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On extremal mappings in complex ellipsoids

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Abstract. Using a generalization of [Pol] we present a description of complex geodesics in arbitrary complex ellipsoids.

1. Introduction and the main results.

Let $\mathcal{E}(p) := \{|z_1|^{2p_1} + \ldots + |z_n|^{2p_n} < 1\} \subset \mathbb{C}^n$, where $p = (p_1, \ldots, p_n)$, $p_j > 0$, $j = 1, \ldots, n$; $\mathcal{E}(p)$ is called a *complex ellipsoid*.

The aim of the paper is to characterize complex $\varkappa_{\mathcal{E}(p)}$- and $\widetilde{k}_{\mathcal{E}(p)}$-geodesics. The case where $\mathcal{E}(p)$ is convex (i.e. $p_1, \ldots, p_n \geq 1/2$) has been solved in [Jar-Pfl-Zei]. The paper is inspired by methods of [Pol].

Let $D \subset \mathbb{C}^n$ be a domain and let $\varphi \in \mathcal{O}(E, D)$, where E denotes the unit disk in $\mathbb{C}$ and $\mathcal{O}(\Omega, D)$ is the set of all holomorphic mappings $\Omega \to D$. Recall that φ is said to be a $\varkappa_D$*-geodesic* if there exists $(z, X) \in D \times \mathbb{C}^n$ such that:

- $\varphi(0) = z$ and $\varphi'(0) = \lambda_\varphi X$ for some $\lambda_\varphi > 0$,
- for any $\psi \in \mathcal{O}(E, D)$ such that $\psi(0) = z$ and $\psi'(0) = \lambda_\psi X$ with $\lambda_\psi > 0$, we have $\lambda_\psi \leq \lambda_\varphi$.

We say that φ is a $\widetilde{k}_D$*-geodesic* if there exists $(z, w) \in D \times D$ such that:

- $\varphi(0) = z$ and $\varphi(\sigma_\varphi) = w$ for some $\sigma_\varphi \in (0, 1)$,
- for any $\psi \in \mathcal{O}(E, D)$ such that $\psi(0) = z$ and $\psi(\sigma_\psi) = w$ with $\sigma_\psi > 0$, we have $\sigma_\varphi \leq \sigma_\psi$; cf. [Pan].

Let us fix some further notations:

- $H^\infty(\Omega, \mathbb{C}^n) :=$ the space of all bounded holomorphic mappings $\Omega \to \mathbb{C}^n$;
- $\|f\|_\infty := \sup\{\|f(z)\| : z \in \Omega\}$, $f \in H^\infty(\Omega, \mathbb{C}^n)$, where $\|\ \|$ denotes the Euclidean norm in $\mathbb{C}^n$;

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• $f^*(\zeta)$:= the non-tangential boundary value of f at $\zeta \in \partial E$, $f \in H^\infty(E,\mathbb{C}^n)$;
• $\mathcal{A}(\Omega,\mathbb{C}^n) := \mathcal{C}(\overline{\Omega},\mathbb{C}^n) \cap \mathcal{O}(\Omega,\mathbb{C}^n)$;
• $z \cdot w := (z_1w_1,\dots,z_nw_n)$, $z \bullet w := \sum_{j=1}^n z_jw_j$, $z=(z_1,\dots,z_n)$, $w=(w_1,\dots,w_n)\in\mathbb{C}^n$;
• $A_\nu := \{z\in\mathbb{C} : \nu < |z| < 1\}$, $\nu\in(0,1)$;
• $PSH(\Omega)$:= the set of all plurisubharmonic functions on Ω.

Fix $w_1,\dots,w_N \in \mathcal{A}(A_\nu,\mathbb{C}^n)$ and define

$$\Phi_j(h) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(h^*(e^{i\theta}) \bullet w_j(e^{i\theta}))\,d\theta, \qquad h\in H^\infty(E,\mathbb{C}^n),\ j=1,\dots,N.$$

We say that the functionals $\Phi_1,\dots,\Phi_N$ are *linearly independent* if for arbitrary $s=(s_1,\dots,s_n), g\in H^\infty(E,\mathbb{C}^n)$, and $\lambda_1,\dots,\lambda_N\in\mathbb{R}$ such that s_k nowhere vanishes on E, $k=1,\dots,n$, and $g(0)=0$ the following implication is true: if $\sum_{j=1}^N \lambda_j w_j\cdot s^* = g^*$ on a subset of ∂E of positive measure, then $\lambda_1=\ldots=\lambda_N=0$.

Later on, we always assume that the functionals $\Phi_1,\dots,\Phi_N$ are linearly independent.

Problem ($\mathcal{P}$). Given a bounded domain $D\subset\mathbb{C}^n$ and numbers $a_1,\dots,a_N\in\mathbb{R}$, find a mapping $f\in\mathcal{O}(E,D)$ such that $\Phi_j(f)=a_j$, $j=1,\dots,N$, and there is no mapping $g\in\mathcal{O}(E,D)$ with

$$\Phi_j(g)=a_j,\quad j=1,\dots,N,\quad g(E)\Subset D.$$

Any solution of ($\mathcal{P}$) is called an *extremal mapping for* ($\mathcal{P}$) or, simply, an *extremal*.

