ANNALES
POLONICI MATHEMATICI
LXII.2 (1995)

Convergence of optimal solutions in control problems for hyperbolic equations

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Abstract. A sequence of optimal control problems for systems governed by linear hyperbolic equations with the nonhomogeneous Neumann boundary conditions is considered. The integral cost functionals and the differential operators in the equations depend on the parameter k. We deal with the limit behaviour, as $k \to \infty$, of the sequence of optimal solutions using the notions of G- and Γ-convergences. The conditions under which this sequence converges to an optimal solution for the limit problem are given.

1. Introduction. In this note, we consider the sequence of optimal control problems for systems described by second-order linear hyperbolic equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x_i}\left(a_{ij}^k(x,t)\frac{\partial y}{\partial x_j}\right) = f$$

with the Cauchy initial data and the nonhomogeneous Neumann boundary conditions. The parameter $k \in \mathbb{N}$ (index of an element of the sequence) appears in the coefficients of the state equations as well as in the cost functionals which have a general integral form.

Our motivation is mainly related to boundary control problems with homogenization in the state equation (see for example [10]); however, the role of controls is played not only by the boundary functions but also by the forcing term in the equation and the initial functions.

We formulate the control problem in the following way (see e.g. [9] and [2]): find

$$(1)_k \qquad \min\{J_k(u,y) + \chi_{\Lambda_k}(u,y) \mid (u,y) \in U \times Y\},$$

1991 *Mathematics Subject Classification*: 49J20, 49J45, 49K40.

Key words and phrases: control problem, hyperbolic equation, G-convergence, Γ-convergence.

This work has been performed as part of Research Project nr 2 1073 91 01 supported by K.B.N. This paper was completed while the author was visiting Scuola Normale Superiore, Pisa, Italy.

where $J_k : U \times Y \to \mathbb{R} \cup \{\pm\infty\}$ are the cost functionals; U, Y are the spaces of control and states, respectively, $\Lambda_k \subset U \times Y$ are the sets of admissible pairs (u, y) and χ_{Λ_k} denotes the indicator function of Λ_k (i.e. $\chi_{\Lambda_k} = 0$ on Λ_k, and ∞ elsewhere). This is of course an equivalent formulation of the problem of minimization of J_k over the sets Λ_k. The elements which realize the minimum in $(1)_k$ are called optimal solutions.

In this paper we consider two problems: (i) we study the existence of optimal solutions for every fixed parameter k and (ii) we investigate the asymptotic behaviour of the sequence of optimal solutions as $k \to \infty$.

We get an affirmative answer to problem (i) by using the direct method of the calculus of variations. As concerns (ii), our approach is based on an abstract framework given in [2] for characterization of the limits of control problems. The abstract scheme requires the Γ-convergence of the cost functionals and of the indicator functions of the sets of admissible pairs (see Proposition 4.1 below). This formulation was applied to study control problems for elliptic equations in [2], [14] and for evolution equations in [8], [15], [16].

Here, we give conditions under which the optimal solutions of the sequence $(1)_k$ converge to an optimal solution of the limit problem of the same kind. In this way, we extend the earlier results (see Theorem 6.2 of [8] and Lemma 2.2 of [13]) to the class of control problems for hyperbolic equations with the nonhomogeneous Neumann boundary conditions. Such equations are treated by using the transposition method described in [11] and [9]. The Γ-convergence of the cost functionals is obtained in a similar way to that used in [2], [8], [14] starting with results of [12]. We prove the Γ-convergence of the sets Λ_k employing the notion of G-convergence introduced in [18] for elliptic operators and extended to parabolic and hyperbolic equations in [3], [4] (for more details we refer to [5], [19], [7], [20], [15]). Finally, we remark that a special case in which G-convergence holds is that of homogenization (in the space variable x), where $a_{ij}^k(x,t) = \alpha_{ij}(kx,t)$ for some $\alpha_{ij}(y,t)$ periodic in y (compare for instance [1] and [17]).

The main result of this paper was announced in its preliminary form in [15].

