) = \gamma(X(t))P(t) \left( \frac{1}{f(x)} \right)^{1/a} + \frac{A}{a} \left( \frac{1}{f(x)} \right)^{2/a} (f(0))^{1/a-1}
\]
\[
\times \int_0^t k_1(s)(H(s))^{1/a-1} \left( \frac{(f(w_2(s)) - f(w_1(s)))h(s)}{(H(s))^{2/a}} \right) ds
\]
\[
\leq \gamma(X(t))P(t) \left( \frac{1}{f(x)} \right)^{1/a} + \frac{A}{a} \left( \frac{1}{f(x)} \right)^{2/a} (f(0))^{1/a-1} L \int_0^t \frac{k_1(s)X(s)}{(H(s))^{1/a}} ds
\]
\[
\leq \gamma(X(t))P(t) \left( \frac{1}{f(x)} \right)^{1/a} + \frac{A}{a} \left( \frac{1}{f(x)} \right)^{2/a} (f(0))^{1/a-1} LX(t)P(t)
\]
for \( t \in [0, b] \) since \( |\nu(t)| \leq f(0)H(t) \) on \([0, b]\). Then
\[
X(t) \leq \gamma(X(t))P(t) \left( \frac{1}{f(x)} \right)^{1/a} + \frac{A}{a} \left( \frac{1}{f(x)} \right)^{2/a} (f(0))^{1/a-1} LX(t)P(t), \quad t \in [0, b],
\]
and hence
\[
1 \leq \frac{\gamma(X(t))}{X(t)} P(t) \left( \frac{1}{f(x)} \right)^{1/a} + \frac{A}{a} \left( \frac{1}{f(x)} \right)^{2/a} (f(0))^{1/a-1} LP(t), \quad t \in (0, b],
\]
which contradicts
\[
\lim_{t \to 0+} \left[ \gamma(X(t)) \frac{P(t)}{X(t)} \left( \frac{1}{f(x)} \right)^{1/a} + \frac{A}{a} \left( \frac{1}{f(x)} \right)^{2/a} (f(0))^{1/a-1} LP(t) \right] = 0. \]
6. Dependence of solutions on a parameter. Consider the differential equation

\[(q(t)k(u)(u')^a)'' = \lambda f(t)h(u)u', \quad \lambda > 0,\]

depending on the positive parameter \(\lambda\) with \(q, k, f\) and \(h\) satisfying assumptions \((H_1)–(H_5)\). Set

\[
\phi(t, \lambda) = P^{-1} \left( \int_0^t \left( \frac{\lambda f(s)}{q(s)} \right)^{1/a} ds \right),
\]

\[
\psi(t, \lambda) = P^{-1} \left( \int_0^t \left( \frac{\lambda f(0)}{q(s)} \right)^{1/a} ds \right)
\]

for \((t, \lambda) \in \mathbb{R}_+ \times (0, \infty)\). Denote by \(u(t, \lambda)\) a solution of \((18_\lambda)\).

By Theorem 3 (with \(\lambda f\) instead of \(f\)), there exist solutions \(\underline{u}(t, \lambda)\) and \(\overline{u}(t, \lambda)\) of \((18_\lambda)\) such that

\[
\underline{\phi}(t, \lambda) \leq \underline{u}(t, \lambda) \leq u(t, \lambda) \leq \overline{u}(t, \lambda) \leq \overline{\phi}(t, \lambda),
\]

for any solution \(u(t, \lambda)\) of \((18_\lambda)\).

**Theorem 6.** Let assumptions \((H_1)–(H_5)\) be satisfied. Then

\[
\overline{\pi}(t, \lambda_1) < \underline{u}(t, \lambda_2), \quad t \in (0, \infty),
\]

for any \(0 < \lambda_1 < \lambda_2\).