Problem ($\mathcal{P}$) is a generalization of Problem (P) from [Pol].

We say that problem ($\mathcal{P}$) is *of m-type* if there exists a polynomial $Q(\zeta)=\prod_{k=1}^m(\zeta-\sigma_k)$ with $\sigma_1,\dots,\sigma_m\in E$ such that Qw_j extends to a mapping of class $\mathcal{A}(E,\mathbb{C}^n)$, $j=1,\dots,N$.

One can prove that (for bounded domains $D\subset\mathbb{C}^n$) any complex $\varkappa_D$- or $\widetilde{k}_D$-geodesic may be characterized as an extremal for a suitable problem ($\mathcal{P}$) of 1-type (cf. §4).

The main result of the paper is the following

Theorem 1. *Let $D\Subset G\Subset\mathbb{C}^n$ be domains and let $u\in PSH(G)\cap\mathcal{C}(G)$ be such that $D=\{u<0\}$, $\partial D=\{u=0\}$. Suppose that $f\in\mathcal{O}(E,D)$ is an extremal for ($\mathcal{P}$). Assume that there exist a set $S\subset\partial E$, a mapping $s=(s_1,\dots,s_n)\in H^\infty(E,\mathbb{C}^n)$, a number $\varepsilon>0$, and a function $v: S\times\mathcal{A}(E,\mathbb{C}^n)\to\mathbb{C}$ such that*:

(a) *$\partial E\setminus S$ has zero measure*,
(b) *$f^*(\zeta)$, $\nabla u(f^*(\zeta))$ and $s^*(\zeta)$ are defined for all $\zeta\in S$*,

(c) s_k *nowhere vanishes on* E *for* $k = 1, \ldots, n$,

(d) $u(f^*(\zeta) + s^*(\zeta) \cdot h(\zeta)) = u(f^*(\zeta)) + 2\operatorname{Re}(\nabla u(f^*(\zeta)) \bullet (s^*(\zeta) \cdot h(\zeta))) + v(\zeta, h)$, $\zeta \in S$, $h \in \mathcal{A}(E, \mathbb{C}^n)$, $\|h\|_\infty \le \varepsilon$,

(e) $\lim_{h\to 0} \sup\{|v(\zeta, h)| : \zeta \in S\}/\|h\|_\infty = 0$.

Then

$$f^*(\zeta) \in \partial D \quad \text{for a.a. } \zeta \in \partial E$$

and there exist $\varrho \in L^\infty(\partial E)$, $\varrho > 0$, $g \in H^\infty(E, \mathbb{C}^n)$, *and* $(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N \setminus \{0\}$ *such that*

$$\sum_{k=1}^{N} \lambda_k w_k(\zeta) \cdot s^*(\zeta) + g^*(\zeta) = \varrho(\zeta) s^*(\zeta) \cdot \nabla u(f^*(\zeta)) \quad \text{for a.a. } \zeta \in \partial E.$$

R e m a r k 2. Under the assumptions of Theorem 1, if $u \in \mathcal{C}^1(G) \cap PSH(G)$, then one can take $s :\equiv (1, \ldots, 1)$.

As an easy corollary to Theorem 1 we obtain

COROLLARY 3. *Under the assumptions of Theorem* 1, *if problem* $(\mathcal{P})$ *is of* m*-type*, *then there exist* $\varrho \in L^\infty(\partial E)$, $\varrho > 0$, *and* $g \in H^\infty(E, \mathbb{C}^n)$ *such that*

$$g^*(\zeta) = Q(\zeta)\varrho(\zeta) s^*(\zeta) \cdot \nabla u(f^*(\zeta)) \quad \text{for a.a. } \zeta \in \partial E.$$

Theorem 1 generalizes Theorems 2 and 3 of [Pol] (cf. Remark 2). The proof of Theorem 1 will be presented in §2.

Corollary 3 gives a tool for describing the extremal mappings for problems $(\mathcal{P})$ of m-type in the case where D is an arbitrary complex ellipsoid $\mathcal{E}(p)$.

