ANNALES
POLONICI MATHEMATICI
LXII.2 (1995)

The algebra generated by a pair of operator weighted shifts

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Abstract. We present a model for two doubly commuting operator weighted shifts. We also investigate general pairs of operator weighted shifts. The above model generalizes the model for two doubly commuting shifts. WOT-closed algebras for such pairs are described. We also deal with reflexivity for such pairs assuming invertibility of operator weights and a condition on the joint point spectrum.

1. Introduction. In what follows $L(\mathbf{H})$ denotes the algebra of all (linear, bounded) operators in a complex separable Hilbert space $\mathbf{H}$ and $I_{\mathbf{H}}$ or I stands for the identity in $\mathbf{H}$. By an algebra of operators we always mean a WOT(= weak operator topology)-closed subalgebra of $L(\mathbf{H})$ with unit $I_{\mathbf{H}}$. If $\mathcal{S}\subset L(\mathbf{H})$, then $\mathcal{W}(\mathcal{S})$ and $\operatorname{Lat}\mathcal{S}$ stand for the WOT-closed algebra generated by $\mathcal{S}$ and the lattice of all (closed) invariant subspaces for $\mathcal{S}$, respectively. $\operatorname{Alg}\operatorname{Lat}\mathcal{S}$ stands for the algebra of all operators on $\mathbf{H}$ which leave invariant all subspaces from $\operatorname{Lat}\mathcal{S}$. An algebra $\mathcal{W}$ is called *reflexive* if $\mathcal{W} = \operatorname{Alg}\operatorname{Lat}\mathcal{W}$. A family $\mathcal{S}\subset L(\mathbf{H})$ is called *reflexive* if so is $\mathcal{W}(\mathcal{S})$. Operators $T_1, T_2 \in \mathbf{H}$ *doubly commmute* if T_1 commutes with T_2 and T_1 commutes with T_2^*.

In the paper we present a model for two doubly commuting operator weighted shifts (Section 2) which generalizes a model for two doubly commuting (but not operator or weighted) shifts (see Theorem 1 of [6]). In [4], the general pair of (neither operator nor weighted) shifts was considered. The main purpose of the paper is to investigate general pairs of operator weighted shifts (for definition see Section 3). This generalizes two doubly commuting operator weighted shifts, in view of the model given in Section 2. On the other hand, a special case of a general pair of scalar weighted shifts was considered in [1]. In what follows we describe the WOT-closed algebra generated by a pair of operator weighted shifts (Theorem 6.2). We also deal

1991 *Mathematics Subject Classification*: Primary 47D25.
Key words and phrases: pairs of operator weighted shifts, reflexive algebras.

with reflexivity for such a pair (Theorem 6.7) assuming invertibility of operator weights and a condition on the joint point spectrum. The corresponding results for a single operator weighted shift were given by Lambert [2].

In Section 4 we present basic properties of the joint point spectrum and joint eigenvalues. In Section 5 we present the case of pairs of operator weighted shifts "defined on the pairs of non-negative integers". In view of the model given in Section 2, it describes two doubly commuting operator weighted shifts. Section 6 discusses the general case. Examples are given in Section 7.

In what follows, let G_0 denote the set of all pairs of non-negative integers and G be the set of all pairs of integers. If $\phi = (\phi^{(1)}, \phi^{(2)})$, $\psi = (\psi^{(1)}, \psi^{(2)}) \in G$, then we write $\phi \le \psi$ if and only if $\phi^{(1)} \le \psi^{(1)}$, $\phi^{(2)} \le \psi^{(2)}$. Let $\varepsilon_1 = (1,0)$, $\varepsilon_2 = (0,1)$.

2. Model for two doubly commuting operator weighted shifts. The main result of this section is

PROPOSITION 2.1 *Let $T_1, T_2 \in L(\mathbf{K})$ be operator weighted shifts whose weights have dense ranges. Assume that T_1, T_2 doubly commute. Then there is a wandering subspace L and families $\{A_\phi^{(l)} : \phi \in G_0\}$ $(l = 1,2)$ such that $\mathbf{K} = \bigoplus_{\phi\in G_0} L_\phi$, where $L_\phi = L$ and for $f = \sum_{\phi\in G_0} f_\phi e_\phi$, $f_\phi \in L$ and e_ϕ indicates that $f_\phi e_\phi$ is an element of L_ϕ, we have*

$$T_i f = \sum_{\phi\in G_0} (A_\phi^{(i)} f_\phi) e_{\phi+\varepsilon_i} \quad \text{for } i = 1,2. \tag{2.1}$$

Moreover,

$$\overline{T_1^i T_2^j L_{(0,0)}} = L_{(i,j)} \quad \text{for all } (i,j) \in G_0 \tag{2.2}$$

and

$$\begin{aligned} A^{(1)}_{\phi+\varepsilon_2} A^{(2)}_\phi &= A^{(2)}_{\phi+\varepsilon_1} A^{(1)}_\phi, \\ A^{(1)}_\phi A^{(2)*}_\phi &= A^{(2)*}_{\phi+\varepsilon_1} A^{(1)}_{\phi+\varepsilon_2}, \\ A^{(2)}_\phi A^{(1)*}_\phi &= A^{(2)*}_{\phi+\varepsilon_2} A^{(1)}_{\phi+\varepsilon_1} \quad \text{for all } \phi \in G_0. \end{aligned} \tag{2.3}$$

Proof. Since T_1 is a weighted shift, there is a subspace H such that $\mathbf{K} = \bigoplus_{i=0}^\infty H_i$, where $H_i = H$, and for $f = \sum_{i=0}^\infty f_i e_i \in \mathbf{K}$, $T_1 f = \sum_{i=0}^\infty B_i^{(1)} f_i e_{i+1}$ for some bounded family of operators $\{B_i^{(1)} \in L(H) : i = 1,2,\ldots\}$. The operator T_2 can be represented as a matrix, say $[X_{ij}]$. Thus by Lemma 2.1 of [3], $X_{ij} = 0$ for $i < j$. The operator T_2^* is represented by $[X_{ji}^*]$. It also commutes with T_1, and thus $X_{ij}^* = 0$ for $i > j$. Hence $X_{ij} = 0$ for $i \ne j$. Therefore H_i reduces T_2 for all i.

The operators $T_2|_{H_i} = X_{ii}$, for all i, are weighted shifts, and thus there are subspaces $L^{(i)}$ such that $H_i = \bigoplus_{j=0}^\infty L_{(i,j)}$, where $L_{(i,j)} = L^{(i)}$, and for $f^{(i)} = \sum_{j=0}^{\infty} f_j^{(i)} e_j^{(i)} \in H_i$, $T_2|_{H_i} f^{(i)} = \sum_{j=0}^{\infty} B_{(i,j)}^{(2)} f_j^{(i)} e_{j+1}^{(i)}$ for some bounded families of operators $\{B_{(i,j)}^{(2)} \in L(L^{(i)}) : j = 1, 2, \ldots\}$.