2. Preliminaries. We shall briefly introduce the essential notations and state some results needed in the sequel.

We consider a Gelfand triple of separable Hilbert spaces $V \subset H \subset V'$ with continuous, dense and compact embeddings. We denote by $\langle \cdot, \cdot \rangle$ the duality of V and its dual V' as well as the inner product on H, and by $\| \cdot \|$, $| \cdot |$, $\| \cdot \|_{V'}$ the norms in V, H and V', respectively. For a fixed real number $T > 0$, we introduce the spaces $\mathcal{V} = L^2(0,T;V)$, $\mathcal{H} = L^2(0,T;H)$ and $\mathcal{V}' = L^2(0,T;V')$. The duality between $\mathcal{V}$ and $\mathcal{V}'$ and the inner product on $\mathcal{H}$ is denoted by

$$\langle\langle f, v\rangle\rangle = \int_0^T \langle f(s), v(s)\rangle\, ds, \qquad f \in \mathcal{V}',\ v \in \mathcal{V}.$$

Moreover, given an open bounded subset Ω in $\mathbb{R}^n$ with Lipschitz continuous boundary Γ, we put $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. The duality between $L^2(0,T;H^{1/2}(\Gamma))$ and its dual (and the inner product on $L^2(\Sigma)$) is denoted by

$$\langle\langle w, z\rangle\rangle_\Sigma = \int_0^T \langle w(s), z(s)\rangle_\Gamma\, ds,$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality of $H^{1/2}(\Gamma)$ and its dual and also for the inner product on $L^2(\Gamma)$.

For a Banach space $\mathcal{X}$, the symbols w-$\mathcal{X}$, s-$\mathcal{X}$ are always used for the weak and the strong topology in $\mathcal{X}$, respectively. Given a sequence $v_n \in L^\infty(0,T;V)$, we will write $v_n \to v$ in w-$*$-$L^\infty(0,T;V)$ if $\langle\langle v_n, g\rangle\rangle \to \langle\langle v, g\rangle\rangle$ as $n \to \infty$, for every $g \in L^1(0,T;V')$. In particular, for $v_n \in L^\infty(0,T)$, $v_n \to v$ in w-$*$-$L^\infty(0,T)$ means that $\int_0^T v_n g\, dt \to \int_0^T vg\, dt$ for all $g \in L^1(0,T)$. Given a convex function $f : \mathbb{R}^d \to \mathbb{R}$, we denote by $f^* : \mathbb{R}^d \to \mathbb{R}$ the *polar* (or *conjugate*) *function* of f, i.e. $f^*(z^*) = \sup\{zz^* - f(z) \mid z \in \mathbb{R}^d\}$. Different constants independent of the parameter k are denoted by the same letter c. We also write $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. In what follows we use the standard summation convention.

We consider a family of linear operators $\mathcal{A}_k : \mathcal{V} \to \mathcal{V}'$, $k \in \mathbb{N}$, of the form

$$\mathcal{A}_k = -\frac{\partial}{\partial x_i}\left(a_{ij}^k(x,t)\frac{\partial}{\partial x_j}\right) \tag{2}$$

with the coefficients $a_{ij}^k \in L^\infty(Q)$ which satisfy in Q, uniformly with respect to k, the following assumptions:

$$a_{ij}^k = a_{ji}^k, \tag{3}$$

$$\lambda_0 \le a_{ij}^k \xi_i \xi_j |\xi|^{-2} \le \lambda_1, \qquad \forall \xi \in \mathbb{R}^n, \tag{4}$$

$$|a_{ij}^k(x,t_2) - a_{ij}^k(x,t_1)| \le M|t_2 - t_1| \tag{5}$$

for some real constants λ_0, λ_1, M such that $0 < \lambda_0 \le \lambda_1$ and $M > 0$.

We denote by $H(\lambda_0,\lambda_1,M)$ the class of hyperbolic operators $H_{\mathcal{A}_k} = \partial^2/\partial t^2 + \mathcal{A}_k$, $H_{\mathcal{A}_k} : \mathcal{V} \supset \mathrm{dom}(H_{\mathcal{A}_k}) \to \mathcal{V}'$ associated with operators $\mathcal{A}_k$ whose coefficients satisfy (3)–(5). Let $\mathcal{E}(\lambda_0,\lambda_1)$ be the class of real measurable functions a_{ij}^k, $k \in \overline{\mathbb{N}}$, on Ω satisfying (3) and (4) uniformly with respect to k.