THEOREM 4. *Let* $\varphi : E \to \mathcal{E}(p)$ *be an extremal for problem* $(\mathcal{P})$ *of* m*-type such that* $\varphi_j \not\equiv 0$, $j = 1, \ldots, n$. *Then*

$$\varphi_j(\lambda) = a_j \prod_{k=1}^{m} \left(\frac{\lambda - \alpha_{kj}}{1 - \overline{\alpha}_{kj}\lambda}\right)^{r_{kj}} \left(\frac{1 - \overline{\alpha}_{kj}\lambda}{1 - \overline{\alpha}_{k0}\lambda}\right)^{1/p_j}, \qquad j = 1, \ldots, n,$$

where

- $a_1, \ldots, a_n \in \mathbb{C} \setminus \{0\}$,
- $\alpha_{kj} \in \overline{E}$, $k = 1, \ldots, m$, $j = 0, \ldots, n$,
- $r_{kj} \in \{0, 1\}$ *and*, *if* $r_{kj} = 1$, *then* $\alpha_{kj} \in E$,
- $\sum_{j=1}^{n} |a_j|^{2p_j} \prod_{k=1}^{m} (\zeta - \alpha_{kj})(1 - \overline{\alpha}_{kj}\zeta) = \prod_{k=1}^{m} (\zeta - \alpha_{k0})(1 - \overline{\alpha}_{k0}\zeta)$, $\zeta \in E$.

In particular, *if* φ *is a complex* $\varkappa_{\mathcal{E}(p)}$*- or* $\widetilde{k}_{\mathcal{E}(p)}$*-geodesic*, *then* φ *is of the above form with* $m = 1$.

Theorem 4 generalizes §6 of [Pol] and Theorem 1 of [Jar-Pfl-Zei]. The proof of Theorem 4 will be given in §§3, 4.

Remark 5. In the case where $\mathcal{E}(p)$ is convex any mapping described in Theorem 4 with $m = 1$ is a complex geodesic in $\mathcal{E}(p)$ ([Jar-Pfl-Zei]). This is no longer true if $\mathcal{E}(p)$ is not convex (cf. [Pfl-Zwo] for the case $n = 2$, $p_1 = 1$, $p_2 < 1/2$).

2. Proof of Theorem 1. Note that there are two possibilities: either $u \circ f^* = 0$ a.e. on ∂E or there exists $\tau > 0$ such that the set $\{\theta : u(f^*(e^{i\theta})) < -\tau\}$ has positive measure. If such a τ exists, fix one of them. We put

$$P_0 := \begin{cases} \emptyset & \text{in the first case,} \\ \{\theta : u(f^*(e^{i\theta})) < -\tau\} & \text{in the second case,} \end{cases}$$

$A_0 := [0, 2\pi) \setminus P_0$, and

$$p_s(h) := \frac{1}{2\pi} \int\limits_{A_0} [\operatorname{Re}(s^*(e^{i\theta}) \cdot \nabla u(f^*(e^{i\theta})) \bullet h(e^{i\theta}))]^+ \, d\theta \quad \text{for } h \in L^1(\partial E, \mathbb{C}^n),$$

where $L^1(\partial E, \mathbb{C}^n)$ denotes the space of all Lebesgue integrable mappings $\partial E \to \mathbb{C}^n$.

Remark 6. (a) Under the assumptions of Theorem 1, there exists $M > 0$ such that

$$\|s^*(\zeta) \cdot \nabla u(f^*(\zeta))\| \le M \quad \text{for a.a. } \zeta \in \partial E.$$

(b) $p_s(h)$ is a seminorm on $H^1(E, \mathbb{C}^n)$ and $p_s(h) \le M\|h\|_1$, where $H^1(E)$ denotes the first Hardy space of holomorphic functions,

$$H^1(E, \mathbb{C}^n) := \{(f_1, \ldots, f_n) : \ f_j \in H^1(E)\},$$

and $\| \ \|_1$ denotes the norm in $H^1(E, \mathbb{C}^n)$.

The proof of Theorem 1 is based on the following result.

Lemma 7 (cf. [Pol], Lemma 6). *Under the assumptions of Theorem* 1 *there exist* $T > 0$, $j \in \{1, \ldots, N\}$, *and* $\delta \in \{-1, 1\}$ *such that*

$$\delta \Phi_j(s \cdot h) \le T p_s(h)$$

for $h \in X_j := \{h \in H^1(E, \mathbb{C}^n) : \Phi_l(s \cdot h) = 0, \ l \ne j\}$.

Let us for a while assume that we already have Lemma 7.

Proof of Theorem 1. By Lemma 7 there exist $T > 0$, $\delta \in \{-1, 1\}$, and $j \in \{1, \ldots, N\}$ such that

$$\delta \Phi_j(s \cdot h) \le T p_s(h) \quad \text{for } h \in X_j.$$

Let $\widetilde{\Phi}(h) := \delta \Phi_j(s \cdot h)$, $h \in X_j$. Using the Hahn–Banach theorem we can extend $\widetilde{\Phi}$ to $L^1(\partial E, \mathbb{C}^n)$ (we denote this extension by Φ) in such a way that

$$\Phi(h) \le T p_s(h) \quad \text{for } h \in L^1(\partial E, \mathbb{C}^n).$$

We know that $p_s(h) \le M|h|_1$, where $|h|_1$ denotes the norm in $L^1(E,\mathbb{C}^n)$. So $\varPhi$ is continuous on $L^1(\partial E,\mathbb{C}^n)$. By Riesz's theorem, $\varPhi$ can be represented as

$$\varPhi(h) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}(h^*(e^{i\theta}) \bullet \widetilde{w}(e^{i\theta}))\,d\theta, \qquad \text{where } \widetilde{w} \in L^\infty(\partial E,\mathbb{C}^n).$$