Now we prove that $T_1 L_{(i,0)} \subset L_{(i+1,0)}$. Assume that, on the contrary, there is $x \in L_{(i,0)}$ such that $(T_1 x, T_2^j|_{H_{i+1}} y) \neq 0$ for some $y \in H_{i+1}$, $j > 0$. Then

$$(T_1 x, T_2^j|_{H_{i+1}} y) = (T_2^{j*} T_1 x, y) = (T_1 T_2^{j*} x, y) = 0$$

since $x \in L_{(i,0)}$ and $T_2^* x = 0$. Hence $T_1 L_{(i,0)} \subset L_{(i+1,0)}$.

Thus $T_1 T_2^j L_{(i,0)} = \underline{T_2^j T_1 L_{(i,0)}} \subset T_2^j L_{(i+1,0)} \subset L_{(i+1,j)}$. The weights of T_2 have dense ranges, so $\overline{T_2^j L_{(i,0)}} = L_{(i,j)}$. Hence $T_1 L_{(i,j)} \subset L_{(i+1,j)}$.

To prove (2.2), assume first that there is $0 \neq y \in L_{(i+1,0)}$ such that $T_1 L_{(i,0)} \perp y$. Then $T_2^* y = 0$ and $T_1^* y \neq 0$ since $y \in H_{i+1}$ and $\ker B_i^{(1)*} = \{0\}$. Also, $T_1^* y \in H_i$ and $T_1^* y \perp L_{(i,0)}$. This implies that $T_2^* T_1^* y \neq 0$ since $\ker B_{(i,j)}^{(2)*} = \{0\}$ for $j = 0, 1, \ldots$ Then $T_1^* T_2^* y \neq 0$, contrary to $T_2^* y = 0$. Thus $\overline{T_1 L_{(i,0)}} = L_{(i+1,0)}$. Hence $\overline{T_1^i L_{(0,0)}} = L_{(i,0)}$. We also have $\overline{T_2 L_{(i,j)}} = L_{(i,j+1)}$, since weights of T_2 have dense ranges. Hence $\overline{T_1^i T_2^j L_{(0,0)}} = L_{(i,j)}$.

Since $H_{i_1} = H_{i_2}$, we have $L^{(i_1)} = L^{(i_2)} =: L$. Define $A_{(i,j)}^{(1)} = B_i^{(1)}|_{L_{(i,j)}}$ and $A_{(i,j)}^{(2)} = B_{(i,j)}^{(2)}$. It is easy to see that (2.1) is satisfied. The operators T_1, T_2 doubly commute, so (2.1) implies (2.3).

In Corollary 5.6 we will consider reflexivity of the above.

3. Definition and elementary properties of a pair of operator weighted shifts. Following [4], we introduce some notation and definitions. Namely, a subset $X \subset G$ is called a *diagram* if $\phi \in X$, $s \in G_0$ implies $\phi + s \in X$. The set of all diagrams is denoted by $\mathbf{X}$. For $\phi \in G$ we define $E_\phi = \{X \in \mathbf{X} : \phi \in X\}$. It is obvious that $E_\phi \subset E_{\phi+s}$ $(\phi \in G,\ s \in G_0)$. Let $\mathcal{B}$ be the σ-algebra generated by E_ϕ $(\phi \in G)$ and μ be a positive finite measure on $(\mathbf{X}, \mathcal{B})$.

Set $\mathcal{H} = \bigoplus_{\phi \in G} H_\phi$, where $H_\phi = \mathbf{H}$ and consider the space $\mathbf{K}$ of all measurable functions $f : \mathbf{X} \to \mathcal{H}$ such that $\int \|f(X)\|^2 \, d\mu \leq \infty$ and $f(X) \in \bigoplus_{\phi \in X} H_\phi$ (we identify functions equal μ-a.e.). Then $\mathbf{K}$ is a Hilbert space with inner product $(f, g) = \int (f(X), g(X))_{\mathcal{H}} \, d\mu(X)$. Each element of $\mathbf{K}$ can be written as $f = \sum_{\phi \in G} f_\phi(\cdot) e_\phi$, where $f_\phi \in \mathcal{H}_\phi := \{f \in L^2(\mathbf{X}, \mathcal{B}, \mu, \mathbf{H}) : f_\phi(X) = 0\ \mu\text{-a.e. on } X - E_\phi\}$, and e_ϕ indicates that $f_\phi(X) e_\phi$ is an element of H_ϕ for $X \in \mathbf{X}$.

Let $\{A_\phi^{(i)}(\cdot) : \mathbf{X} \to L(\mathbf{H}),\ A_\phi^{(i)}(X) = 0\ \mu\text{-a.e. on } X - E_\phi,\ \phi \in G\}$ $(i = 1, 2)$ be sets of uniformly bounded operator functions $(\sup\{\operatorname{supess} \|A_\phi^{(i)}(\cdot)\| : \phi \in G,\ i = 1, 2\} = C < \infty)$. We will consider $A_\phi^{(i)}(\cdot)$ as operators on $\mathcal{H}_\phi$. In what follows we assume that $\ker A_\phi^{(i)}(\,\cdot\,) = \{0\}$, $\phi \in G$, $i = 1, 2$. Hence we can define operators T_1, T_2 on $\mathbf{K}$. If $f = \sum_{\phi\in G} f_\phi(\,\cdot\,)e_\phi$ then

$$T_i f = \sum_{\phi\in G} A_\phi^{(i)}(\,\cdot\,) f_\phi(\,\cdot\,) e_{\phi+\varepsilon_i}, \quad i = 1, 2. \tag{3.1}$$

Since $\mathbf{X}$ is the set of diagrams, we have $T_i f \in \mathbf{K}$, $i = 1, 2$. The sets $\{A_\phi^{(i)}(\,\cdot\,)\}$, $i = 1, 2$, are uniformly bounded, and thus $T_1, T_2 \in L(\mathbf{K})$. Let us call $\{T_1, T_2\}$ a *pair of operator weighted shifts*.

LEMMA 3.1. *The operators T_1, T_2 commute if and only if*

$$A_{\phi+\varepsilon_2}^{(1)}(\,\cdot\,)A_\phi^{(2)} = A_{\phi+\varepsilon_1}^{(2)}(\,\cdot\,)A_\phi^{(1)}(\,\cdot\,) \quad \textit{for all } \phi \in G. \tag{3.2}$$

To prove the lemma, it is enough to compare T_1T_2 with T_2T_1 on an element of $\mathbf{K}$. From now on we assume that T_1, T_2 commute.

For $s = (s_1, s_2) \in G_0$ and $\phi \in G$ let $T^s = T_1^{s_1}T_2^{s_2}$ and

$$\begin{aligned} S_\phi^s(\,\cdot\,) = {} & A_{\phi+s-\varepsilon_2}^{(2)}(\,\cdot\,)\ldots A_{\phi+s_1\varepsilon_1+\varepsilon_2}^{(2)}(\,\cdot\,)A_{\phi+s_1\varepsilon_1}^{(2)}(\,\cdot\,) \\ & \cdot A_{\phi+(s_1-1)\varepsilon_1}^{(1)}(\,\cdot\,)\ldots A_{\phi+\varepsilon_1}^{(1)}(\,\cdot\,)A_\phi^{(1)}(\,\cdot\,). \end{aligned} \tag{3.3}$$

Hence, for $f = \sum_{\phi\in G} f_\phi(\,\cdot\,)e_\phi$,

$$T^s f = \sum_{\phi\in G} S_\phi^s(\,\cdot\,) f_\phi(\,\cdot\,) e_{\phi+s}. \tag{3.4}$$

Now, if $\phi \in G$ then we write $G_\phi = \phi + G_0 = \{\phi + s : s \in G_0\}$ and $L_\phi = \{f \in \mathbf{K} : f(X) \in \bigoplus_{\psi\in G_\phi} H_\psi$ μ-a.e. and $f(X) = 0$ μ-a.e. on $X - E_\phi\}$.

