On extremal mappings in complex ellipsoids

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1. Introduction and the main results. Let $E(p) := \{ |z_1|^{2p_1} + \ldots + |z_n|^{2p_n} < 1 \} \subset \mathbb{C}^n$, where $p = (p_1, \ldots, p_n)$, $p_j > 0$, $j = 1, \ldots, n$; $E(p)$ is called a complex ellipsoid.

The aim of the paper is to characterize complex $\kappa_{E(p)}$- and $\tilde{k}_{E(p)}$-geodesics. The case where $E(p)$ is convex (i.e. $p_1, \ldots, p_n \geq 1/2$) has been solved in [Jar-Pfl-Zei]. The paper is inspired by methods of [Pol].

Let $D \subset \mathbb{C}^n$ be a domain and let $\varphi \in O(E, D)$, where $E$ denotes the unit disk in $\mathbb{C}$ and $O(\Omega, D)$ is the set of all holomorphic mappings $\Omega \to D$. Recall that $\varphi$ is said to be a $\kappa_D$-geodesic if there exists $(z, X) \in D \times \mathbb{C}^n$ such that:

- $\varphi(0) = z$ and $\varphi'(0) = \lambda_\varphi X$ for some $\lambda_\varphi > 0$,
- for any $\psi \in O(E, D)$ such that $\psi(0) = z$ and $\psi'(0) = \lambda_\psi X$ with $\lambda_\psi > 0$, we have $\lambda_\psi \leq \lambda_\varphi$.

We say that $\varphi$ is a $\tilde{k}_D$-geodesic if there exists $(z, w) \in D \times D$ such that:

- $\varphi(0) = z$ and $\varphi(\sigma_\varphi) = w$ for some $\sigma_\varphi \in (0, 1)$,
- for any $\psi \in O(E, D)$ such that $\psi(0) = z$ and $\psi(\sigma_\psi) = w$ with $\sigma_\psi > 0$, we have $\sigma_\psi \leq \sigma_\varphi$; cf. [Pan].

Let us fix some further notations:

- $H^\infty(\Omega, \mathbb{C}^n) :=$ the space of all bounded holomorphic mappings $\Omega \to \mathbb{C}^n$;
- $\|f\| := \sup\{ \|f(z)\| : z \in \Omega \}$, $f \in H^\infty(\Omega, \mathbb{C}^n)$, where $\|\|$ denotes the Euclidean norm in $\mathbb{C}^n$.

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\[ f^*(\zeta) := \text{the non-tangential boundary value of } f \text{ at } \zeta \in \partial E, \; f \in \mathcal{H}^\infty(E, \mathbb{C}^n); \]

- \[ A(\Omega, \mathbb{C}^n) := \mathcal{C}(\overline{\Omega}, \mathbb{C}^n) \cap \mathcal{O}(\Omega, \mathbb{C}^n); \]
- \[ z \cdot w := (z_1 w_1, \ldots, z_n w_n), \; z \cdot w := \sum_{j=1}^n z_j w_j, \; z = (z_1, \ldots, z_n), \; w = (w_1, \ldots, w_n) \in \mathbb{C}^n; \]
- \[ A_v := \{ z \in \mathbb{C} : \nu < |z| < 1 \}, \; \nu \in (0, 1); \]
- \[ PSH(\Omega) := \text{the set of all plurisubharmonic functions on } \Omega. \]

Fix \( w_1, \ldots, w_N \in A(A_v, \mathbb{C}^n) \) and define

\[ \Phi_j(h) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re}(h^*(e^{i\theta} \cdot w_j(e^{i\theta}))) \, d\theta, \quad h \in \mathcal{H}^\infty(E, \mathbb{C}^n), \; j = 1, \ldots, N. \]

We say that the functionals \( \Phi_1, \ldots, \Phi_N \) are linearly independent if for arbitrary \( s = (s_1, \ldots, s_n), g \in \mathcal{H}^\infty(E, \mathbb{C}^n), \) and \( \lambda_1, \ldots, \lambda_N \in \mathbb{R} \) such that \( s_k \) nowhere vanishes on \( E, \; k = 1, \ldots, n, \) and \( g(0) = 0 \) the following implication is true: if \( \sum_{j=1}^N \lambda_j w_j \cdot s^* = g^* \) on a subset of \( \partial E \) of positive measure, then \( \lambda_1 = \ldots = \lambda_N = 0. \)

Later on, we always assume that the functionals \( \Phi_1, \ldots, \Phi_N \) are linearly independent.

**Problem (P).** Given a bounded domain \( D \subset \mathbb{C}^n \) and numbers \( a_1, \ldots, a_N \in \mathbb{R} \), find a mapping \( f \in \mathcal{O}(E, D) \) such that \( \Phi_j(f) = a_j, \; j = 1, \ldots, N, \) and there is no mapping \( g \in \mathcal{O}(E, D) \) with

\[ \Phi_j(g) = a_j, \quad j = 1, \ldots, N, \quad g(E) \in D. \]

Any solution of (P) is called an extremal mapping for (P) or, simply, an extremal.

Problem (P) is a generalization of Problem (P) from [Pol].

We say that problem (P) is of \( m \)-type if there exists a polynomial \( Q(\zeta) = \prod_{k=1}^m (\zeta - \sigma_k) \) with \( \sigma_1, \ldots, \sigma_m \in E \) such that \( Qw_j \) extends to a mapping of class \( \mathcal{A}(E, \mathbb{C}^n), \; j = 1, \ldots, N. \)

One can prove that (for bounded domains \( D \subset \mathbb{C}^m \)) any complex \( \kappa_D \)-geodesic may be characterized as an extremal for a suitable problem (P) of 1-type (cf. §4).