Following [18], [7], [20], we make the following

Definition 2.1. We say that a sequence $(a_{ij}^k) \in \mathcal{E}(\lambda_0, \lambda_1)$ *G-converges* to a_{ij}^∞ on Ω as $k \to \infty$ (and write $a_{ij}^k \xrightarrow{G} a_{ij}^\infty$ on Ω) iff for every $f \in H^{-1}(\Omega)$, we have $u_k \to u_\infty$ weakly in $H_0^1(\Omega)$, where u_k, $k \in \overline{\mathbb{N}}$, denotes the solution of the problem

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(a_{ij}^k(x)\dfrac{\partial u_k}{\partial x_j}\right) = f \quad \text{in } \Omega, \\ u_k \in H_0^1(\Omega). \end{cases}$$

For the definition and properties of G-convergence in the abstract case see [18]–[20].

For the reader's convenience, let us also recall the notion of sequential Γ-convergence for function(al)s of two variables. The case of one variable is trivial since it suffices to omit the other variable. Let $\mathcal{X}_i$, $i = 1, 2$, be topological spaces, let $x_i \in \mathcal{X}_i$ and $S_i = S(x_i) = \{(x_i^k) \subset \mathcal{X}_i \mid x_i^k \to x_i \text{ in } \mathcal{X}_i \text{ as } k \to \infty\}$. Given the functionals $J_k : \mathcal{X}_1 \times \mathcal{X}_2 \to \overline{\mathbb{R}}$, $k \in \overline{\mathbb{N}}$, we adopt

Definition 2.2. We say that the sequence J_k is $\Gamma_{\mathrm{seq}}(\mathcal{X}_1^-, \mathcal{X}_2)$-*convergent* to J_∞ (and write $J_\infty = \Gamma_{\mathrm{seq}}(\mathcal{X}_1^-, \mathcal{X}_2) \lim J_k$) iff for every $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$, the four extended-real numbers

$$\begin{aligned}
\Gamma_{\mathrm{seq}}(\mathcal{X}_1^-, \mathcal{X}_2^-) \liminf J_k(x_1, x_2) &= \inf_{S_1} \inf_{S_2} \liminf_{k\to\infty} J_k(x_1^k, x_2^k), \\
\Gamma_{\mathrm{seq}}(\mathcal{X}_1^-, \mathcal{X}_2^-) \limsup J_k(x_1, x_2) &= \inf_{S_1} \inf_{S_2} \limsup_{k\to\infty} J_k(x_1^k, x_2^k), \\
\Gamma_{\mathrm{seq}}(\mathcal{X}_1^-, \mathcal{X}_2^+) \liminf J_k(x_1, x_2) &= \inf_{S_1} \sup_{S_2} \liminf_{k\to\infty} J_k(x_1^k, x_2^k), \\
\Gamma_{\mathrm{seq}}(\mathcal{X}_1^-, \mathcal{X}_2^+) \limsup J_k(x_1, x_2) &= \inf_{S_1} \sup_{S_2} \limsup_{k\to\infty} J_k(x_1^k, x_2^k)
\end{aligned}$$

are equal to $J_\infty(x_1, x_2)$.

In the sequel, since we only use the sequential Γ-convergence, we shall omit the subscript "seq" appearing in the above definition.

Remark 2.1. If $\mathcal{X}$ is a topological space and $f_k : \mathcal{X} \to \overline{\mathbb{R}}$ satisfy

(i) for every $x \in \mathcal{X}$ and every sequence $x_k \to x$,

$$f_\infty(x) \le \liminf_{k\to\infty} f_k(x_k);$$

(ii) for every $x \in \mathcal{X}$, there exists a sequence $x_k \to x$ such that

$$f_\infty(x) = \lim_{k\to\infty} f_k(x_k),$$

then $f_\infty = \Gamma(\mathcal{X}^-) \lim f_k$.

The property which motivates the introduction of Γ-convergence in the calculus of variations is the following

Proposition 2.1. *Let* $f_k : \mathcal{X} \to \overline{\mathbb{R}}$ *and* $f_\infty = \Gamma(\mathcal{X}^-)\lim f_k$. *Assume that there exists a sequence* $(\widehat{x}_k) \subset \mathcal{X}$ *such that* $\widehat{x}_k \to \widehat{x}$ *and*

$$\liminf_{k\to\infty} f_k(\widehat{x}_k) = \liminf_{k\to\infty}(\inf_{\mathcal{X}} f_k). \tag{6}$$

Then

$$f_\infty(\widehat{x}) = \min_{\mathcal{X}} f_\infty = \lim_{k\to\infty}(\inf_{\mathcal{X}} f_k).$$

For further information on Γ-convergence we refer to [5], [6] and the references cited there.