Let us recall from [4] that

Remark 3.2. $\mathbf{K} = \overline{\operatorname{span}\{L_\phi : \phi \in G\}}$.

The following lemma can be proved similarly to Lemma 5 of [4].

LEMMA 3.3. *The subspace L_ϕ is invariant for T^s if $s \in G_0$ and $\phi \in G$.*

Let us state the basic examples.

EXAMPLE 3.4. Let X be a diagram and $\mu = \delta_X$ (the point mass at X). Then the sets $\{A_\phi^{(i)}\}_{\phi\in X} \subset L(\mathbf{H})$, $i = 1, 2$, generate operators T_1, T_2 on the spaces $\mathbf{K}_X := \{f : f = \sum_{\phi\in X} f_\phi e_\phi,\ f_\phi \in \mathbf{H}\}$.

EXAMPLE 3.5. In view of Proposition 2.1, notice the special case of Example 3.4 with $X = G_0$, since it is a model for two doubly commuting weigted shifts. In that case, we write $\mathbf{K}_0$ instead of $\mathbf{K}_{G_0}$.

4. Joint eigenvalues. We start with recalling the definition of the *joint eigenvalue*. Let $B_1, B_2 \in L(\mathbf{H})$. Then we write $\lambda = (\lambda_1, \lambda_2) \in \sigma_{\mathrm{p}}(B_1, B_2)$ if there exists a non-zero vector $x \in \mathbf{H}$ such that $(B_i - \lambda_i)x = 0$ for $i = 1, 2$. It is easy to see that $\mu(E_\phi) \neq 0$ implies $0 \in \sigma_{\mathrm{p}}((T_1|_{L_\phi}^*, (T_2|_{L_\phi})^*)$ for all $\phi \in G$.

Now we turn our attention to Example 3.5 and consider the operators $T_1, T_2 \in L(\mathbf{K}_0)$ given there. As for a single shift, we can show

LEMMA 4.1. *Let $\lambda = (\lambda_1, \lambda_2)$ be a non-zero joint eigenvalue for T_1^*, T_2^*. Then $\lambda' = (\lambda_1', \lambda_2')$ is a joint eigenvalue for T_1^*, T_2^* if $|\lambda_i'| \le |\lambda_i|$ for $i = 1, 2$.*

It is easy to show the following

LEMMA 4.2. *Let T_1, T_2 be as in Example* 3.5 *and let $f = \sum_{\psi \in G_0} f_\psi e_\psi \in \mathbf{K}_0$. Then $\lambda = (\lambda_1, \lambda_2)$ is a joint eigenvalue for T_1^*, T_2^* with a joint eigenvector f if and only if*

$$A_\psi^{(i)*} f_{\psi+\varepsilon_i} = \lambda_i f_\psi \quad \text{for } \psi \in G_0,\ i = 1, 2.$$

An immediate consequence is

LEMMA 4.3. *Let T_1, T_2 be as in Example* 3.5 *and let $f = \sum_{\psi \in G_0} f_\psi e_\psi \in \mathbf{K}_0$. Then $\lambda = (\lambda_1, \lambda_2)$ is a joint eigenvalue for T_1^*, T_2^* with a joint eigenvector f if and only if*

$$(S_\psi^{\phi-\psi})^* f_\phi = \lambda^{\phi-\psi} f_\psi \quad \text{for } \phi \ge \psi \in G_0.$$

As a consequence, for T_1, T_2 as in (3.1), we have

Remark 4.4. Let $\phi \in G$ and $f = \sum_{\psi \in G_\phi} f_\psi(\cdot) e_\psi \in L_\phi$. The operators $(T_1|_{L_\phi})^*$, $(T_2|_{L_\phi})^*$ have a joint eigenvalue λ with a joint eigenvector f if and only if

$$S_\eta^{\psi-\eta}(\cdot)^* f_\psi(\cdot) = \lambda^{\psi-\eta} f_\eta(\cdot) \quad \text{for } \psi \ge \eta,\ \psi, \eta \in G_\phi.$$

For the proof it is enough to note that

$$L_\phi = L^2\Big(\bigoplus_{\psi \in G_\phi} H_\psi, \mathcal{B}|_{E_\phi}, \mu|_{E_\phi}\Big) \cong \bigoplus_{\psi \in G_\phi} L^2(H_\psi, \mathcal{B}|_{E_\phi}, \mu|_{E_\phi}). \tag{4.1}$$

The following lemma will be of use later.

LEMMA 4.5. *Let $\lambda \in \sigma_{\mathrm{p}}((T_1|_{L_{\phi_0}})^*, (T_2|_{L_{\phi_0}})^*)$. If $\phi \ge \phi_0$ then $\lambda \in \sigma_{\mathrm{p}}((T_1|_{L_\phi})^*, (T_2|_{L_\phi})^*)$.*

Proof. Let $f = \sum_{\psi \in G_{\phi_0}} f_\psi(\cdot) e_\psi$ be a joint eigenvector for the given eigenvalue. We define a vector $\bar{f} = \sum_{\psi \in G_\phi} \bar{f}_\psi(\cdot) e_\psi \in L_\phi$ as follows: if $\psi \in G_\phi$ then

$$\bar{f}_\psi(X) = \begin{cases} f_\psi(X) & \text{if } X \in E_{\phi_0}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\bar{f} \in L_\phi$. Remark 4.4 shows that if f is a joint eigenvector for the joint eigenvalue λ in the space L_{ϕ_0} then so is $\bar{f}$ in the space L_ϕ.

5. Shifts defined on $\mathbf{K}_0$. In this section we will consider the shifts $T_1, T_2 \in L(\mathbf{K}_0)$ defined in Example 3.5. Namely, if $f = \sum_{\phi\in G_0} f_\phi e_\phi$ then

$$T_i f = \sum_{\phi\in G_0} (A_\phi^{(i)} f_\phi) e_{\phi+\varepsilon_i} \quad \text{for } i = 1, 2. \tag{5.1}$$

We will use the following notation: if $D \in L(\mathbf{K}_0)$, then there is the associated matrix $[D_{\alpha\beta}]_{\alpha,\beta\in G_0}$ of operators on $\mathbf{H}$ such that for $f = \sum_{\phi\in G_0} f_\phi e_\phi$,

$$Df = \sum_{\alpha\in G_0} \Big(\sum_{\beta\in G_0} D_{\alpha\beta} f_\beta \Big) e_\alpha. \tag{5.2}$$