The main result of the paper is the following:

**Theorem 1.** Let \( D \in G \subset \mathbb{C}^n \) be domains and let \( u \in PSH(G) \cap \mathcal{C}(G) \) be such that \( D = \{ u < 0 \}, \; \partial D = \{ u = 0 \}. \) Suppose that \( f \in \mathcal{O}(E, D) \) is an extremal for (P). Assume that there exist a set \( S \subset \partial E, \) a mapping \( s = (s_1, \ldots, s_n) \in \mathcal{H}^\infty(E, \mathbb{C}^n), \) a number \( \varepsilon > 0, \) and a function \( \nu : S \times A(E, \mathbb{C}^n) \to \mathbb{C} \) such that:

(a) \( \partial E \setminus S \) has zero measure,
(b) \( f^*(\zeta), \nabla u(f^*(\zeta)) \) and \( s^*(\zeta) \) are defined for all \( \zeta \in S, \)
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(c) \( s_k \) nowhere vanishes on \( E \) for \( k = 1, \ldots, n \),
(d) \( u(f^*(\zeta)) + s^*(\zeta) \cdot h(\zeta) = u(f^*(\zeta)) + 2 \Re(\nabla u(f^*(\zeta)) \cdot (s^*(\zeta) \cdot h(\zeta))) + v(\zeta, h), \zeta \in S, h \in A(E, \mathbb{C}^n), \|h\|_\infty \leq \varepsilon 
(e) \lim_{h \to 0} \sup\{|v(\zeta, h)|: \zeta \in S\}/\|h\|_\infty = 0.

Then
\[ f^*(\zeta) \in \partial D \quad \text{for a.a. } \zeta \in \partial E \]
and there exist \( g \in L^\infty(\partial E), g > 0, g \in H^\infty(E, \mathbb{C}^n), \) and \( (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N \setminus \{0\} \) such that
\[ \sum_{k=1}^N \lambda_k w_k(\zeta) \cdot s^*(\zeta) + g^*(\zeta) = g(\zeta) s^*(\zeta) \cdot \nabla u(f^*(\zeta)) \quad \text{for a.a. } \zeta \in \partial E. \]

Remark 2. Under the assumptions of Theorem 1, if \( u \in C^1(G) \cap PSH(G) \), then one can take \( s \equiv (1, \ldots, 1) \).

As an easy corollary to Theorem 1 we obtain

Corollary 3. Under the assumptions of Theorem 1, if problem \((P)\) is of \( m \)-type, then there exist \( g \in L^\infty(\partial E), g > 0, g \in H^\infty(E, \mathbb{C}^n) \) such that
\[ g^*(\zeta) = Q(\zeta) g(\zeta) s^*(\zeta) \cdot \nabla u(f^*(\zeta)) \quad \text{for a.a. } \zeta \in \partial E. \]

Theorem 1 generalizes Theorems 2 and 3 of [Pol] (cf. Remark 2). The proof of Theorem 1 will be presented in \( \S 2 \).

Corollary 3 gives a tool for describing the extremal mappings for problems \((P)\) of \( m \)-type in the case where \( D \) is an arbitrary complex ellipsoid \( E(p) \).

Theorem 4. Let \( \varphi : E \to E(p) \) be an extremal for problem \((P)\) of \( m \)-type such that \( \varphi_j \not\equiv 0, j = 1, \ldots, n \). Then
\[ \varphi_j(\lambda) = a_j \prod_{k=1}^m \left( \frac{\lambda - \alpha_{kj}}{1 - \alpha_{kj} \lambda} \right)^{r_{kj}} \left( \frac{1 - \sigma_{kj} \lambda}{1 - \sigma_{kj} \lambda} \right)^{1/p_j}, \quad j = 1, \ldots, n, \]
where
• \( a_1, \ldots, a_n \in \mathbb{C} \setminus \{0\} \),
• \( \alpha_{kj} \in E, k = 1, \ldots, m, j = 0, \ldots, n \),
• \( r_{kj} \in \{0, 1\} \) and, if \( r_{kj} = 1 \), then \( \alpha_{kj} \in E \),
• \( \sum_{j=1}^n |a_j|^{2p_j} \prod_{k=1}^m (\zeta - \alpha_{kj})(1 - \sigma_{kj} \zeta) = \prod_{k=1}^m (\zeta - \alpha_{kj})(1 - \sigma_{kj} \zeta), \zeta \in E. \)

In particular, if \( \varphi \) is a complex \( \kappa_{E(p)} \)- or \( \tilde{k}_{E(p)} \)-geodesic, then \( \varphi \) is of the above form with \( m = 1 \).

Theorem 4 generalizes \( \S 6 \) of [Pol] and Theorem 1 of [Jar-Pfl-Zei]. The proof of Theorem 4 will be given in \( \S \S 3, 4. \)
Remark 5. In the case where $E(p)$ is convex any mapping described in Theorem 4 with $m = 1$ is a complex geodesic in $E(p)$ ([Jar-Pfl-Zei]). This is no longer true if $E(p)$ is not convex (cf. [Pfl-Zwo] for the case $n = 2$, $p_1 = 1$, $p_2 < 1/2$).

2. Proof of Theorem 1. Note that there are two possibilities: either $u \circ f^* = 0$ a.e. on $\partial E$ or there exists $\tau > 0$ such that the set $\{ \theta : u(f^*(e^{i\theta})) < -\tau \}$ has positive measure. If such a $\tau$ exists, fix one of them. We put

$$P_0 := \begin{cases} \emptyset & \text{in the first case,} \\ \{ \theta : u(f^*(e^{i\theta})) < -\tau \} & \text{in the second case,} \end{cases}$$

$$A_0 := [0, 2\pi) \setminus P_0,$$

and

$$p_s(h) := \frac{1}{2\pi} \int_{A_0} [\operatorname{Re}(s^*(e^{i\theta}) \cdot \nabla u(f^*(e^{i\theta})) \cdot h(e^{i\theta}))]^+ d\theta \quad \text{for } h \in L^1(\partial E, \mathbb{C}^n),$$

where $L^1(\partial E, \mathbb{C}^n)$ denotes the space of all Lebesgue integrable mappings $\partial E \to \mathbb{C}^n$.