3. Main result. Consider first the initial value problem

$$\begin{cases} H_{\mathcal{A}} y = f & \text{in } Q,\\ \partial y/\partial\nu_{\mathcal{A}} = v & \text{on } \Sigma,\\ y(0) = \phi & \text{in } \Omega,\\ y'(0) = \psi & \text{in } \Omega, \end{cases} \tag{7}$$

where $\partial/\partial\nu_{\mathcal{A}} = a_{ij}(x,t)\cos(\nu, x_i)\partial/\partial x_j$ denotes the outward conormal derivative on Γ corresponding to the coefficients $a_{ij}(x,t)$ (see e.g. [11]). As mentioned earlier, in order to study (7), we use the method of transposition (cf. [11] or [9]). To this end, taking $V = H^1(\Omega)$, $H = L^2(\Omega)$, we recall (see Theorem 1.1 in Chapter IV of [9]) that given $\bar{f} \in \mathcal{H}$, there exists a unique $z \in \mathcal{V}$ with $z' \in \mathcal{H}$ satisfying

$$\begin{cases} H_{\mathcal{A}} z = \bar{f} & \text{in } Q,\\ \partial z/\partial\nu_{\mathcal{A}} = 0 & \text{on } \Sigma,\\ z(T) = 0 & \text{in } \Omega,\\ z'(T) = 0 & \text{in } \Omega. \end{cases}$$

Let $\mathcal{Z}$ denote the space spanned by z as $\bar{f}$ ranges over $\mathcal{H}$.

Definition 3.1 A function $y \in \mathcal{H}$ is called a *weak solution* to (7) iff

$$\langle\langle y, H_{\mathcal{A}} z\rangle\rangle = \langle\langle f, z\rangle\rangle - \langle \phi, z'(0)\rangle + \langle \psi, z(0)\rangle + \langle\langle v, z\rangle\rangle_\Sigma$$

for every $z \in \mathcal{Z}$.

It is well known (see Chapter IV of [9]) that for every $H_{\mathcal{A}} \in H(\lambda_0, \lambda_1, M)$, $f \in \mathcal{H}$, $v \in L^2(\Sigma)$, $\phi \in H$, $\psi \in V'$ there exists a unique weak solution to (7) in the sense of Definition 3.1. Moreover, this solution depends continuously on the data f, v, ϕ, ψ.

Now, given a sequence of operators $(H_{\mathcal{A}_k})$, $k \in \overline{\mathbb{N}}$, of class $H(\lambda_0, \lambda_1, M)$, consider the family of initial value problems (the state equations in control problems):

$$\begin{cases} H_{\mathcal{A}_k} y = f & \text{in } Q,\\ \partial y/\partial\nu_{\mathcal{A}_k} = v & \text{on } \Sigma,\\ y(0) = \phi & \text{in } \Omega,\\ y'(0) = \psi & \text{in } \Omega. \end{cases} \tag*{$(8)_k$}$$

We define the sets of admissible control-state pairs by

(9) $$\Lambda_k = \{(u,y) \in U \times Y \mid (u,y) \text{ satisfies } (8)_k\},$$

where $u = (f, v, \phi, \psi)$, $U = \mathcal{H} \times L^2(\Sigma) \times V \times H$ and $Y = \mathcal{H}$.

We study the sequence of control problems

$(10)_k$ $$\min\{J_k(u,y) + \chi_{\Lambda_k}(u,y) \mid (u,y) \in U \times Y\}, \qquad k \in \overline{\mathbb{N}}.$$