As in [2], we can obtain

LEMMA 5.1. *Let $B = [B_{\alpha\beta}]_{\alpha,\beta\in G_0}$ be an operator on $\mathbf{K}_0$ and $[\gamma_{\alpha\beta}]_{\alpha,\beta\in G_0}$ be a scalar matrix such that $[\gamma_{\alpha\beta}]_{\alpha,\beta=(0,0)}^{(n,n)}$ defines a positive operator on $\mathbb{C}^n\times\mathbb{C}^n$ such that $\gamma = \sup_{\alpha\in G_0} \gamma_{\alpha\alpha} < \infty$. Then the matrix $[\gamma_{\alpha\beta} B_{\alpha\beta}]_{\alpha,\beta\in G_0}$ defines an operator D on $\mathbf{K}_0$ satisfying $\|D\| \le \gamma\|B\|$.*

We also have the following

LEMMA 5.2. *The matrix $C_n = [\gamma_{\alpha\beta}]_{\alpha,\beta\in G_0}$, for $\alpha = (k,l)$, $\beta = (i,j)$, with*

$$\gamma_{\alpha\beta} = \begin{cases} \Big(1 - \dfrac{|k-l|}{n+1}\Big)\Big(1 - \dfrac{|i-j|}{n+1}\Big) & \text{if } |k-l| \le n \text{ and } |i-j| \le n, \\ 0 & \text{otherwise}, \end{cases}$$

is positive definite.

Proof. The matrix $B_n = [b_{ij}]$, $i, j \ge 0$, with

$$b_{ij} = \begin{cases} 1 - \dfrac{|i-j|}{n+1} & \text{if } |i-j| \le n, \\ 0 & \text{otherwise}, \end{cases}$$

is positive definite by [5]. The matrix C_n is the tensor product of B_n by itself, thus it is also positive definite.

The next lemma is a consequence of Lebesgue's Dominated Convergence Theorem for a discrete measure.

LEMMA 5.3. *Let $\lambda_{kl} \ge 0$ and $\sum_{k,l=0}^{\infty} \lambda_{kl} < \infty$. Then*

$$\sum_{k,l=0}^{n} \Big(\Big(1 - \frac{k}{n+1}\Big)\Big(1 - \frac{l}{n+1}\Big) - 1 \Big) \lambda_{kl} \to 0 \quad \text{as } n \to \infty.$$

In view of Proposition 2.1, the following theorem describes the WOT-closed algebra generated by two doubly commuting operator weighted shifts.

THEOREM 5.4. *An operator $D \in L(\mathbf{K}_0)$ belongs to $\mathcal{W}(T_1, T_2)$ if and only if $D_{\alpha\beta} = 0$ for $\alpha \not\ge \beta$ and there is a sequence $\{\lambda_\alpha\}_{\alpha\in G_0}$ of scalars such that $D_{\alpha\beta} = \lambda_{\alpha-\beta} S_\beta^{\alpha-\beta}$ for $\alpha \ge \beta$.*

P r o o f. The statements (3.3) and (3.4) imply that T^s, $s \in G_0$, has the following matrix: $(T^s)_{\phi+s,\phi} = S^s_\phi$ for $\phi \in G_0$ and $(T^s)_{\phi\psi} = 0$ otherwise. Hence, for each polynomial p there is a sequence $\{\lambda_\alpha(p)\}_{\alpha\in G_0}$ of (finitely non-zero) scalars such that $p(T_1,T_2)_{\alpha\beta} = \lambda_{\alpha-\beta}(p)S^{\alpha-\beta}_\beta$ for $\alpha \ge \beta$ and $p(T_1,T_2)_{\alpha\beta} = 0$ otherwise. Let $D \in \mathcal{W}(T_1,T_2)$. Then there is a net $\{p_\omega(T_1,T_2)\}$ of polynomials in T_1, T_2 (T_1, T_2 commute) converging in the Weak Operator Topology to D. Thus $(p_\omega(T_1,T_2))_{\alpha\beta}$ converges to $D_{\alpha\beta}$. Hence D has the desired matrix.

Conversely, assume that there is a sequence $\{\lambda_\alpha\}_{\alpha\in G_0}$ of scalars such that $D_{\alpha\beta} = \lambda_{\alpha-\beta}S^{\alpha-\beta}_\beta$ for $\alpha \ge \beta$. Consider a sequence of polynomials in T_1, T_2,

$$(5.4)\qquad \begin{aligned} p_n(T_1,T_2) &= \sum_{k,l=0}^{n} \gamma^{(n)}_{kl}\lambda_{(k,l)}T_1^kT_2^l, \qquad \text{where} \\ \gamma^{(n)}_{kl} &= \Big(1-\frac{k}{n+1}\Big)\Big(1-\frac{l}{n+1}\Big). \end{aligned}$$

Lemmas 5.1 and 5.2 show that

$$(5.5)\qquad \|p_n(T_1,T_2)\| \le \|D\|.$$

Let $x \in \mathbf{H}$, $\zeta \in G_0$ and $\delta_{\zeta\beta}$ be the Kronecker δ. Then

$$\begin{aligned}
&\|p_n(T_1,T_2)xe_\zeta - Dxe_\zeta\|^2 \\
&\quad = \Big\| \sum_{\alpha\in G_0}\Big(\sum_{\beta\in G_0}\delta_{\zeta\beta}(p_n(T_1,T_2)_{\alpha\beta})x\Big)e_\alpha - \sum_{\alpha\in G_0}\Big(\sum_{\beta\in G_0}\delta_{\zeta\beta}D_{\alpha\beta}x\Big)e_\alpha\Big\|^2 \\
&\quad = \sum_{\alpha\in G_0}\|(p_n(T_1,T_2)_{\alpha\zeta})x - D_{\alpha\zeta}x\|^2 \\
&\quad = \sum_{\alpha\in G_0}\|(p_n(T_1,T_2)_{\alpha+\zeta,\zeta})x - D_{\alpha+\zeta,\zeta}x\|^2 \\
&\quad = \sum_{k,l=0}^{n}\|\gamma^{(n)}_{kl}\lambda_{(k,l)}S^{(k,l)}_\zeta x - \lambda_{(k,l)}S^{(k,l)}_\zeta x\|^2 + \sum_{\alpha\not\le(n,n)}\|D_{\alpha+\zeta,\zeta}x\|^2 \\
&\quad = \sum_{k,l=0}^{n}(\gamma^{(n)}_{kl}-1)\|\lambda_{(k,l)}S^{(k,l)}_\zeta x\|^2 + \sum_{\alpha\not\le(n,n)}\|D_{\alpha+\zeta,\zeta}x\|^2 \\
&\quad = \sum_{k,l=0}^{n}(\gamma^{(n)}_{kl}-1)\|D_{(k,l)+\zeta,\zeta}x\|^2 + \sum_{\alpha\not\le(n,n)}\|D_{\alpha+\zeta,\zeta}x\|^2.
\end{aligned}$$

Since $\sum_{\alpha\in G_0}\|D_{\alpha+\zeta,\zeta}x\|^2 = \|Dxe_\zeta\|^2 \le \infty$, by Lemma 5.3 we have $\|p_n(T_1,T_2)xe_\zeta - Dxe_\zeta\| \to 0$ $(n\to\infty)$. Finally, $p_n(T_1,T_2)f \to Df$ on a dense set, and by (5.5), $p_n(T_1,T_2)\to D$ in the Strong Operator Topology.