Remark 6. (a) Under the assumptions of Theorem 1, there exists $M > 0$ such that

$$\|s^*(\zeta) \cdot \nabla u(f^*(\zeta))\| \leq M \quad \text{for a.a. } \zeta \in \partial E.$$

(b) $p_s(h)$ is a seminorm on $H^1(E, \mathbb{C}^n)$ and $p_s(h) \leq M \|h\|_1$, where $H^1(E)$ denotes the first Hardy space of holomorphic functions,

$$H^1(E, \mathbb{C}^n) := \{(f_1, \ldots, f_n) : f_j \in H^1(E)\},$$

and $\|\|_1$ denotes the norm in $H^1(E, \mathbb{C}^n)$.

The proof of Theorem 1 is based on the following result.

Lemma 7 (cf. [Pol], Lemma 6). Under the assumptions of Theorem 1 there exist $T > 0$, $j \in \{1, \ldots, N\}$, and $\delta \in \{-1, 1\}$ such that

$$\delta \Phi_j(s \cdot h) \leq T p_s(h)$$

for $h \in X_j := \{ h \in H^1(E, \mathbb{C}^n) : \Phi_l(s \cdot h) = 0, l \neq j \}$.

Let us for a while assume that we already have Lemma 7.

Proof of Theorem 1. By Lemma 7 there exist $T > 0$, $\delta \in \{-1, 1\}$, and $j \in \{1, \ldots, N\}$ such that

$$\delta \Phi_j(s \cdot h) \leq T p_s(h) \quad \text{for } h \in X_j.$$

Let $\tilde{\Phi}(h) := \delta \Phi_j(s \cdot h)$, $h \in X_j$. Using the Hahn–Banach theorem we can extend $\tilde{\Phi}$ to $L^1(\partial E, \mathbb{C}^n)$ (we denote this extension by $\Phi$) in such a way that

$$\Phi(h) \leq T p_s(h) \quad \text{for } h \in L^1(\partial E, \mathbb{C}^n).$$
We know that \( p_s(h) \leq M|h|_1 \), where \( |h|_1 \) denotes the norm in \( L^1(E,\mathbb{C}^n) \). So \( \Phi \) is continuous on \( L^1(\partial E,\mathbb{C}^n) \). By Riesz’s theorem, \( \Phi \) can be represented as
\[
\Phi(h) = \frac{1}{2\pi} \int_0^{2\pi} \Re(h^*(e^{i\theta}) \cdot \tilde{w}(e^{i\theta})) \, d\theta, \quad \text{where } \tilde{w} \in L^\infty(\partial E,\mathbb{C}^n).
\]

It is easy to see that there are \( \lambda_1, \ldots, \lambda_N \), not all zero, such that \( \Phi(h) = \sum_{k=1}^N \lambda_k \Phi_k(s \cdot h) \) for \( h \in H^1(E,\mathbb{C}^n) \). We denote by \( G \) the linear functional on \( L^1(\partial E,\mathbb{C}^n) \) defined by the formula
\[
G(h) := \frac{1}{2\pi} \int_0^{2\pi} \Re \left( \sum_{k=1}^N \lambda_k w_k(e^{i\theta}) \cdot s^*(e^{i\theta}) \cdot h(e^{i\theta}) \right) \, d\theta.
\]

Then \( \Phi(h) - G(h) = 0 \) for \( h \in H^1(E,\mathbb{C}^n) \). By the theorem of F. & M. Riesz it follows that there exists \( g \in H^\infty(E,\mathbb{C}^n) \), \( g(0) = 0 \), such that
\[
\tilde{w} - s^* \cdot \sum_{k=1}^N \lambda_k w_k = g^*.
\]

We have
\[
\Phi(h) = \frac{1}{2\pi} \int_0^{2\pi} \Re \left( \left( \sum_{k=1}^N \lambda_k w_k(e^{i\theta}) \cdot s^*(e^{i\theta}) + g^*(e^{i\theta}) \right) \cdot h^*(e^{i\theta}) \right) \, d\theta \leq T \frac{1}{2\pi} \int_{\lambda_0} \Re(s^*(e^{i\theta}) \cdot \nabla u(f^*(e^{i\theta})) \cdot h^*(e^{i\theta})) \, d\theta
\]
for any \( h \in H^1(E,\mathbb{C}^n) \). We see that the right-hand side is zero for any \( h \in H^1(E,\mathbb{C}^n) \) (hence, for any \( h \in L^1(\partial E,\mathbb{C}^n) \)) such that
\[
\Re(s^*(e^{i\theta}) \cdot \nabla u(f^*(e^{i\theta})) \cdot h^*(e^{i\theta})) \leq 0
\]
on \( \partial E \setminus (P_0 \cup \{ \zeta \in \partial E : s^*(\zeta) \cdot \nabla u(f^*(\zeta)) = 0 \}) \).

Hence
\[
\sum_{k=1}^N \lambda_k w_k \cdot s^* + g^* = 0 \quad \text{a.e. on } P_0 \cup \{ \zeta \in \partial E : s^*(\zeta) \cdot \nabla u(f^*(\zeta)) = 0 \}.
\]

We know that \( \Phi_1, \ldots, \Phi_N \) are linearly independent, so the Lebesgue measures of \( P_0 \) and of \( \{ \zeta \in \partial E : s^*(\zeta) \cdot \nabla u(f^*(\zeta)) = 0 \} \) are zero. Hence
\[
\sum_{k=1}^N \lambda_k w_k(\zeta) \cdot s^*(\zeta) + g^*(\zeta) = g(\zeta) s^*(\zeta) \cdot \nabla u(f^*(\zeta)),
\]
where \( g(\zeta) \in \mathbb{C} \setminus \{0\} \) for a.a. \( \zeta \in \partial E \). Now, it is enough to remark that condition (1) implies that \( 0 < q \leq T \) a.e. on \( \partial E \).