It is easy to see that there are $\lambda_1,\dots,\lambda_N$, not all zero, such that $\varPhi(h) = \sum_{k=1}^N \lambda_k \varPhi_k(s\cdot h)$ for $h \in H^1(E,\mathbb{C}^n)$. We denote by G the linear functional on $L^1(\partial E,\mathbb{C}^n)$ defined by the formula

$$G(h) := \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\Big(\sum_{k=1}^N \lambda_k w_k(e^{i\theta}) \bullet s^*(e^{i\theta})\cdot h(e^{i\theta})\Big)\,d\theta.$$

Then $\varPhi(h) - G(h) = 0$ for $h \in H^1(E,\mathbb{C}^n)$. By the theorem of F. & M. Riesz it follows that there exists $g \in H^\infty(E,\mathbb{C}^n)$, $g(0)=0$, such that

$$\widetilde{w} - s^* \cdot \sum_{k=1}^N \lambda_k w_k = g^*.$$

We have

$$\begin{aligned}(1)\qquad \varPhi(h) &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\Big[\Big(\sum_{k=1}^N \lambda_k w_k(e^{i\theta})\cdot s^*(e^{i\theta}) + g^*(e^{i\theta})\Big) \bullet h^*(e^{i\theta})\Big]\,d\theta \\ &\le T\frac{1}{2\pi}\int_{A_0} [\operatorname{Re}(s^*(e^{i\theta})\cdot \nabla u(f^*(e^{i\theta})) \bullet h^*(e^{i\theta}))]^+\,d\theta\end{aligned}$$

for any $h \in H^1(E,\mathbb{C}^n)$. We see that the right-hand side is zero for any $h \in H^1(E,\mathbb{C}^n)$ (hence, for any $h \in L^1(\partial E,\mathbb{C}^n)$) such that

$$\operatorname{Re}(s^*(e^{i\theta})\cdot \nabla u(f^*(e^{i\theta})) \bullet h^*(e^{i\theta})) \le 0$$

on $\partial E \setminus (P_0 \cup \{\zeta \in \partial E : s^*(\zeta)\cdot\nabla u(f^*(\zeta)) = 0\})$.

Hence

$$\sum_{k=1}^N \lambda_k w_k \cdot s^* + g^* = 0 \qquad \text{a.e. on } P_0 \cup \{\zeta \in \partial E : s^*(\zeta)\cdot\nabla u(f^*(\zeta)) = 0\}.$$

We know that $\varPhi_1,\dots,\varPhi_N$ are linearly independent, so the Lebesgue measures of P_0 and of $\{\zeta \in \partial E :\ s^*(\zeta)\cdot\nabla u(f^*(\zeta)) = 0\}$ are zero. Hence

$$\sum_{k=1}^N \lambda_k w_k(\zeta)\cdot s^*(\zeta) + g^*(\zeta) = \varrho(\zeta) s^*(\zeta)\cdot \nabla u(f^*(\zeta)),$$

where $\varrho(\zeta) \in \mathbb{C}\setminus\{0\}$ for a.a. $\zeta \in \partial E$. Now, it is enough to remark that condition (1) implies that $0 < \varrho \le T$ a.e. on ∂E. ∎

Now, we are going to prove Lemma 7.

Proof of Lemma 7. Suppose that the lemma is not true. Then for each $j \in \{1, \dots, N\}$ and $m \in \mathbb{N}$ there are $h^+_{jm}, h^-_{jm} \in X_j$ such that

$$\Phi_j(s \cdot h^+_{jm}) > mp_s(h^+_{jm}), \quad -\Phi_j(s \cdot h^-_{jm}) > mp_s(h^-_{jm}).$$

We may assume that $h^+_{jm}, h^-_{jm} \in \mathcal{A}(E, \mathbb{C}^n)$ and that

$$\Phi_j(s \cdot h^+_{jm}) = 1, \quad \Phi_j(s \cdot h^-_{jm}) = -1.$$

For any $q = (q^+_1, q^-_1, \dots, q^+_N, q^-_N) \in \mathbb{R}^{2N}_+$ we define the function

$$f_{qm} = f + \sum_{j=1}^N (q^+_j s \cdot h^+_{jm} + q^-_j s \cdot h^-_{jm}) = f + s \cdot h_{qm}$$

and the linear mapping $A : \mathbb{R}^{2N}_+ \to \mathbb{R}^N$, $A(q) := (q^+_1 - q^-_1, \dots, q^+_N - q^-_N)$. Note that $\Phi_j(f_{qm}) - \Phi_j(f) = A(q)_j$.