**Proof.** Let \(0 < \lambda_1 < \lambda_2\). Since

\[
\lim_{t \to 0^+} \frac{\int_0^t \left( \frac{\lambda_2 f(s)}{q(s)} \right)^{1/a} ds}{\int_0^t \left( \frac{\lambda_1 f(s)}{q(s)} \right)^{1/a} ds} = \lim_{t \to 0^+} \left( \frac{\lambda_2 f(t)}{\lambda_1 f(0)} \right)^{1/a} = (\lambda_2 / \lambda_1)^{1/a} > 1,
\]

there exists an \(\varepsilon > 0\) such that \(\phi(t, \lambda_2) > \psi(t, \lambda_1)\) for \(t \in (0, \varepsilon]\), and consequently,

\[
\overline{\pi}(t, \lambda_1) < \underline{u}(t, \lambda_2) \quad \text{for} \quad t \in (0, \varepsilon]
\]

by \((19)\). Assume \(\overline{\pi}(t, \lambda_1) < \underline{u}(t, \lambda_2)\) on \((0, t_0)\) while \(\overline{\pi}(t_0, \lambda_1) = \underline{u}(t_0, \lambda_2)\) for a \(t_0 \in (\varepsilon, \infty)\). Then
0 = K_1(u(t_0, \lambda_2)) - K_1(u(t_0, \lambda_1))
\begin{align*}
&= \int_0^{t_0} \left( \frac{\lambda_2}{q(t)} \int_0^{u(t, \lambda_2)} f(u^{-1}(s, \lambda_2)) h(s) \, ds \right)^{1/a} \, dt \\
&\quad - \int_0^{t_0} \left( \frac{\lambda_1}{q(t)} \int_0^{u(t, \lambda_1)} f(u^{-1}(s, \lambda_1)) h(s) \, ds \right)^{1/a} \, dt,
\end{align*}
which contradicts
\[
\left( \frac{\lambda_2}{q(t)} \int_0^{u(t, \lambda_2)} f(u^{-1}(s, \lambda_2)) h(s) \, ds \right)^{1/a} > \left( \frac{\lambda_1}{q(t)} \int_0^{u(t, \lambda_1)} f(u^{-1}(s, \lambda_1)) h(s) \, ds \right)^{1/a} > \frac{1}{a} > 0 \quad \text{for } 0 < t \leq t_0.
\]

**Corollary 2.** Let assumptions (H_1)–(H_5) be satisfied. Then there exists an at most countable set \( R \subset [0, \infty) \) such that equation \((18_\lambda)\) has a unique solution for every \( \lambda \in (0, \infty) \setminus R \).

**Proof.** Let \( t_0 \in (0, \infty) \) and set \( g(\lambda) = u(t_0, \lambda) \) for \( \lambda \in (0, \infty) \). Then \( g \) is strictly increasing on \((0, \infty)\) by Theorem 6, and
\[
\lim_{\lambda \to \infty} g(\lambda) = \lim_{\lambda \to \infty} u(t_0, \lambda) = \lim_{\lambda \to \infty} \varphi(t_0, \lambda) = P^{-1} \left( \int_0^{t_0} \left( \frac{\lambda f(s)}{g(s)} \right)^{1/a} \, ds \right) = \infty.
\]
Evidently, if \( g \) is continuous at a point \( \lambda = \lambda_0 \) then \((18_\lambda)\) has a unique solution for \( \lambda = \lambda_0 \). For each \( n \in \mathbb{N} \) denote by \( R_n \) the set of points of discontinuity of \( g \) on the interval \([1/n, n]\). By Theorem 1 of [5, p. 229], the set \( R_n \) is at most countable. Hence \( R = \bigcup_{n=1}^{\infty} R_n \) is the set of points of discontinuity of \( g \) and since \( R \) is at most countable, the proof of Corollary 2 is finished.