Now, we are going to prove Lemma 7.
Proof of Lemma 7. Suppose that the lemma is not true. Then for each \( j \in \{1, \ldots, N\} \) and \( m \in \mathbb{N} \) there are \( h_{jm}^+, h_{jm}^- \in X_j \) such that
\[
\Phi_j(s \cdot h_{jm}^+) > mp_A(h_{jm}^+), \quad -\Phi_j(s \cdot h_{jm}^-) > mp_A(h_{jm}^-).
\]
We may assume that \( h_{jm}^+, h_{jm}^- \in \mathcal{A}(E, \mathbb{C}^n) \) and that
\[
\Phi_j(s \cdot h_{jm}^+) = 1, \quad \Phi_j(s \cdot h_{jm}^-) = -1.
\]
For any \( q = (q_1^+, q_1^-, \ldots, q_N^+, q_N^-) \in \mathbb{R}^{2N} \) we define the function
\[
f_{qm} = f + \sum_{j=1}^N (q_j^+ s \cdot h_{jm}^+ + q_j^- s \cdot h_{jm}^-) = f + s \cdot h_{qm}
\]
and the linear mapping \( A : \mathbb{R}_+^{2N} \to \mathbb{R}^N, A(q) := (q_1^+ - q_1^-, \ldots, q_N^+ - q_N^-) \).
Note that \( \Phi_j(f_{qm}) - \Phi_j(f) = A(q) \).

Lemma 8 (see [Pol], Lemma 7). Let \( u \) be a non-positive subharmonic function in \( E \) and let \( \Delta u \) be the Riesz measure of \( u \). Suppose that one of the following conditions is true:

(a) \( \Delta u(r_0 E) > a > 0 \) for some \( r_0 \in (0, 1) \),
(b) for some set \( Z \subset [0, 2\pi) \) with positive measure, the upper radial limit of \( u \) at \( \zeta \in Z \) does not exceed \(-a < 0 \) (i.e. \( \limsup_{r \to 1} u(r\zeta) \leq -a \)).

Then \( u(\zeta) \leq -C(1 - |\zeta|) \), where \( C > 0 \) is a constant depending only on \( r_0, a, \) and \( Z \).

Let \( u_0 := u \circ f \).

Lemma 9. There exist a constant \( C > 0 \) and constants \( t_m > 0, m \in \mathbb{N} \), such that for \( \|q\| < t_m \) we have

(a) \( f_{qm} \in \mathcal{O}(E, G) \) (so, we define \( u_{qm} := u \circ f_{qm} \)),
(b) \( u_{qm}(\zeta) \leq v_{qm}(\zeta) := C \ln |\zeta| + \frac{1}{2\pi} \int_{A_0} [u_{qm}(e^{i\theta})]^+ P(\zeta, \theta) d\theta \)

for \( |\zeta| > 1/2 \).

Proof. (a) follows from the assumption that \( D \subset G \).
(b) Suppose that there exists \( r_0 \in (0, 1) \) such that \( \Delta u_0(r_0 E) > a > 0 \).
The continuity of \( u \) implies that for
\[
\tilde{u}_{qm}(\zeta) := u_{qm}(\zeta) - \frac{1}{2\pi} \int_{A_0} [u_{qm}(e^{i\theta})]^+ P(\zeta, \theta) d\theta, \quad \zeta \in E,
\]
if \( t_m \) is small enough then \( \Delta \tilde{u}_{qm}(rE) > a/2 \). Hence, from Lemma 8 we get the required result.

If \( \Delta u_0(rE) = 0 \) for any \( r \in (0, 1) \) and \( u_0^\circ(\zeta) = 0 \) for a.a. \( \zeta \in \partial E \), then by the Riesz representation theorem ([Hay-Ken], Ch. 3.5) we see that \( u_0 \) is harmonic in \( E \). But this is a contradiction, since \( u_0 \neq 0 \). Hence, \( P_0 \) has
positive measure. From the continuity of \( u \) we conclude that if \( t_m \) are small enough, then \( \{ \zeta : \tilde{u}_{q_m}(\zeta) < -\tau/2 \} \) has positive measure. By Lemma 8 we get the required result.  

Let us introduce some new notation:  
\[ E_{qm} := \{ \zeta \in E : v_{qm}(\zeta) < 0 \} \]

\[ g_{qm}(\zeta) := \zeta \exp \left\{ \frac{1}{2\pi C} \int_{A_0} [u_{qm}^*(e^{i\theta})]^+ S(\zeta, \theta) \, d\theta \right\}. \]

Here \( S(\zeta, \theta) := (\zeta + e^{i\theta})/(\zeta - e^{i\theta}) \) is the Schwarz kernel.

**Remark 10.** Note that \( C \ln |g_{qm}| = v_{qm} \), \( v_{qm}(\zeta) \geq C \ln |\zeta| \) (hence, \( |g_{qm}(\zeta)| \geq |\zeta| \)), and \( E_{qm} = g_{qm}^{-1}(E) \).

**Lemma 11** (cf. [Pol], Statement 2). (a) \( E_{qm} \) is connected, \( 0 \in E_{qm} \), (b) \( g_{qm} \) maps \( E_{qm} \) conformally onto \( E \).