Lemma 8 (see [Pol], Lemma 7). *Let u be a non-positive subharmonic function in E and let $\triangle u$ be the Riesz measure of u. Suppose that one of the following conditions is true*:

(a) $\triangle u(r_0 E) > a > 0$ *for some* $r_0 \in (0, 1)$,

(b) *for some set* $Z \subset [0, 2\pi)$ *with positive measure*, *the upper radial limit of u at $\zeta \in Z$ does not exceed* $-a < 0$ (*i.e.* $\limsup_{r \to 1} u(r\zeta) \le -a$).

Then $u(\zeta) \le -C(1 - |\zeta|)$, *where $C > 0$ is a constant depending only on* r_0, a, *and* Z.

Let $u_0 := u \circ f$.

Lemma 9. *There exist a constant $C > 0$ and constants $t_m > 0$, $m \in \mathbb{N}$, such that for $\|q\| < t_m$ we have*

(a) $f_{qm} \in \mathcal{O}(E, G)$ (*so, we define* $u_{qm} := u \circ f_{qm}$),

(b) $u_{qm}(\zeta) \le v_{qm}(\zeta) := C \ln|\zeta| + \dfrac{1}{2\pi} \displaystyle\int_{A_0} [u^*_{qm}(e^{i\theta})]^+ P(\zeta, \theta)\, d\theta$

for $|\zeta| > 1/2$.

Proof. (a) follows from the assumption that $D \Subset G$.

(b) Suppose that there exists $r_0 \in (0, 1)$ such that $\triangle u_0(r_0 E) > a > 0$. The continuity of u implies that for

$$\widetilde{u}_{qm}(\zeta) := u_{qm}(\zeta) - \frac{1}{2\pi} \int_{A_0} [u^*_{qm}(e^{i\theta})]^+ P(\zeta, \theta)\, d\theta, \quad \zeta \in E,$$

if t_m is small enough then $\triangle \widetilde{u}_{qm}(rE) > a/2$. Hence, from Lemma 8 we get the required result.

If $\triangle u_0(rE) = 0$ for any $r \in (0, 1)$ and $u^*_0(\zeta) = 0$ for a.a. $\zeta \in \partial E$, then by the Riesz representation theorem ([Hay-Ken], Ch. 3.5) we see that u_0 is harmonic in E. But this is a contradiction, since $u_0 \not\equiv 0$. Hence, P_0 has positive measure. From the continuity of u we conclude that if t_m are small enough, then $\{\zeta : \widetilde{u}_{qm}(\zeta) < -\tau/2\}$ has positive measure. By Lemma 8 we get the required result. ■

Let us introduce some new notation: $E_{qm} := \{\zeta \in E : v_{qm}(\zeta) < 0\}$ and

$$g_{qm}(\zeta) := \zeta \exp\left\{\frac{1}{2\pi C}\int\limits_{A_0} [u^*_{qm}(e^{i\theta})]^+ S(\zeta,\theta)\,d\theta\right\}.$$

Here $S(\zeta,\theta) := (\zeta + e^{i\theta})/(\zeta - e^{i\theta})$ is the Schwarz kernel.

Remark 10. Note that $C\ln|g_{qm}| = v_{gm}$, $v_{qm}(\zeta) \ge C\ln|\zeta|$ (hence, $|g_{qm}(\zeta)| \ge |\zeta|$), and $E_{qm} = g^{-1}_{qm}(E)$.

Lemma 11 (cf. [Pol], Statement 2). (a) *E_{qm} is connected*, $0 \in E_{qm}$, (b) *g_{qm} maps E_{qm} conformally onto E.*

Proof. (a) Note that $E_{qm} = \bigcup_{\delta>0}\{\zeta : v_{qm}(\zeta) < -\delta\}$ and

$$\{\zeta : v_{qm}(\zeta) < -\delta\} \subset \{\zeta : |\zeta| < e^{-\delta/C}\}.$$

Since v_{qm} is harmonic outside 0 and $v^*_{qm}(e^{i\theta}) \ge 0$, any connected component of $\{\zeta : v_{qm}(\zeta) < -\delta\}$ must contain 0.

(b) First let us see that $g_{qm} : E_{qm} \to E$ is proper. Let $\zeta_k \to \zeta_0 \in \partial E_{qm}$. If $\zeta_0 \in \partial E$, then $|g_{qm}(\zeta_k)| \to 1$ (since $|g_{qm}| \ge |\zeta|$). If $\zeta_0 \in E$, then $|g_{qm}(\zeta_k)| \to |g_{qm}(\zeta_0)| = 1$.