THEOREM 5.5. *Let T_1, T_2 be the operator weighted shifts* (5.1) *such that* $\ker A_\alpha^{(i)} = \{0\} = \ker A_\alpha^{(i)*}$ *for $\alpha\in G_0$, $i=1,2$. Assume also that T_1^*, T_2^* have a non-zero joint eigenvalue. Then $\mathcal{W}(T_1,T_2)$ is reflexive.*

Proof. The main idea of the proof is taken from [2]. However, we present some parts of the proof because they are different. Moreover, we now have a pair of operator weighted shifts instead of a single one and we assume less about them.

Let $D\in L(\mathbf{K}_0)$ and $\mathrm{Lat}(T_1,T_2)\subset \mathrm{Lat}\,D$. The subspaces $L^2(\bigoplus_{t\in G_\psi} H_t)$ (for all $\psi\in G_0$) are invariant for T_1, T_2. Hence they are also invariant for D, and thus $D_{\phi\psi}=0$ if $\psi\not\le\phi$.

Let $f\in H_{(0,0)}=\mathbf{H}$ and let $[f]$ denote the one-dimensional subspace generated by f. Then $\bigoplus_{\phi\in G_0} S_0^\phi[f]$ is invariant for T_1, T_2. Thus, if $\Lambda = \{\lambda_\phi\}_{\phi\in G_0}$ is a sequence of scalars such that $\sum_{\phi\in G_0}|\lambda_\phi|^2\|S_0^\phi f\|^2<\infty$, then there is a sequence $\{\gamma_\phi(f)\}_{\phi\in G_0}$ depending on Λ and f such that

$$D\Big(\bigoplus_{\phi\in G_0}\lambda_\phi S_0^\phi f\Big) = \bigoplus_{\phi\in G_0}\gamma_\phi(f)S_0^\phi f. \tag{5.6}$$

Let $\phi\in G_0$ and let $\Lambda_\phi = \{\delta_{\phi\psi}\}_{\psi\in G_0}$, where $\delta_{\phi\psi}$ is the Kronecker δ. Then there is $\Gamma_\phi = \{\gamma_{\phi\psi}f\}$ defined as above. As in [2], it can be shown that

$$D_{\phi\psi}S_0^\psi f = \gamma_{\phi\psi}(f)S_0^\phi f \quad \text{for } f\in\mathbf{H} \text{ and } \psi\le\phi. \tag{5.7}$$

Let $f,g\in\mathbf{H}$ be non-zero elements. Then using (5.7) we can prove as in [2] that

$$\gamma_{\phi\psi}(f+g)S_0^\phi(f+g) = S_0^\phi(\gamma_{\phi\psi}(f)f + \gamma_{\phi\psi}(g)g).$$

Since $\ker S_0^\phi \ne \{0\}$, we obtain

$$\gamma_{\phi\psi}(f+g)(f+g) = \gamma_{\phi\psi}(f)f + \gamma_{\phi\psi}(g)g.$$

Hence, if f, g are linearly independent, then

$$\gamma_{\phi\psi}(f) = \gamma_{\phi\psi}(g). \tag{5.8}$$

If $f=\alpha g$, then using (5.7), we can show that

$$\gamma_{\phi\psi}(f)S_0^\phi f = \gamma_{\phi\psi}(g)S_0^\phi f.$$

Thus, in this case we also have (5.8). Hence, the $\gamma_{\phi\psi}(f)$ do not depend on f, so $D_{\phi\psi}S_0^\psi = \gamma_{\phi\psi}S_0^\phi$. Now we will show that

$$\gamma_{\phi\psi} = \gamma_{\phi-\psi,0}. \tag{5.9}$$

We know that $\mathrm{Lat}(T_1^*, T_2^*) \subset \mathrm{Lat}\, D^*$. Let $g = \sum_{\phi\in G_0} g_\phi e_\phi$ be a joint eigenvector for an eigenvalue $\lambda = (\lambda_1, \lambda_2)$ for T_1^*, T_2^*. Then, for $\psi \in G_0$ and $f \in H_{(0,0)} = \mathbf{H}$, using Lemma 4.3, (3.3) and (5.2), we have

$$\begin{aligned}(D^* g, S_0^\psi f) &= \Big(\sum_{\psi\le\phi} D^*_{\phi\psi} g_\phi, S_0^\psi f\Big) = \sum_{\psi\le\phi} (D^*_{\phi\psi} g_\phi, S_0^\psi f)\\ &= \sum_{\psi\le\phi} (g_\phi, D_{\phi\psi} S_0^\psi f) = \sum_{\psi\le\phi} (g_\phi, \gamma_{\psi\phi} S_0^\phi f)\\ &= \sum_{\psi\le\phi} (\overline{\gamma}_{\psi\phi} g_\phi, S_\psi^{\phi-\psi} S_0^\psi f) = \sum_{\psi\le\phi} (\overline{\gamma}_{\psi\phi} (S_\psi^{\phi-\psi})^* g_\phi, S_0^\psi f)\\ &= \sum_{\psi\le\phi} (\overline{\gamma}_{\psi\phi} \lambda^{\phi-\psi} g_\psi, S_0^\psi f) = \sum_{\psi\le\phi} \overline{\gamma}_{\psi\phi} \lambda^{\phi-\psi} (g_\psi, S_0^\psi f).\end{aligned}$$

On the other hand, $[g]$ is invariant for D^*, and so there is $\gamma \in \mathbb{C}$ such that $D^* g = \gamma g$. Hence

$$(D^* g, S_0^\psi f) = \gamma(g, S_0^\psi f) = \gamma(g_\psi, S_0^\psi f).$$

Since $\ker S_0^{\psi *} = \{0\}$, we have $\overline{\mathcal{R}(S_0^\psi)} = \mathbf{H}$. Hence $\sum_{\psi\le\phi} \overline{\gamma}_{\psi\phi} \lambda^{\phi-\psi} = \gamma$. Thus

$$\sum_{\phi\in G_0} \overline{\gamma}_{\psi+\phi,\psi} \lambda^\phi = \sum_{\phi\in G_0} \overline{\gamma}_{\psi 0}\ \lambda^\phi.$$

Let $\alpha = (\alpha_1, \alpha_2)$ be an existing non-zero eigenvalue for T_1^*, T_2^*. Then Lemma 4.1 shows that the above equality holds for all $\lambda = (\lambda_1, \lambda_2)$, $|\lambda_i| \le |\alpha_i|$, $i = 1, 2$. Hence (5.9) is shown and

$$D_{\phi\psi} = \gamma_{\phi-\psi} S_\psi^{\phi-\psi}. \tag{5.10}$$

So, Theorem 5.4 implies that $D \in \mathcal{W}(T_1, T_2)$.