**Proof.** (a) Note that \( E_{qm} = \bigcup_{\delta > 0} \{ \zeta : v_{qm}(\zeta) < -\delta \} \) and

\[ \{ \zeta : v_{qm}(\zeta) < -\delta \} \subset \{ \zeta : |\zeta| < e^{-\delta/C} \}. \]

Since \( v_{qm} \) is harmonic outside \( 0 \) and \( v_{qm}^*(e^{i\theta}) \geq 0 \), any connected component of \( \{ \zeta : v_{qm}(\zeta) < -\delta \} \) must contain \( 0 \).

(b) First let us see that \( g_{qm} : E_{qm} \to E \) is proper. Let \( \zeta_k \to 0 \in \partial E_{qm} \). If \( \zeta_0 \in \partial E \), then \( |g_{qm}(\zeta_k)| \to 1 \) (since \( |g_{qm}| \geq |\zeta| \)). If \( \zeta_0 \in E \), then \( |g_{qm}(\zeta_k)| \to |g_{qm}(0)| = 1 \).

Since \( g_{qm}'(0) \neq 0 \) and \( g_{qm}^{-1}(0) = \{ 0 \} \), \( g_{qm} \) is conformal.  

We define \( \tilde{f}_{qm}(\zeta) = f_{qm}(g_{qm}^{-1}(\zeta)) \), \( \tilde{f}_{qm}(\zeta) = \tilde{f}_{qm}(e^{-\|\eta\|/m}\zeta) \),

\[ \tilde{A}_m(q) = (\Phi_1(\tilde{f}_{qm}) - \Phi(f), \ldots, \Phi_N(\tilde{f}_{qm}) - \Phi_N(f)), \]

and

\[ \tilde{A}_m(q) = (\Phi_1(\tilde{f}_{qm}) - \Phi(f), \ldots, \Phi_N(\tilde{f}_{qm}) - \Phi_N(f)). \]

**Remark 12.** It is easy to see that \( \tilde{f}_{qm}(E) \subset \tilde{D} \), \( \tilde{f}_{qm}(E) \subset D \), and \( \tilde{A}(0) = \hat{A}(0) = 0 \).

The following result explains why we have used functionals of the special form.

**Lemma 13.** Suppose that

\[ \Phi(h) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re}(h^*(e^{i\theta}) \cdot w(e^{i\theta})) \, d\theta, \]

where \( w \in A(A_r, C^n) \) for some \( r \in (0, 1) \), \( f \in H^\infty(E, C^n) \), and that \( g \in O(E, E) \), \( g(0) = 0 \). Then

\[ |\Phi(f \circ g) - \Phi(f)| \leq K \|f\|_{\infty} \sup_{\zeta \in E} |g(\nu\zeta) - \nu\zeta|, \]

where \( K > 0 \) depends only on \( \Phi \).
Proof. We have
\begin{equation}
\Phi(h) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re}(h(\nu e^{i\theta}) \bullet w(\nu e^{i\theta})) \, d\theta.
\end{equation}
Hence
\[ |\Phi(h)| \leq (\max_{\zeta \in \partial E} \|w(\nu \zeta)\|)(\max_{\zeta \in \partial E} \|h(\nu \zeta)\|). \]
But
\[ \|f(g(\nu \zeta)) - f(\nu \zeta)\| \leq (\sup_{\xi \in E} |f'(\nu \xi)||g(\nu \zeta) - \nu \zeta|, \]
and \( \sup_{\xi \in E} |f'(\nu \xi)| \leq \|f\|_{\infty}/(1 - \nu^2). \)

Lemma 14 (cf. [Pol], Statement 3). The mappings \( \tilde{A}_m, \hat{A}_m \) are continuous in \( q \) for \( \|q\| < t_m \).

Proof. It is enough to remark that if \( q_k \to q \), then \( u_{q_km}^* \to u_{qm}^* \) uniformly on \( \partial E \). Hence \( g_{q_km} \to g_{qm} \) uniformly on compact subsets of \( E \). It is evident from the last assertion that also \( g_{q_km}^{-1} \to g_{qm}^{-1} \), \( \tilde{f}_{q_km} \to \tilde{f}_{qm} \), and \( \hat{f}_{q_km} \to \hat{f}_{qm} \) uniformly on compact sets. Since the \( \Phi_j \) are continuous with respect to this convergence (this follows easily from (2)), we conclude the proof.

Lemma 15. For each \( b > 0 \) there is \( m_0 \in \mathbb{N} \) such that for any \( m \geq m_0 \) there is \( q_m > 0 \) such that \( \|A(q) - \tilde{A}_m(q)\| \leq b\|q\| \) whenever \( \|q\| \leq q_m \).

Proof. It follows from the definition of \( A, \tilde{A}_m \) that it is enough to prove the inequality
\[ |\Phi(\tilde{f}_{qm}) - \Phi(f_{qm})| \leq b\|q\| \]
for small \( q \), where \( \Phi \) is a functional of our special form. By Lemma 14 it is enough to show that
\[ \sup_{\zeta \in \nu E} |g_{qm}^{-1}(\zeta) - \zeta| \leq b\|q\| \]
for small \( q \). Note that
\[ \sup_{\zeta \in \nu E} |g_{qm}^{-1}(\zeta) - \zeta| \leq \sup_{\zeta \in \nu E} |g_{qm}(\zeta) - \zeta| \]
and for small \( q_m \) (such that \( |1 - \exp q_m| \leq 2q_m \)) and \( \|q\| \leq q_m \),
\[ \left| 1 - \exp \left( \frac{1}{2\pi C} \int_{A_0} [u_{qm}^*(e^{i\theta})]^+ S(\zeta, \theta) \, d\theta \right) \right| \leq 2 \frac{1 + \nu}{1 - \nu} \left( \frac{1}{2\pi C} \int_{A_0} [u_{qm}^*(e^{i\theta})]^+ \, d\theta \right) \]
for $\zeta \in \nu E$. Hence, it is enough to show that
\[
\int_{A_0} [u_{qm}(e^{i\theta})]^+ \, d\theta \leq \int_{A_0} 2[\Re u(f^*(e^{i\theta})) \cdot s^*(e^{i\theta}) \cdot h_{qm}(e^{i\theta})]^+ \, d\theta + o(\|h_{qm}\|_\infty).
\]
But $p_s(h_{qm}) \leq \|q\| \max\{p_s(h_{jm}) : j = 1, \ldots, N\} \leq \|q\|/m$. Hence, if $m$ is large and $q_m$ is small enough, we get the required result.