Since $g'_{qm}(0) \ne 0$ and $g^{-1}_{qm}(0) = \{0\}$, g_{qm} is conformal. ■

We define $\widetilde{f}_{qm}(\zeta) = f_{qm}(g^{-1}_{qm}(\zeta))$, $\widehat{f}_{qm}(\zeta) = \widetilde{f}_{qm}(e^{-\|q\|/m}\zeta)$,

$$\widetilde{A}_m(q) = (\Phi_1(\widetilde{f}_{qm}) - \Phi_1(f), \ldots, \Phi_N(\widetilde{f}_{qm}) - \Phi_N(f)),$$

and

$$\widehat{A}_m(q) = (\Phi_1(\widehat{f}_{qm}) - \Phi_1(f), \ldots, \Phi_N(\widehat{f}_{qm}) - \Phi_N(f)).$$

Remark 12. It is easy to see that $\widetilde{f}_{qm}(E) \subset D$, $\widehat{f}_{qm}(E) \Subset D$, and $\widetilde{A}_m(0) = \widehat{A}_m(0) = 0$.

The following result explains why we have used functionals of the special form.

Lemma 13. *Suppose that*

$$\Phi(h) = \frac{1}{2\pi}\int\limits_0^{2\pi} \mathrm{Re}(h^*(e^{i\theta}) \bullet w(e^{i\theta}))\,d\theta,$$

where $w \in \mathcal{A}(A_\nu, \mathbb{C}^n)$ for some $\nu \in (0,1)$, $f \in H^\infty(E,\mathbb{C}^n)$, and that $g \in \mathcal{O}(E,E)$, $g(0) = 0$. Then

$$|\Phi(f\circ g) - \Phi(f)| \le K\|f\|_\infty \sup_{\zeta\in E}|g(\nu\zeta) - \nu\zeta|,$$

where $K > 0$ depends only on Φ.

Proof. We have

$$\Phi(h) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}(h(\nu e^{i\theta}) \bullet w(\nu e^{i\theta}))\,d\theta. \tag{2}$$

Hence

$$|\Phi(h)| \le (\max_{\zeta\in\partial E}\|w(\nu\zeta)\|)(\max_{\zeta\in\partial E}\|h(\nu\zeta)\|).$$

But

$$\|f(g(\nu\zeta)) - f(\nu\zeta)\| \le (\sup_{\xi\in E}|f'(\nu\xi)|)|g(\nu\zeta)-\nu\zeta|,$$

and $\sup_{\xi\in E}|f'(\nu\xi)| \le \|f\|_\infty/(1-\nu^2)$. ∎

Lemma 14 (cf. [Pol], Statement 3). *The mappings $\widetilde{A}_m$, $\widehat{A}_m$ are continuous in q for $\|q\| < t_m$.*

Proof. It is enough to remark that if $q_k \to q$, then $u^*_{q_k m} \to u^*_{qm}$ uniformly on ∂E. Hence $g_{q_k m} \to g_{qm}$ uniformly on compact subsets of E. It is evident from the last assertion that also $g^{-1}_{q_k m} \to g^{-1}_{qm}$, $\widetilde{f}_{q_k m} \to \widetilde{f}_{qm}$, and $\widehat{f}_{q_k m} \to \widehat{f}_{qm}$ uniformly on compact sets. Since the Φ_j are continuous with respect to this convergence (this follows easily from (2)), we conclude the proof. ∎

Lemma 15. *For each $b > 0$ there is $m_0 \in \mathbb{N}$ such that for any $m \ge m_0$ there is $q_m > 0$ such that $\|A(q) - \widetilde{A}_m(q)\| \le b\|q\|$ whenever $\|q\| \le q_m$.*

Proof. It follows from the definition of A, $\widetilde{A}_m$ that it is enough to prove the inequality

$$|\Phi(\widetilde{f}_{qm}) - \Phi(f_{qm})| \le b\|q\|$$

for small q, where Φ is a functional of our special form. By Lemma 14 it is enough to show that

$$\sup_{\zeta\in\nu E}|g^{-1}_{qm}(\zeta) - \zeta| \le b\|q\|$$

for small q. Note that

$$\sup_{\zeta\in\nu E}|g^{-1}_{qm}(\zeta)-\zeta| \le \sup_{\zeta\in\nu E}|g_{qm}(\zeta)-\zeta|$$

and for small q_m (such that $|1-\exp q_m| \le 2q_m$) and $\|q\| \le q_m$,

$$\left|1 - \exp\left(\frac{1}{2\pi C}\int_{A_0}[u^*_{qm}(e^{i\theta})]^+ S(\zeta,\theta)\,d\theta\right)\right| \le 2\frac{1+\nu}{1-\nu}\left(\frac{1}{2\pi C}\int_{A_0}[u^*_{qm}(e^{i\theta})]^+\,d\theta\right)$$

for $\zeta \in \nu E$. Hence, it is enough to show that

$$\int\limits_{A_0} [u^*_{qm}(e^{i\theta})]^+ \, d\theta \le \int\limits_{A_0} 2[\mathrm{Re}(\nabla u(f^*(e^{i\theta})) \bullet s^*(e^{i\theta}) \cdot h_{qm}(e^{i\theta}))]^+ \, d\theta + o(\|h_{qm}\|_\infty).$$