The cost functionals in $(10)_k$ are given by

(11) $$J_k((f,v,\phi,\psi),y) = J_k^{(1)}(f,y) + J_k^{(2)}(v) + J_k^{(3)}(\phi) + J_k^{(4)}(\psi), \qquad k \in \overline{\mathbb{N}},$$

where

$$J_k^{(1)}(f,y) = \int_Q F_k^{(1)}(x,t,y(x,t),f(x,t))\,dx\,dt, \qquad (f,y) \in \mathcal{H} \times \mathcal{H},$$

$$J_k^{(2)}(v) = \int_\Sigma F_k^{(2)}(x,t,v(x,t))\,d\sigma\,dt, \qquad v \in L^2(\Sigma),$$

$$J_k^{(3)}(\phi) = \int_\Omega F_k^{(3)}(x,\phi(x),\nabla\phi(x))\,dx, \qquad \phi \in V,$$

$$J_k^{(4)}(\psi) = \int_\Omega F_k^{(4)}(x,\psi(x))\,dx, \qquad \psi \in H.$$

We need the following hypotheses on the integrands $F_k^{(i)}$, $i = 1,2,3,4$, $k \in \overline{\mathbb{N}}$ ($F_k^{(i)*}$ denotes the polar function to $F_k^{(i)}$ with respect to the starred variables):

(H_1) (a) $F_k^{(1)} : Q \times \mathbb{R}^2 \to \mathbb{R}$ are Borel functions and $F_k^{(1)}(x,t,y,\cdot)$ are convex for all $(x,t) \in Q$ and $y \in \mathbb{R}$,

(b) there exists $c \geq 1$ such that
$$|z|^2 \leq F_k^{(1)}(x,t,y,z) \leq c(1 + |y|^2 + |z|^2),$$

(c) $|F_k^{(1)}(x,t,y,z) - F_k^{(1)}(x,t,y_1,z)| \leq \varrho(|y-y_1|)(1 + |y|^2 + |z|^2)$ for $(x,t) \in Q$, $z \in \mathbb{R}$, and $y, y_1 \in \mathbb{R}$ such that $|y - y_1| \leq 1$, and $\varrho : [0,1) \to \mathbb{R}$ is a continuous, increasing function with $\varrho(0) = 0$,

(d) $F_k^{(1)*}(\cdot,\cdot,y,z^*) \to F_\infty^{(1)*}(\cdot,\cdot,y,z^*)$ in w-$L^1(Q)$ as $k \to \infty$, for all $y, z^* \in \mathbb{R}$;

(H_2) (a) $F_k^{(2)} : \Sigma \times \mathbb{R} \to \mathbb{R}$ are Borel functions, $F_k^{(2)}(x,t,\cdot)$ are convex for all $(x,t) \in \Sigma$,

(b) there exists $c > 0$ such that $c|z|^2 \leq F_k^{(2)}(x,t,z)$ for $(x,t) \in \Sigma$ and $z \in \mathbb{R}$,

(c) $F_k^{(2)*}(\cdot,\cdot,z^*) \to F_\infty^{(2)*}(\cdot,\cdot,z^*)$ in w-$*$-$L^\infty(\Sigma)$ as $k \to \infty$, for all $z^* \in \mathbb{R}$;

(H$_3$) (a) $F_k^{(3)} : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ are Borel functions, $F_k^{(3)}(x,\cdot)$ are convex for all $x \in \Omega$,

(b) $c_1|z|^2 \le F_k^{(3)}(x,z) \le c_2|z|^2$, with $0 < c_1 \le c_2$, for all $x \in \Omega$ and $z \in \mathbb{R}^{n+2}$,

(c) both $F_k^{(3)}(\cdot,z) \to F_\infty^{(3)}(\cdot,z)$ and $F_k^{(3)*}(\cdot,z^*) \to F_\infty^{(3)*}(\cdot,z^*)$ in w-$*$-$L^\infty(\Omega)$ as $k \to \infty$ for all $z, z^* \in \mathbb{R}^{n+1}$;

(H$_4$) (a) $F_k^{(4)} : \Omega \times \mathbb{R} \to \mathbb{R}$ are Borel functions, $F_k^{(4)}(x,\cdot)$ are convex for all $x \in \Omega$,

(b) there exists $c > 0$ such that $c|v|^2 \le F_k^{(4)}(x,v)$ for $x \in \Omega$ and $v \in \mathbb{R}$,

(c) $F_k^{(4)*}(\cdot,v^*) \to F_\infty^{(4)*}(\cdot,v^*)$ in w-$*$-$L^\infty(\Omega)$ as $k \to \infty$ for all $v^* \in \mathbb{R}$.

As regards the hyperbolic operators, we assume that

(H$_5$) (a) $H_{\mathcal{A}_k} \in H(\lambda_0,\lambda_1,M)$ for $k \in \overline{\mathbb{N}}$,

(b) $a_{ij}^k(\cdot,t) \xrightarrow{G} a_{ij}^\infty(\cdot,t)$ on Ω, for all $t \in [0,T]$, as $k \to \infty$.