A consequence of the above and Proposition 2.1 is

Corollary 5.6. *Let T_1, T_2 be operator weighted shifts whose weights and their adjonts have trivial kernels. Assume that T_1, T_2 doubly commute and $\sigma_{\mathrm{p}}(T_1^*, T_2^*) \ne \{0\}$. Then $\mathcal{W}(T_1, T_2)$ is reflexive.*

6. The general situation. As an immediate consequence of Theorem 5.4, by (4.1) we obtain

Proposition 6.1. *An operator $D \in L(L_\phi)$ ($\mu(E_\phi) \ne 0$) belongs to $\mathcal{W}(T_1|_{L_\phi}, T_2|_{L_\phi})$ if and only if there is $D(\,\cdot\,) : E_\phi \to \bigoplus_{\psi\in G_\phi} H_\psi$ such that $D_{\alpha\beta}(\,\cdot\,) = 0$ for $\alpha \not\ge \beta$ and $\alpha, \beta \ge \phi$ and there is a sequence $\{\lambda_\alpha\}_{\alpha\in G_0}$ of scalars such that $D_{\alpha\beta}(\,\cdot\,) = \lambda_{\alpha-\beta} S_\beta^{\alpha-\beta}(\,\cdot\,)$ for $\alpha \ge \beta \ge \phi$.*

We are thus led to the following strengthening of Theorem 5.4:

THEOREM 6.2. *An operator $D \in L(K)$ belongs to $\mathcal{W}(T_1, T_2)$ if and only if there is $D(\,\cdot\,) : \mathbf{X} \to \mathcal{H}$ such that $D_{\alpha\beta}(\,\cdot\,) = 0$ for $\alpha \not\geq \beta$ and there is a sequence $\{\lambda_\alpha\}_{\alpha \in G_0}$ of scalars such that $D_{\alpha\beta}(\,\cdot\,) = \lambda_{\alpha-\beta} S_\beta^{\alpha-\beta}(\,\cdot\,)$ for $\alpha \geq \beta$.*

Proof. If $D \in \mathcal{W}(T_1, T_2)$ then $L_\beta \in \operatorname{Lat} D$. If $\alpha \geq \beta$ and $\mu(E_\beta) \neq 0$, then, by Proposition 6.1 for $\psi = \beta$, $D|_{L_\beta}$ is represented by a function $D^\beta(\,\cdot\,) : E_\beta \to \bigoplus_{t \in G_\beta} H_t$ and there is a sequence $\{\lambda_\alpha^\beta\}_{\alpha \in G_0}$ of scalars such that $D_{\alpha\beta}^\beta(\,\cdot\,) = \lambda_{\alpha-\beta}^\beta S_\beta^{\alpha-\beta}(\,\cdot\,)|_{L_\beta}$. We need to show that the sequence does not depend on the ψ chosen in Proposition 6.1. Let $\alpha \geq \beta \geq \psi$, $\mu(E_\psi) \neq 0$. Then $D|_{L_\psi}$ is represented by a function $D^\psi(\,\cdot\,) : E_\psi \to \bigoplus_{t \in G_\psi} H_t$ and there is also a sequence $\{\lambda_\alpha^\psi\}_{\alpha \in G_0}$ of scalars such that $D_{\alpha\beta}^\psi(\,\cdot\,) = \lambda_{\alpha-\beta}^\psi S_\beta^{\alpha-\beta}(\,\cdot\,)|_{L_\psi}$. Let $0 \neq f(\,\cdot\,) \in \mathcal{H}_\beta$. Then

$$\begin{aligned}\lambda_{\alpha-\beta}^\beta S_\beta^{\alpha-\beta}(\,\cdot\,) f(\,\cdot\,) &= D_{\alpha\beta}^\beta(\,\cdot\,) f(\,\cdot\,) = P_{H_\alpha} D|_{L_\beta} f(\,\cdot\,) e_\beta = P_{H_\alpha} D f(\,\cdot\,) e_\beta \\ &= P_{H_\alpha} D|_{L_\psi} f(\,\cdot\,) e_\beta = D_{\alpha\beta}^\psi(\,\cdot\,) f(\,\cdot\,) = \lambda_{\alpha-\beta}^\psi S_\beta^{\alpha-\beta}(\,\cdot\,) f(\,\cdot\,).\end{aligned}$$

Thus $\lambda_\alpha^\beta = \lambda_\alpha^\psi$ and the function $D_{\alpha\beta}^\psi$ does not depend on ψ in fact. On the other hand, if $\alpha \not\geq \beta$, then $D_{\alpha\beta} = 0$. In that case $D_{\alpha\beta}$ is decomposable and we can define $D_{\alpha\beta}(\,\cdot\,) = 0$. Thus, it is easy to see that we can construct a function $D(\,\cdot\,) : \mathbf{X} \to \mathcal{H}$ with the desired properties.

For the proof of the inverse implication we can construct a uniformly bounded sequence of polynomials $p(T_1, T_2)$ as in (5.4), (5.5). It converges on each L_ϕ, $\phi \in G$. As in the proof of Lemma 5.4 and by the uniform boundedness (5.5) and Remark 3.2 it converges on the whole $\mathbf{K}$.

Now we will present the reflexivity results. We will consider the conditions:

(∗) there is ϕ such that $\sigma_{\mathrm{p}}((T_1|_{L_\phi})^*, (T_2|_{L_\phi})^*) \neq \{0\}$

(∗∗) for each $\phi \in G$ such that $\mu(E_\phi) \neq 0$, there is $\phi_0 \leq \phi$ such that $\mu(E_{\phi_0}) \neq 0$ and $\sigma_{\mathrm{p}}((T_1|_{L_{\phi_0}})^*, (T_2|_{L_{\phi_0}})^*) \neq \{0\}$.

By Lemma 4.5, (∗∗) is equivalent to the condition

(∗∗∗) for each $\phi \in G$ such that $\mu(E_\phi) \neq 0$, $\sigma_{\mathrm{p}}((T_1|_{L_\phi})^*, (T_2|_{L_\phi})^*) \neq \{0\}$.

Firstly, we strengthen (∗) to imply (∗∗):

(∗∗∗∗) there is $\phi_0 \in G$ and a non-zero joint eigenvalue λ for $(T_1|_{L_{\phi_0}})^*$, $(T_2|_{L_{\phi_0}})^*$ such that there is an eigenvector $f \in L_{\phi_0}$ for the eigenvalue λ satisfying

$$\forall \phi \in G \quad \mu(E_\phi) \neq 0 \ \Rightarrow \ \|f\|_{L^2(E_\phi, \mu|_{E_\phi})} \neq 0. \tag{6.1}$$

The following lemma, together with Lemma 4.5, shows that (∗∗∗∗) implies (∗∗).

LEMMA 6.3. *Let condition* $(****)$ *be satisfied. If* $\phi \le \phi_0$ *and* $\mu(E_\phi) \neq 0$, *then* $\sigma_{\rm p}(T_1|_{L_\phi})^*, (T_2|_{L_\phi})^* \neq \{0\}$.