**Theorem 7.** Let assumptions (H_1)–(H_6) be satisfied and, moreover, \( \int_0^{\infty} (1/q(t))^{1/a} \, dt < \infty \). Then for each \( c \in (0, \infty) \) there exists a unique \( \lambda_c \in (0, \infty) \) such that equation \((18_\lambda)\) for \( \lambda = \lambda_c \) has a (necessarily unique) solution \( u(t, \lambda_c) \) with
\[
\lim_{t \to \infty} u(t, \lambda_c) = c.
\]
Proof. By Theorem 5, equation (18) has a unique solution \( u(t, \lambda) \) for each \( \lambda \in (0, \infty) \). This solution is strictly increasing (by Corollary 1) and bounded on \( \mathbb{R}^+ \) (by Theorem 4). Define \( g(\lambda) = \lim_{t \to \infty} u(t, \lambda) \) for all \( \lambda > 0 \). The function \( g : (0, \infty) \to (0, \infty) \) is increasing by Theorem 6. To prove our theorem it is sufficient to show that \( g \) is continuous, strictly increasing and maps \( (0, \infty) \) onto itself. Assume \( g(\lambda_1) = g(\lambda_2) \) for some \( 0 < \lambda_1 < \lambda_2 \). Then \( u(t, \lambda_1) < u(t, \lambda_2) \) on \( (0, \infty) \) and thus

\[
\begin{align*}
g(\lambda_1) &= \lim_{t \to \infty} \left( \frac{\lambda_1}{q(t)} \int_0^t f(u^{-1}(s, \lambda_1))h(s) \, ds \right)^{1/a} dt \\
&< \lim_{t \to \infty} \left( \frac{\lambda_2}{q(t)} \int_0^t f(u^{-1}(s, \lambda_2))h(s) \, ds \right)^{1/a} dt = g(\lambda_2),
\end{align*}
\]
a contradiction. Assume

\[
\lim_{\lambda \to \lambda_0} g(\lambda) = \lim_{\lambda \to \lambda_0} g(\lambda) > 0 \quad \text{for a } \lambda_0 \in (0, \infty).
\]

Set

\[
\alpha(t) = \lim_{\lambda \to \lambda_0^+} u(t, \lambda), \quad \beta(t) = \lim_{\lambda \to \lambda_0^-} u(t, \lambda) \quad \text{for } t \in \mathbb{R}^+.
\]

Then

\[
(21) \quad \lim_{t \to \infty} \inf (\alpha(t) - \beta(t)) > 0.
\]

Using the Lebesgue dominated convergence theorem as \( \lambda \to \lambda_0^+ \) and \( \lambda \to \lambda_0^- \) in the equality

\[
u(t, \lambda) = K^{-1}_0 \left( \int_0^t \left( \frac{\lambda}{q(s)} \int_0^s f(u^{-1}(\tau, \lambda))h(\tau) \, d\tau \right)^{1/a} \, ds \right),
\]

we see (cf. Lemma 2) that \( \alpha \) and \( \beta \) are solutions of (18). Consequently, \( \alpha(t) = \beta(t) = u(t, \lambda_0) \) for \( t \in \mathbb{R}^+ \), which contradicts (21). Finally,

\[
\begin{align*}
\lim_{\lambda \to \infty} \lim_{t \to \infty} \varphi(t, \lambda) &= \lim_{\lambda \to \infty} P^{-1} \left( \int_0^\infty \left( \frac{\lambda f(s)}{q(s)} \right)^{1/a} ds \right) = \infty, \\
\lim_{\lambda \to 0^+} \lim_{t \to \infty} \varphi(t, \lambda) &= \lim_{\lambda \to 0^+} P^{-1} \left( \int_0^\infty \left( \frac{\lambda f(0)}{q(s)} \right)^{1/a} ds \right) = 0,
\end{align*}
\]

since \( \lim_{t \to \infty} P^{-1}(t) = \infty, \lim_{t \to 0^+} P^{-1}(t) = 0, \)

\[
0 < \int_0^\infty \left( \frac{f(s)}{q(s)} \right)^{1/a} ds < \int_0^\infty \left( \frac{f(0)}{q(s)} \right)^{1/a} ds < \infty
\]

and therefore (cf. (19)) \( \lim_{\lambda \to \infty} g(\lambda) = \infty \) and \( \lim_{\lambda \to 0^+} g(\lambda) = 0. \)

\]
References