We have the following

THEOREM 3.1. (i) *If hypotheses* (a), (b), (c) *of* (H$_1$), (a), (b) *of* (H$_2$), (a), (b) *of* (H$_3$), (a), (b) *of* (H$_4$) *and* (a) *of* (H$_5$) *hold*, *then for every fixed* $k \in \overline{\mathbb{N}}$, *there exists an optimal solution* $(\widetilde{u}_k,\widetilde{y}_k) \in U \times Y$ *to control problem* $(10)_k$ *with* J_k *defined by* (11).

(ii) *If hypotheses* (H$_1$)–(H$_5$) *are satisfied*, *then the sequence* $(\widetilde{u}_k,\widetilde{y}_k)$ *has a subsequence converging in* $(w\text{-}U) \times (s\text{-}Y)$ *topology to an optimal solution* $(\widetilde{u}_\infty,\widetilde{y}_\infty)$ *of the limit problem* $(10)_\infty$. *Moreover*, *the sequence of minimal values* $(J_k(\widetilde{u}_k,\widetilde{y}_k))$ *converges to the minimal value* $J_\infty(\widetilde{u}_\infty,\widetilde{y}_\infty)$.

(iii) *If the limit problem has a unique optimal solution*, *then the sequence* $(\widetilde{u}_k,\widetilde{y}_k)$ *itself converges to* $(\widetilde{u}_\infty,\widetilde{y}_\infty)$.

4. Proof of the main result. The proof of Theorem 3.1 is based on the following abstract result concerning the asymptotic behaviour of $(1)_k$ and obtained in [2] as the sequential version of Proposition 2.1.

PROPOSITION 4.1. *Let* U, Y *be topological spaces and let* (u_k,y_k) *be optimal* (*or quasi-optimal in the sense of* (6)) *solutions to* $(1)_k$, $k \in \overline{\mathbb{N}}$, *such that* $(u_k,y_k) \to (u_\infty,y_\infty)$ *in* $U \times Y$. *Let*

$$J_\infty = \Gamma(U^-,Y)\lim J_k, \tag{12}$$

$$\chi_{\Lambda_\infty} = \Gamma(U,Y^-)\lim \chi_{\Lambda_k}. \tag{13}$$

Then (u_∞,y_∞) *is an optimal solution to* $(1)_\infty$.

We first remark (following [6], [2], [8]) that for $\Lambda_k \subset U \times Y$, $k \in \mathbb{N}$, the Γ-limit of the indicator function of Λ_k is also the indicator function of a set Λ_∞ and (13) is equivalent to the following two conditions:

(14) if $u_k \to u_\infty, y_k \to y_\infty$ and $(u_k, y_k) \in \Lambda_k$ for infinitely many k then $(u_\infty, y_\infty) \in \Lambda_\infty$,

(15) if $u_k \to u_\infty, (u_\infty, y_\infty) \in \Lambda_\infty$ then there are $y_k \to y_\infty$ and $k_0 \in \mathbb{N}$ such that $(u_k, y_k) \in \Lambda_k$ for $k \geq k_0$.

Next, in order to apply Proposition 4.1, we will show two lemmas which elaborate on conditions (12) and (14), (15) in the case when Λ_k and J_k are defined by (9) and (11), respectively.

Lemma 4.1. *If hypotheses* (H_1)–(H_4) *hold*, *then*

$$J_\infty = \Gamma(w\text{-}U^-, s\text{-}Y) \lim J_k,$$

where J_k, $k \in \overline{\mathbb{N}}$, *are defined by* (11).

Proof. First of all, we can show that the functionals $J_k^{(i)}$ are Γ-convergent to $J_\infty^{(i)}$, $i = 1, 2, 3, 4$. Namely adding t to the independent variable x and augmenting the space variable from $\mathbb{R}$ to $\mathbb{R}^2$ in Lemma 3.1 of [2] and Theorem 3.4 of [12], we get

$$J_\infty^{(1)} = \Gamma(w\text{-}\mathcal{H}^-, s\text{-}\mathcal{H}) \lim J_k^{(1)}, \qquad J_\infty^{(2)} = \Gamma(w\text{-}L^2(\Sigma)^-) \lim J_k^{(2)},$$

respectively. Under our assumptions, from Theorems 3.3 and 3.4 of [12] we have (see also [8])

$$J_\infty^{(3)} = \Gamma(w\text{-}V^-) \lim J_k^{(3)}.$$

Again, by Theorem 3.4 of [12], we directly deduce that

$$J_\infty^{(4)} = \Gamma(w\text{-}H^-) \lim J_k^{(4)}.$$

Now, using Lemma 2.8 of [14], we calculate the sum of these four Γ-limits and hence we obtain the result. ∎