Proof. Let $f = \sum_{\psi\in G_{\phi_0}} f_\psi(\cdot)e_\psi$ be an eigenvector for the eigenvalue $\lambda=(\lambda_1,\lambda_2)$ for $(T_1|_{L_{\phi_0}})^*$, $(T_2|_{L_{\phi_0}})^*$, existing by $(****)$. We define the vector $\bar f = \sum_{\psi\in G_\phi} \bar f_\psi(\cdot)e_\psi$ as follows:

$$\bar f_\psi(X) = \begin{cases} f_\psi(X) & \text{if } X\in E_\phi \text{ and } \psi\in G_{\phi_0},\\ (\lambda^{\phi_0-\psi})^{-1} S_\psi^{\phi_0-\psi}(X)^* f_{\phi_0}(X) & \\ & \text{if } X\in E_\phi \text{ and } \psi\le\phi_0,\\ (\lambda_2^{\phi_0^{(2)}-\psi^{(2)}})^{-1} S_\psi^{(0,\phi_0^{(2)}-\psi^{(2)})}(X)^* f_{(\psi^{(1)},\phi_0^{(2)})}(X) & \\ & \text{if } X\in E_\phi \text{ and } \psi\not\le\phi_0,\ \psi^{(2)}<\phi_0^{(2)},\\ (\lambda_1^{\phi_0^{(1)}-\psi^{(1)}})^{-1} S_\psi^{(\phi_0^{(1)}-\psi^{(1)},0)}(X)^* f_{(\phi_0^{(1)},\psi^{(2)})}(X) & \\ & \text{if } X\in E_\phi \text{ and } \psi\not\le\phi_0,\ \psi^{(1)}<\phi_0^{(1)},\\ 0 & \text{if } X\notin E_\phi. \end{cases} \tag{6.2}$$

First, we show that

$$\sum_{\psi\in G_\phi} \|\bar f_\psi(\cdot)\|^2 < \infty. \tag{6.3}$$

Since $f\in \mathbf{K}$, for given $\varepsilon>0$ there is m such that

$$\sum_{\substack{\psi\in G_{\phi_0}\\ \psi\not\le \mathbf{m}}} \|\bar f_\psi(\cdot)\|^2 < \varepsilon \quad \text{where } \mathbf{m}=(m,m). \tag{6.4}$$

Let $G^i = \{\psi\in G_\phi : \psi\notin G_{\phi_0},\ \psi\not\le\mathbf{m},\ \psi^{(i)}\le m\}$, $i=1,2$. We have

$$\sum_{\substack{\psi\in G_\phi\\ \psi\not\le\mathbf{m}}} \|\bar f_\psi(\cdot)\|^2 \sum_{\substack{\psi\in G_{\phi_0}\\ \psi\not\le\mathbf{m}}} \|\bar f_\psi(\cdot)\|^2 + \sum_{\psi\in G^1} \|\bar f_\psi(\cdot)\|^2 + \sum_{\psi\in G^2} \|\bar f_\psi(\cdot)\|^2.$$

Let us estimate the last sum:

$$\sum_{\psi\in G^2} \|\bar f_\psi(\cdot)\|^2 = \sum_{k=m}^{\infty} \sum_{l=\phi^{(2)}}^{\phi_0^{(2)}-1} \|\bar f_{(k,l)}(\cdot)\|^2$$
$$\le \sum_{k=m}^{\infty} \sum_{l=\phi^{(2)}}^{\phi_0^{(2)}-1} \|(\lambda_2^{\phi_0^{(2)}-l})^{-1} A_{(k,l)}^{(2)}(\cdot)^* \ldots A_{(k,\phi_0^{(2)}-1)}^{(2)}(\cdot)^* f_{(k,\phi_0^{(2)})}\|^2$$

$$\leq \sum_{k=m}^{\infty} \sum_{l=\phi^{(2)}}^{\phi_0^{(2)}-1} |\lambda_2|^{2(l-\phi_0^{(2)})} \|A_{(k,l)}^{(2)}\|^2 \ldots \|A_{(k,\phi_0^{(2)}-1)}^{(2)}(\,\cdot\,)\|^2 \|f_{(k,\phi_0^{(2)})}(\,\cdot\,)\|^2$$

$$\leq \sum_{k=m}^{\infty} \sum_{l=\phi^{(2)}}^{\phi_0^{(2)}-1} (|\lambda|^{l-\phi_0^{(2)}-1} C^{\phi_0^{(2)}-l})^2 \|f_{(k,\phi_0^{(2)})}(\,\cdot\,)\|^2.$$

Thus (6.4) implies that

$$\sum_{\psi \in G^2} \|\bar{f}_\psi(\,\cdot\,)\|^2 \leq M_1 \varepsilon,$$

where M_1 is a suitable constant. In the same way we can estimate the last but one sum. Hence, if M is a suitable constant then

$$\begin{aligned}\sum_{\substack{\psi \in G_\phi \\ \psi \not\leq \mathbf{m}}} \|\bar{f}_\psi(\,\cdot\,)\|^2 &= \sum_{\substack{\psi \in G_{\phi_0} \\ \psi \not\leq \mathbf{m}}} \|\bar{f}_\psi(\,\cdot\,)\|^2 + M\varepsilon \\ &= \sum_{\substack{\psi \in G_{\phi_0} \\ \psi \not\leq \mathbf{m}}} \|f_\psi((\,\cdot\,))\|^2 + M\varepsilon \leq (M+1)\varepsilon.\end{aligned}$$

Thus $\bar{f} \in \mathbf{K}$ and by (6.1), $\bar{f} \neq 0$.

Now, it is easy to see by (6.2) that $\bar{f} \in L_\phi$. Using Remark 4.4, one can see from (6.2) that λ is a joint eigenvalue for $(T_1|_{L_\phi})^*$, $(T_2|_{L_\phi})^*$.

The pair $\{T_1, T_2\}$ of operator weighted shifts is called *of IW type* if the operators $A_\phi^{(i)}(\,\cdot\,)$, $\phi \in G$, $i = 1, 2$, are invertible. We have the following

Theorem 6.4. *Let $\{T_1, T_2\}$ be a pair of operator weighted shifts of IW type. Assume that $(*)$ is satisfied. Then $\mathcal{W}(T_1, T_2) = \operatorname{Alg}\operatorname{Lat}(T_1, T_2) \cap \{T_1, T_2\}'$.*

Proof. Let $A \in \{T_1, T_2\} \subset L(\mathbf{K})$ and $\operatorname{Lat}(T_1, T_2) \subset \operatorname{Lat}(A)$. Let $\phi_0 \in G$ be as in assumption $(*)$ and take $\phi \geq \phi_0$. Lemma 3.3 shows that $L_\phi \in \operatorname{Lat}(T_1, T_2) \subset \operatorname{Lat}(A)$ and Theorem 5.5 and Lemma 4.5 imply that there is a sequence of polynomials η_n^ϕ of the operators $T_1|_{L_\phi}$, $T_2|_{L_\phi}$ WOT-converging to $A|_{L_\phi}$. Moreover, (5.5) shows that $\|\eta_n^\phi\| \leq \|A|_{L_\phi}\| \leq \|A\| = C$. The following lemma is needed:

Lemma 6.5. *If $\phi_0 \leq \phi \leq \psi$ and $n \in \mathbb{N}$, then $\eta_n^\psi|_{L_\phi} = \eta_n^\phi|_{L_\psi}$.*

Proof. This is a consequence of the equalities (5.9) and (5.10). According to (5.4) the coefficients of the polynomials $\eta_n^\psi|_{L_\phi}$, $\eta_n^\phi|_{L_\psi}$ depend directly on γ_α^ϕ, γ_α^ψ given in (5.10) (ϕ in the superscript means that γ_α was constructed on L_η). If $\psi \leq \beta \leq \alpha$ and $f : \mathbf{X} \to H_\beta$, $f(X) = 0$ for $X \notin E_\phi$, then

$$\gamma^{\psi}_{\alpha-\beta} S^{\alpha-\beta}_{\beta}(\,\cdot\,) f(\,\cdot\,) = P_{H_\alpha} A|_{L_\psi} f(\,\cdot\,) e_\beta = P_{H_\alpha} A f(\,\cdot\,) e_\beta$$
$$= P_{H_\alpha} A|_{L_\phi} f(\,\cdot\,) e_\beta = \gamma^{\phi}_{\alpha-\beta} S^{\alpha-\beta}_{\beta}(\,\cdot\,) f(\,\cdot\,).$$

Hence $\gamma^\phi_\alpha = \gamma^\psi_\alpha$, which finishes the proof of the lemma.