**Lemma 16.** For each $b \geq 0$ there is $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$ there is $q_m > 0$ such that $\|\hat{A}_m(q) - \hat{A}_m(q)\| \leq b\|q\|$ whenever $\|q\| \leq q_m$.

**Proof.** As in Lemma 15, by Lemma 13 it is enough to prove the inequality
\[
\sup_{\zeta \in \nu E} |e^{-\|q\|/m} \zeta - \zeta| \leq b\|q\|
\]
for small $\|q\|$. But for small $\|q\|/m$ we have $|1 - e^{-\|q\|/m}| \leq 2\|q\|/m$. Hence, we get the required result.

**Lemma 17** (cf. [Pol], Lemma 8). For any continuous mapping $F : \mathbb{R}_+^{2N} \to \mathbb{R}^N$, if
\[
\|F(x) - A(x)\| \leq b\|x\| \quad \text{for } x \in B(0, r) \cap \mathbb{R}_+^{2N},
\]
where $b = 1/(2\sqrt{N})$, then there exists $q \in B(0, r) \cap \mathbb{R}_+^{2N} \setminus \{0\}$ such that $F(q) = 0$.

**Proof.** Define
\[
\mathcal{Q} := \{(x_1, \ldots, x_N) : 0 < x_j < t_0, j = 1, \ldots, N\}
\]
and
\[
\pi : \mathbb{R}^N \ni (x_1, \ldots, x_N) \mapsto (x_1, t_0 - x_1, \ldots, x_N, t_0 - x_N) \in \mathbb{R}_+^{2N},
\]
where $t_0 = (2\sqrt{N})^{-1} \min\{1, r\}$. It is easy to check that $\|\pi(l)\| \leq t_0 \sqrt{N}$ for $l \in \mathcal{Q}$ and $\pi(\mathcal{Q}) \subset B(0, r) \cap \mathbb{R}_+^{2N}$. Note that
\[
\|F \circ \pi(l) - A \circ \pi(l)\| \leq b\|\pi(l)\| \leq t_0/2 \quad \text{for } l \in \mathcal{Q}.
\]
Consider the homotopy defined by the formula $\tilde{F}_t = t F \circ \pi + (1 - t) A \circ \pi$. It is enough to show that $0 \notin \tilde{F}_t(\partial \mathcal{Q})$. Then from the homotopical invariance of the degree of mappings [Zei] we have $\deg(F \circ \pi, \mathcal{Q}, 0) = \deg(A \circ \pi, \mathcal{Q}, 0) \neq 0$, hence $0 \in F \circ \pi(\mathcal{Q})$.

It is easy to see that for any $l \in \partial \mathcal{Q}$,
\[
t_0 \leq \|A \circ \pi(l)\| \leq \|\tilde{F}_t(l)\| + t\|F \circ \pi(l) - A \circ \pi(l)\| \leq \|\tilde{F}_t(l)\| + t_0/2.
\]
Hence, we get the required result.

Let us return to the proof of Lemma 9. By Lemmas 14–16 it follows that $\hat{A}_m$ is continuous in $\mathbb{R}_+^{2N}$ and for each $b > 0$ there are $m \in \mathbb{N}$ and $q_m > 0$.
such that $\|\hat{A}_m(q) - A(q)\| \leq b\|q\|$ for $\|q\| \leq q_m$. By Lemma 17, for some $m$ we can find $q_0$ which is a solution of the equation $\hat{A}_m(q_0) = 0$. Hence, we have

$$\Phi_j(\hat{f}_{q_0m}) = a_j \quad \text{for } j = 1, \ldots, N.$$ 

But this contradicts the extremality of $f$, since $\hat{f}_{q_0m}(E) \in D$. ■

3. Proof of Theorem 4. Before we prove the theorem we recall some auxiliary results.

**Lemma 18.** Let $\varphi \in H^1(E)$ be such that

$$\varphi^*(\zeta) \prod_{k=1}^m (\zeta - \sigma_k) \in \mathbb{R}_{>0} \quad \text{for a.a. } \zeta \in \partial E,$$

where $\sigma_k \in \mathbb{C}$, $k = 1, \ldots, m$. Then there exist $r \in \mathbb{R}$ and $\alpha_k \in \mathbb{E}$, $k = 1, \ldots, m$, such that

$$\varphi(\zeta) = r \prod_{k=1}^m (\zeta - \alpha_k)(1 - \sigma_k \zeta) \prod_{k=1}^m (1 - \bar{\sigma}_k \zeta), \quad \zeta \in E.$$

This lemma is a generalization of Lemma 8.4.6 of [Jar-Pfl].

**Proof.** Put $\tilde{\varphi}(\zeta) = \varphi(\zeta) \prod_{k=1}^m (1 - \sigma_k \zeta)$. Then $\tilde{\varphi} \in H^1(E)$ and

$$\frac{1}{\zeta^m} \tilde{\varphi}^*(\zeta) \in \mathbb{R}_{>0} \quad \text{for a.a. } \zeta \in \partial E.$$

Hence, it is enough to prove the lemma for $\sigma_k = 0$, $k = 1, \ldots, m$. Set

$$P(\zeta) = \sum_{k=0}^m \frac{\varphi(k)(0)}{k!} \zeta^k + \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(0)}{k!} \zeta^{2m-k}.$$

It is easy to see that if $\psi(\zeta) := (\varphi(\zeta) - P(\zeta)) / \zeta^m$, then $\psi \in H^1(E)$ and $\psi^*(\zeta) \in \mathbb{R}$ for a.a. $\zeta \in \partial E$. Hence $\psi \equiv 0$.