But $p_s(h_{qm}) \le \|q\| \max\{p_s(h_{jm}) : j = 1, \ldots, N\} \le \|q\|/m$. Hence, if m is large and q_m is small enough, we get the required result. ∎

LEMMA 16. *For each $b > 0$ there is $m_0 \in \mathbb{N}$ such that for any $m \ge m_0$ there is $q_m > 0$ such that $\|\widetilde{A}_m(q) - \widehat{A}_m(q)\| \le b\|q\|$ whenever $\|q\| \le q_m$.*

Proof. As in Lemma 15, by Lemma 13 it is enough to prove the inequality

$$\sup_{\zeta \in \nu E} |e^{-\|q\|/m}\zeta - \zeta| \le b\|q\|$$

for small $\|q\|$. But for small $\|q\|/m$ we have $|1 - e^{-\|q\|/m}| \le 2\|q\|/m$. Hence, we get the required result. ∎

LEMMA 17 (cf. [Pol], Lemma 8). *For any continuous mapping $F : \mathbb{R}^{2N}_+ \to \mathbb{R}^N$, if*

$$\|F(x) - A(x)\| \le b\|x\| \quad \text{for } x \in B(0, r) \cap \mathbb{R}^{2N}_+,$$

where $b = 1/(2\sqrt{N})$, then there exists $q \in B(0, r) \cap \mathbb{R}^{2N}_+ \setminus \{0\}$ such that $F(q) = 0$.

Proof. Define

$$\mathcal{Q} := \{(x_1, \ldots, x_N) : \ 0 < x_j < t_0, \ j = 1, \ldots, N\}$$

and

$$\pi : \mathbb{R}^N \ni (x_1, \ldots, x_N) \to (x_1, t_0 - x_1, \ldots, x_N, t_0 - x_N) \in \mathbb{R}^{2N},$$

where $t_0 = (2\sqrt{N})^{-1} \min\{1, r\}$. It is easy to check that $\|\pi(l)\| \le t_0\sqrt{N}$ for $l \in \overline{\mathcal{Q}}$ and $\pi(\mathcal{Q}) \subset B(0, r) \cap \mathbb{R}^{2N}_+$. Note that

$$\|F \circ \pi(l) - A \circ \pi(l)\| \le b\|\pi(l)\| \le t_0/2 \quad \text{for } l \in \overline{\mathcal{Q}}.$$

Consider the homotopy defined by the formula $\widetilde{F}_t = tF \circ \pi + (1-t)A \circ \pi$. It is enough to show that $0 \notin \widetilde{F}_t(\partial \mathcal{Q})$. Then from the homotopical invariance of the degree of mappings [Zei] we have $\deg(F \circ \pi, \mathcal{Q}, 0) = \deg(A \circ \pi, \mathcal{Q}, 0) \ne 0$, hence $0 \in F \circ \pi(\mathcal{Q})$.

It is easy to see that for any $l \in \partial \mathcal{Q}$,

$$t_0 \le \|A \circ \pi(l)\| \le \|\widetilde{F}_t(l)\| + t\|F \circ \pi(l) - A \circ \pi(l)\| \le \|\widetilde{F}_t(l)\| + t_0/2.$$

Hence, we get the required result. ∎

Let us return to the proof of Lemma 9. By Lemmas 14–16 it follows that $\widehat{A}_m$ is continuous in $\mathbb{R}^{2N}_+$ and for each $b > 0$ there are $m \in \mathbb{N}$ and $q_m > 0$

such that $\|\widehat{A}_m(q) - A(q)\| \le b\|q\|$ for $\|q\| \le q_m$. By Lemma 17, for some m we can find q_0 which is a solution of the equation $\widehat{A}_m(q_0) = 0$. Hence, we have

$$\Phi_j(\widehat{f}_{q_0 m}) = a_j \quad \text{for } j = 1, \ldots, N.$$

But this contradicts the extremality of f, since $\widehat{f}_{q_0 m}(E) \Subset D$. ∎

3. Proof of Theorem 4. Before we prove the theorem we recall some auxiliary results.

LEMMA 18. *Let $\varphi \in H^1(E)$ be such that*

$$\frac{\varphi^*(\zeta)}{\prod_{k=1}^m (\zeta - \sigma_k)} \in \mathbb{R}_{>0} \quad \textit{for a.a. } \zeta \in \partial E,$$

where $\sigma_k \in \mathbb{C}$, $k = 1, \ldots, m$. Then there exist $r \in \mathbb{R}$ and $\alpha_k \in \overline{E}$, $k = 1, \ldots, m$, such that

$$\varphi(\zeta) = r \frac{\prod_{k=1}^m (\zeta - \alpha_k)(1 - \overline{\alpha}_k \zeta)}{\prod_{k=1}^m (1 - \overline{\sigma}_k \zeta)}, \quad \zeta \in E.$$

This lemma is a generalization of Lemma 8.4.6 of [Jar-Pfl].