Now, as in [4], for $\phi \in G$ we consider the subspaces $M_\phi = \{f \in \mathbf{K} : f(X) = 0\ \mu\text{-a.e. on } \mathbf{X} - E_\phi\}$. We extend η^ϕ_n to the whole M_ϕ considering the spaces $M^l_\phi = \{f \in M_\phi : f(X) \in \bigoplus_{\alpha \in G_1} H_\alpha\ \mu\text{-a.e.}\}$ for all non-negative integers l and $\mathbf{l} = (l, l)$. We define

$$\eta^\phi_n f = T^{-1}_{\mathbf{l}+\phi} \eta^\phi_n T_{\mathbf{l}+\phi} f \quad \text{for } f \in M^l_\phi. \tag{6.5}$$

As in [4] it can be shown that the extension is well defined and we can extend η^ϕ_n to M_ϕ with $\|\eta^\phi_n\| \le C$. The equality (6.5) and Lemma 9 of [4] show that Lemma 6.5 holds not only for L_ϕ, but also for M_ϕ. Hence, if $\phi \le \psi$ and $n \in \mathbb{N}$, then $M_\phi \subset M_\psi$ and $\eta^\phi_n = \eta^\psi_n|_{M_\psi}$.

Now, choose any strongly increasing sequence $\{\phi_n\} \subset G$ ($\phi^{(i)}_n < \phi^{(i)}_{n+1}$, $i = 1, 2$, and the $\phi^{(i)}_k$ are coordinates of ϕ_k) with first element ϕ_0. Then, by Lemma 9 of [4], $\overline{\bigcup_{n\in\mathbb{N}} M_{\phi_n}} = \mathbf{K}$. Hence we can define $\bigcup_m \eta^{\phi_m}_m$ and extend it to an operator η_m on the whole $\mathbf{K}$, as in [4], with $\|\eta_m\| \le C$. We have assumed that $A \in \{T_1, T_2\}'$, hence we can prove, as in [4], that η_n WOT-converges to A, and $A \in \mathcal{W}(T_1, T_2)$.

Next we will show the following:

PROPOSITION 6.6. *Let T_1, T_2 be a pair of operator weighted shifts of IW type. If* $(**)$ *is satisfied*, *then* $\operatorname{Alg}\operatorname{Lat}(T_1, T_2) \subset \{T_1, T_2\}'$.

P r o o f. Let $\phi \in G$, and $\mu(E_\phi) \ne 0$. By Lemma 3.3, $L_\phi \in \operatorname{Lat}(T_1, T_2) \subset \operatorname{Lat} A$ for $\phi \in G$, and thus it is enough to show that

$$A|_{L_\phi} T_i|_{L_\phi} = T_i|_{L_\phi} A|_{L_\phi}. \tag{6.6}$$

We also have $\operatorname{Lat}(T_1|_{L_\phi}, T_2|_{L_\phi}) \subset \operatorname{Lat} A|_{L_\phi}$. The assumption $(**)$ is equivalent to $(***)$. Hence, by Theorem 5.5, the algebra $\mathcal{W}(T_1|_{L_\phi}, T_2|_{L_\phi})$ is reflexive. So, (6.6) holds.

Hence we can state a consequence of Theorem 6.5 and Proposition 6.6:

THEOREM 6.7. *Let T_1, T_2 be a pair of operator weighted shifts of IW type. If* $(**)$ (*or* $(***)$) *is satisfied*, *then T_1, T_2 is reflexive.*

R e m a r k 6.8. Theorem 6.7 can be proved directly from Theorem 6.2 without Theorem 6.5, which is of independent interest.

R e m a r k 6.9. For the sake of simplicity of notation, our main theorems are formulated for pairs. However, they can be easily generalized to N-tuples.

7. Examples. We present a few examples.

Example 7.1. Let $A_\phi^{(i)} = I$, $\phi \in G$, $i = 1, 2$, in (3.1). Then we obtain the unweighted shifts considered in [4]. The assumption $(****)$ holds and the reflexivity of $\mathcal{W}(T_1, T_2)$ is shown as in [4].

Example 7.2. Let $X_1, \ldots, X_n$ be diagrams and consider the operators $T_1^{(j)}$, $T_2^{(j)}$ given in Example 3.4 for $X = X_j$. Let $T_i = T_i^{(1)} \oplus \ldots \oplus T_i^{(n)}$, $i = 1, 2$. Assume that $T_1^{(i)}$, $T_2^{(i)}$, $i = 1, \ldots, n$, are of IW type and there is ϕ_i such that $\sigma_{\mathrm{p}}(T_1^{(j)}|_{K_{G_{\phi_i}}})^*, (T_2^{(j)}|_{K_{G_{\phi_i}}})^* \neq \{0\}$, where $K_{G_{\phi_i}} = \bigoplus_{\psi \in G_{\phi_i}} H_\psi$. Hence $(**)$ holds for T_1, T_2, and thus $\mathcal{W}(T_1, T_2)$ is reflexive.

Example 7.3. Let X be a bounded diagram (i.e. for each $\phi \in X$ there is n such that $\phi - n\varepsilon_i \notin X$, $i = 1, 2$). Let T_1, T_2 be given by Example 3.4 for the diagram X. Assume that T_1, T_2 are of IW type and $\sigma_{\mathrm{p}}(T_1^*, T_2^*) \neq \{0\}$. It is easy to see that $(**)$ holds, and so $\mathcal{W}(T_1, T_2)$ is reflexive.

Example 7.4. Let $X_0 = \{(i, j) : i \geq 0 \text{ or } j \geq 0\}$ and $\mu = \delta_{X_0}$. Let $A_\phi^{(i)} = a_i^{|\phi_i|} I$, $0 < a_i < 1$ for $\phi = (\phi_1, \phi_2) \in X_0$, $i = 1, 2$. Then one can calculate that $\sigma_{\mathrm{p}}(T_1|_{K_0})^*, (T_2|_{K_0})^* \neq \{0\}$, because the pair $T_1|_{K_0}$, $T_2|_{K_0}$ is unitarily equivalent to the pair $B_1 \otimes I$, $I \otimes B_2$, where B_i, $i = 1, 2$, are weighted shifts with $\sigma_{\mathrm{p}}(B_i^*) \neq \{0\}$. Hence Example 7.2 implies the reflexivity of $\mathcal{W}(T_1, T_2)$.

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Reçu par la Rédaction le 13.4.1992