Let $t(\theta) := P(e^{i\theta}) / e^{i\theta m}$. We know that $t$ is $\mathbb{R}$-analytic and $t(\theta) \geq 0$ for $\theta \in \mathbb{R}$. If for some $\theta_0 \in \mathbb{R}$ we have $t(\theta_0) = 0$ then $t(\theta) = (\theta - \theta_0)^k i(\theta)$, where $k$ is even.

Note that $\tilde{P}(1/\zeta) = P(\zeta) / \zeta^2$ and if $P(0) = 0$, then $P(\zeta) = \zeta^k \tilde{P}(\zeta)$, $\tilde{P}(0) \neq 0$, $\deg \tilde{P} = 2m - 2k$, and $P(1/\zeta) = \tilde{P}(\zeta) / \zeta^{2(m-k)}$. Now, it is enough to note that if $P(\zeta_0) = 0$, $\zeta_0 \neq 0$, then $P(1/\zeta_0) = 0$ and if

$$Q(\zeta) := \frac{P(\zeta)}{(\zeta - \zeta_0)(1 - \bar{\zeta}_0 \zeta)},$$

then $Q(1/\zeta) = Q(\zeta) / \zeta^{2(m-1)}$. ■

**Lemma 19.** Let $S_1$, $S_2$ be singular inner functions and let $S_1 S_2 \equiv 1$. Then $S_1, S_2 \equiv 1$. 


Proof. Suppose that

$$S_j(z) = \exp \left( - \int_0^z \frac{e^{\alpha t} + z}{e^{\alpha t} - z} \, d\mu_j(t) \right), \quad j = 1, 2,$$

where $\mu_1$ and $\mu_2$ are non-negative Borel measures, singular w.r.t. Lebesgue measure. Then $S_1S_2 \equiv 1$ is equivalent to $\mu_1 + \mu_2 = 0$. Since $\mu_j \geq 0$, $j = 1, 2$, we get $\mu_1 = \mu_2 = 0$. □

Proof of Theorem 4. We know that $\varphi_j = B_jS_jF_j$, where $B_j$ is a Blaschke product, $S_j$ is a singular inner function and $F_j$ is an outer function. Take $s := (F_1, \ldots, F_n)$. Note that $|\varphi_j^*(\zeta)/F_j^*(\zeta)| = 1$ for a.a. $\zeta \in \partial E$ and

$$\frac{\partial u}{\partial \bar{z}_j}(z) = \frac{|\varphi_j|^2}{\bar{\varphi}_j} \quad \text{for } j = 1, \ldots, n.$$

We want to show that the assumptions of Theorem 1 are satisfied. Let $\varphi_j \neq 0$, $j = 1, \ldots, n$. Hence $\nabla u(\varphi^*(\zeta))$ exists for a.a. $\zeta \in \partial E$. We have

$$|\varphi_j + F_jh_j|^{2p_j} - |\varphi_j|^{2p_j} - 2 \Re \left( \frac{\varphi_j}{\bar{\varphi}_j} F_jh_j \right) |h_j| = |\varphi_j|^{2p_j} \frac{|1 + \frac{F_j}{\bar{\varphi}_j} h_j|^{2p_j} - 1 - 2 \Re \left( \frac{F_j}{\bar{\varphi}_j} h_j \right) |h_j|}{|\bar{\varphi}_j|}.$$

From the equality

$$\lim_{z \to 0} \frac{|1 + z|^\alpha - 1 - \alpha \Re z}{|z|} = 0, \quad \alpha > 0,$$

we see that all the assumptions of Theorem 1 are satisfied.

Hence, by Corollary 3, there exist $g \in H^\infty(E, \mathbb{C}^n)$ and $g \in L^\infty(\partial E)$, $\varphi > 0$, such that

$$Q(\zeta)g(\zeta)F_j^*(\zeta)|\varphi_j^*(\zeta)|^{2p_j} = g_j^*(\zeta) \quad \text{for a.a. } \zeta \in \partial E, \ j = 1, \ldots, n,$$

where $Q(\zeta) = \prod_{k=1}^n (\zeta - \sigma_k)$ is a polynomial witnessing the $m$-type. This is equivalent to

$$Q(\zeta)g(\zeta)|F_j^*(\zeta)|^{2p_j} = B_j^*(\zeta)S_j^*(\zeta)g_j^*(\zeta) \quad \text{for a.a. } \zeta \in \partial E, \ j = 1, \ldots, n.$$

By Lemma 18 there exist $r_j > 0$ and $\alpha_{kj} \in \mathcal{E}$ such that

$$B_j^*(\zeta)S_j^*(\zeta)g_j^*(\zeta) = r_j \prod_{k=1}^n (\zeta - \alpha_{kj})(1 - \bar{\sigma}_k\zeta) \prod_{k=1}^n (1 - \bar{\sigma}_k\zeta)$$