Let $(f_k, v_k, \phi_k, \psi_k) \in \mathcal{H} \times L^2(\Sigma) \times H \times V'$, $k \in \overline{\mathbb{N}}$, and suppose hypothesis (H_5)(a) holds. We denote by $y_k \in \mathcal{H}$, $k \in \overline{\mathbb{N}}$, the unique solutions (in the sense of Definition 3.1) of the problem $(8)_k$ with right hand sides f_k, v_k, ϕ_k and ψ_k.

Lemma 4.2. *If hypothesis* (H_5) *holds and*

$$\text{(16)} \qquad \begin{aligned} &f_k \to f_\infty \quad \text{in } w\text{-}\mathcal{H}, &&\phi_k \to \phi_\infty \quad \text{in } s\text{-}H,\\ &v_k \to v_\infty \quad \text{in } w\text{-}L^2(\Sigma), &&\psi_k \to \psi_\infty \quad \text{in } s\text{-}V', \end{aligned}$$

as $k \to \infty$, *then*

$$\text{(17)} \qquad y_k \to y_\infty \quad \text{in } s\text{-}\mathcal{H},$$

where y_∞ *is the solution* (*unique in the same sense*) *of the problem* $(8)_\infty$ *corresponding to* $f_\infty, v_\infty, \phi_\infty$ *and* ψ_∞.

Proof. From the hypotheses and from the uniform *a priori* estimate (see [9])

$$\|y_k\|_{\mathcal{H}} \le c(\|f_k\|_{\mathcal{H}} + \|v_k\|_{L^2(\Sigma)} + |\phi_k| + \|\psi_k\|_{V'}), \tag{18}$$

where c is independent of k, we deduce that (y_k) lies in a bounded subset of $\mathcal{H}$. Therefore passing to a subsequence if necessary, again called y_k, we may assume

$$y_k \to y_0 \quad \text{in } w\text{-}\mathcal{H} \tag{19}$$

with some $y_0 \in \mathcal{H}$. In what follows, we shall show that $y_0 = y_\infty$.

To this end, fix $\bar{f} \in \mathcal{H}$. By definition, y_k satisfies the equality

$$\langle\!\langle y_k, H_{\mathcal{A}_k} z_k \rangle\!\rangle = \langle\!\langle f_k, z_k \rangle\!\rangle - \langle \phi_k, z_k'(0) \rangle + \langle \psi_k, z_k(0) \rangle + \langle\!\langle v_k, z_k \rangle\!\rangle_\Sigma, \tag{20}$$

where $z_k \in \mathcal{V}$, $k \in \mathbb{N}$, is the solution to

$$(21)_k \qquad \begin{cases} H_{\mathcal{A}_k} z_k = \bar{f} & \text{in } Q, \\ \partial z_k / \partial \nu_{\mathcal{A}_k} = 0 & \text{on } \Sigma, \\ z_k(T) = 0 & \text{in } \Omega, \\ z_k'(T) = 0 & \text{in } \Omega. \end{cases}$$

Now, by hypothesis (H_5), from Lemma 2.2 of [13] (see also [4]), we have

$$(22) \qquad \begin{array}{ll} z_k \to z_\infty & \text{in } w\text{-}*\text{-}L^\infty(0,T;V) \text{ and in } s\text{-}C([0,T];H), \\ z_k' \to z_\infty' & \text{in } w\text{-}*\text{-}L^\infty(0,T;H) \text{ and in } s\text{-}C([0,T];V'), \end{array}$$

where z_∞ is a solution of the limit problem $(21)_\infty$.

Since (z_k) and (z_k') are bounded in $\mathcal{V}$ and $\mathcal{H}$, respectively, according to a well known compactness theorem (see e.g. [11], [9]), (z_k) is a precompact subset of some $L^2(0,T;H^\beta(\Omega))$, where $\beta \in (1/2, 1)$. Thus by the trace theorem we conclude that $(z_k|_\Sigma)$ is precompact in $L^2(\Sigma)$. Hence without loss of generality, we can suppose that

$$z_k|_\Sigma \to z_\infty|_\Sigma \quad \text{in } s\text{-}L^2(\Sigma). \tag{23}$$

By (16), (19), (22) and (23) we can pass to the limit in (20) as $k \to \infty$ to get

$$\langle\!\langle y_0, \bar{f} \rangle\!\rangle = \langle\!\langle f_\infty, z_\infty \rangle\!\rangle - \langle \phi_\infty, z_\infty'(0) \rangle + \langle \psi_\infty, z_\infty(0) \rangle + \langle\!\langle v_\infty, z_\infty \rangle\!\rangle_\Sigma. \tag{24}$$

Taking into account that z_∞ satisfies $(21)_\infty$, we conclude from the arbitrariness of $\bar{f}$ that y_0 is a weak solution of $(8)_\infty$ corresponding to $f_\infty, v_\infty, \phi_\infty$ and ψ_∞. By the uniqueness of solutions to this problem we get $y_0 = y_\infty$ and $y_k \to y_\infty$ in w-$\mathcal{H}$. To conclude, it is enough to show that the last convergence is strong. Putting in $(21)_k$, $\bar{f} = y_k$ and $\bar{f} = y_\infty$, respectively, and using (24), a short computation gives

$$\|y_k - y_\infty\|^2_{\mathcal{H}} = \langle\!\langle f_\infty, z_\infty \rangle\!\rangle - \langle\!\langle f_k, z_k \rangle\!\rangle + \langle \psi_\infty, z_\infty(0)\rangle - \langle \psi_k, z_k(0)\rangle + \langle \phi_k, z'_k(0)\rangle - \langle \phi_\infty, z'_\infty(0)\rangle + \langle\!\langle v_\infty, z_\infty \rangle\!\rangle_\Sigma - \langle\!\langle v_k, z_k \rangle\!\rangle_\Sigma.$$

From (16), (22) and (23) it follows that each term on the right hand side tends to zero as $k \to \infty$, showing (17). ■

Proof of Theorem 3.1. For the proof of assertion (i) of Theorem 3.1 we apply the direct method. Fix $k \in \overline{\mathbb{N}}$ and let $\{(u_n, y_n)\}$ be a minimizing sequence in $U \times Y$, i.e.

$$J_k(u_n, y_n) \to \inf\{J_k(u, y) \mid (u, y) \in \Lambda_k\},$$

where $u_n = (f_n, v_n, \phi_n, \psi_n)$. Note that from assumptions (H_1)(b), (H_2)(b), (H_3)(b), (H_4)(b) we have that $\|u_n\|_U \le c$, where $U = \mathcal{H} \times L^2(\Sigma) \times V \times H$ and c is independent of n. Next, in view of the compactness of the embeddings $V \subset H \subset V'$ and from the reasoning analogous to that in Lemma 4.2, we deduce that $\{(u_n, y_n)\}$ is compact in $(w\text{-}U) \times (s\text{-}Y)$ topology. Since the Γ-limit of a constant sequence of functionals gives the l.s.c. envelope of the functional (see e.g. [6], [5], [8]), for every fixed $k \in \overline{\mathbb{N}}$, we have the sequential l.s.c. of J_k in the same topology. This completes the proof of (i).

Proof of (ii). As above, we find that the sequence of optimal solutions to $(10)_k$ is compact in $(w\text{-}U) \times (s\text{-}Y)$ topology. Furthermore, from Lemma 4.2 and from the compactness of the embedding $V \subset H$, we find that the conditions (14) and (15) hold for Λ_k defined by (9). This proves that

$$\chi_{\Lambda_\infty} = \Gamma(w\text{-}U, s\text{-}Y^-) \lim \chi_{\Lambda_k}.$$

From this relation and from Lemma 4.1, it follows that we may apply Proposition 4.1, which in turn immediately implies the assertion (ii) of the theorem. The convergence of the minimal values is a consequence of Proposition 2.1. Finally, (iii) follows directly from (i) and (ii). ■

Acknowledgements. The author would like to thank Professor Z. Denkowski for helpful conversations and the referee for his (her) remarks.

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Reçu par la Rédaction le 8.1.1993
Révisé le 15.12.1994