$$B_j^*(\zeta)S_j^*(\zeta)g_j^*(\zeta) = r_j \prod_{k=1}^n (\zeta - \alpha_{kj})(1 - \bar{\sigma}_k\zeta) \prod_{k=1}^n (1 - \bar{\sigma}_k\zeta)$$
and there exist \( r_0 > 0 \) and \( \alpha_{k0} \in \mathbb{F} \) such that
\[
Q(\zeta) g(\zeta) = \sum_{j=1}^{n} B_j^*(\zeta) S_j^*(\zeta) g_j^*(\zeta) = r_0 \frac{\prod_{k=1}^{m} (\zeta - \alpha_{k0})(1 - \overline{\alpha}_{k0} \zeta)}{\prod_{k=1}^{m} (1 - \overline{\alpha}_k \zeta)}.
\]
We have
\[
r_0 \prod_{k=1}^{m} (\zeta - \alpha_{k0})(1 - \overline{\alpha}_{k0} \zeta)|F_j(\zeta)|^{2p_j} = r_j \prod_{k=1}^{m} (\zeta - \alpha_{kj})(1 - \overline{\alpha}_{kj} \zeta).
\]
Hence
\[
F_j(\zeta) = a_j \prod_{k=1}^{m} \left( \frac{1 - \overline{\alpha}_{kj} \zeta}{1 - \overline{\alpha}_{k0} \zeta} \right)^{1/p_j},
\]
where \( a_j \in \mathbb{C} \setminus \{0\} \). From (6) it follows that
\[
B_j(\zeta) = \prod_{k=1}^{m} \left( \frac{\zeta - \alpha_{kj}}{1 - \overline{\alpha}_{kj} \zeta} \right)^{r_{kj}}, \quad \text{where } r_{kj} \in \{0, 1\}.
\]
Hence
\[
S_j(\zeta) g_j(\zeta) = r_j \prod_{k=1}^{m} (\zeta - \alpha_{kj})^{1-r_{kj}} (1 - \overline{\alpha}_{kj} \zeta)^{1+r_{kj}} \prod_{k=1}^{m} (1 - \overline{\alpha}_k \zeta).
\]
Since the right-hand side is an outer function, from Lemma 19 we conclude that \( S_j \equiv 1, j = 1, \ldots, n \).

From (5) and (6) we see that \( |a_j|^{2p_j} = r_j/r_0 \) and from (3) and (4) it follows that
\[
\sum_{j=1}^{n} |a_j|^{2p_j} \prod_{k=1}^{m} (\zeta - \alpha_{kj})(1 - \overline{\alpha}_{kj} \zeta) = \prod_{k=1}^{m} (\zeta - \alpha_{k0})(1 - \overline{\alpha}_{k0} \zeta), \quad \zeta \in \mathbb{E}.
\]
So, we get the required result.

4. The case of complex geodesics

**Lemma 20.** Any \( \kappa_D \)- and \( \tilde{\kappa}_D \)-geodesic is extremal for an appropriate problem \((P)\) of 1-type.

**Proof.** The case of a \( \kappa_D \)-geodesic. Consider problem \((P)\) with linear functionals such that:

- \( N = 4n \),
- \( w_j := (0, \ldots, 1, \ldots, 0) \) and \( a_j := \text{Re } z_j \) for \( j = 1, \ldots, n \),
- \( w_j := (0, \ldots, -i, \ldots, 0) \) and \( a_j := \text{Im } z_j \) for \( j = n + 1, \ldots, 2n \),
- \( w_j := (0, \ldots, 1/\zeta, \ldots, 0) \) and \( a_j := \text{Re } X_j \) for \( j = 2n + 1, \ldots, 3n \),
- \( w_j := (0, \ldots, -i/\zeta, \ldots, 0) \) and \( a_j := \text{Im } X_j \) for \( j = 3n + 1, \ldots, 4n \),

where \( z \in D \) and \( X \in \mathbb{C}^n \setminus \{0\} \).
Extremal mappings in complex ellipsoids

It is easy to see that the corresponding linear functionals are linearly independent and problem (P) is of 1-type.

Let us show that any $\kappa_D$-geodesic $f$ for $(z,X)$ is extremal for this problem (P). Suppose that there exists a mapping $g \in \mathcal{O}(E,D)$ such that $g(0) = z$, $g'(0) = X$, and $g(E) \subseteq D$. Write $\tilde{g}(\zeta) := g(\zeta) + \zeta tX$, where $t > 0$ will be defined later. Then $\tilde{g}(0) = g(0) = z$ and $\tilde{g}'(0) = g'(0) + tX = (1+t)X$. If we take $t$ such that $\tilde{g}(E) \subseteq D$ (that is possible, because $g(E) \subseteq D$), then we have a contradiction with $f$ being a $\kappa_D$-geodesic.

The case of a $\tilde{k}_D$-geodesic. Consider problem (P) with linear functionals such that $f \in \mathcal{O}(E,D)$ is extremal iff $f(0) = z$, $f(\sigma) = w$, where $\sigma > 0$, and there is no mapping $g \in \mathcal{O}(E,D)$ such that

1. $g(0) = z$, $g(\sigma) = w$,
2. $g(E) \subseteq D$.

(The functions $w_j$ in this case can be constructed similarly to the case of a $\kappa_D$-geodesic. It is enough to replace $1/\zeta$ by $1/(\zeta-\sigma)$ and $-i/\zeta$ by $-i/(\zeta-\sigma)$.) It is easy to see that the relevant linear functionals are linearly independent and that the problem (P) is of 1-type.

Let us show that any $\tilde{k}_D$-geodesic $f$ is extremal for this problem. Suppose that there exists a mapping $g \in \mathcal{O}(E,D)$ such that $g(0) = z$, $g(\sigma) = w$, and $g(E) \subseteq D$. Define

$\tilde{g}(\zeta) := g(\zeta) + \frac{\zeta}{\sigma} (g(\sigma) - g(t\sigma))$,

where $0 < t < 1$ will be defined later. Then $\tilde{g}(0) = g(0) = z$ and $\tilde{g}(t\sigma) = g(\sigma) = w$. If we take $t$ such that $\tilde{g}(E) \subseteq D$ (use $g(E) \subseteq D$), then we have a contradiction, because $f$ is a $\tilde{k}_D$-geodesic.

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References