Proof. Put $\widetilde{\varphi}(\zeta) = \varphi(\zeta) \prod_{k=1}^m (1 - \overline{\sigma}_k \zeta)$. Then $\widetilde{\varphi} \in H^1(E)$ and

$$\frac{1}{\zeta^m} \widetilde{\varphi}^*(\zeta) \in \mathbb{R}_{>0} \quad \text{for a.a. } \zeta \in \partial E.$$

Hence, it is enough to prove the lemma for $\sigma_k = 0$, $k = 1, \ldots, m$. Set

$$P(\zeta) = \sum_{k=0}^m \frac{\varphi^{(k)}(0)}{k!} \zeta^k + \sum_{k=0}^{m-1} \frac{\overline{\varphi^{(k)}(0)}}{k!} \zeta^{2m-k}.$$

It is easy to see that if $\psi(\zeta) := (\varphi(\zeta) - P(\zeta))/\zeta^m$, then $\psi \in H^1(E)$ and $\psi^*(\zeta) \in \mathbb{R}$ for a.a. $\zeta \in \partial E$. Hence $\psi \equiv 0$.

Let $t(\theta) := P(e^{i\theta})/e^{i\theta m}$. We know that t is $\mathbb{R}$-analytic and $t(\theta) \ge 0$ for $\theta \in \mathbb{R}$. If for some $\theta_0 \in \mathbb{R}$ we have $t(\theta_0) = 0$ then $t(\theta) = (\theta - \theta_0)^k \widetilde{t}(\theta)$, where k is even.

Note that $\overline{P(1/\overline{\zeta})} = P(\zeta)/\zeta^{2m}$ and if $P(0) = 0$, then $P(\zeta) = \zeta^k \widetilde{P}(\zeta)$, $\widetilde{P}(0) \ne 0$, $\deg \widetilde{P} = 2m - 2k$, and $\overline{\widetilde{P}(1/\overline{\zeta})} = \widetilde{P}(\zeta)/\zeta^{2(m-k)}$. Now, it is enough to note that if $P(\zeta_0) = 0$, $\zeta_0 \ne 0$, then $P(1/\overline{\zeta}_0) = 0$ and if

$$Q(\zeta) := \frac{P(\zeta)}{(\zeta - \zeta_0)(1 - \overline{\zeta}_0 \zeta)},$$

then $\overline{Q(1/\overline{\zeta})} = Q(\zeta)/\zeta^{2(m-1)}$. ∎

LEMMA 19. *Let S_1, S_2 be singular inner functions and let $S_1 S_2 \equiv 1$. Then $S_1, S_2 \equiv 1$.*

P r o o f. Suppose that

$$S_j(z) = \exp\Big(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,d\mu_j(t)\Big), \qquad j = 1, 2,$$

where μ_1 and μ_2 are non-negative Borel measures, singular w.r.t. Lebesgue measure. Then $S_1S_2 \equiv 1$ is equivalent to $\mu_1+\mu_2 = 0$. Since $\mu_j \ge 0$, $j = 1, 2$, we get $\mu_1 = \mu_2 = 0$. ∎

P r o o f o f T h e o r e m 4. We know that $\varphi_j = B_jS_jF_j$, where B_j is a Blaschke product, S_j is a singular inner function and F_j is an outer function. Take $s := (F_1, \ldots, F_n)$. Note that $|\varphi_j^*(\zeta)/F_j^*(\zeta)| = 1$ for a.a. $\zeta \in \partial E$ and

$$\frac{\partial u}{\partial z_j}(\varphi) = p_j \frac{|\varphi_j|^{2p_j}}{\varphi_j} \qquad \text{for } j = 1, \ldots, n.$$

We want to show that the assumptions of Theorem 1 are satisfied. Let $u(z) := \sum_{j=1}^n |z_j|^{2p_j} - 1$ be the defining function for $\mathcal{E}(p)$.

We know that $\varphi_j \not\equiv 0$, $j = 1, \ldots, n$. Hence $\nabla u(\varphi^*(\zeta))$ exists for a.a. $\zeta \in \partial E$. We have

$$\frac{|\varphi_j + F_jh_j|^{2p_j} - |\varphi_j|^{2p_j} - 2\,\mathrm{Re}\big(p_j \frac{|\varphi_j|^{2p_j}}{\varphi_j} F_jh_j\big)}{|h_j|}$$
$$= |\varphi_j|^{2p_j} \frac{\big|1 + \frac{F_j}{\varphi_j}h_j\big|^{2p_j} - 1 - 2p_j\,\mathrm{Re}\big(\frac{F_j}{\varphi_j}h_j\big)}{\big|h_j\frac{F_j}{\varphi_j}\big|}.$$

From the equality

$$\lim_{z\to 0} \frac{|1+z|^\alpha - 1 - \alpha\,\mathrm{Re}\,z}{|z|} = 0, \qquad \alpha > 0,$$

we see that all the assumptions of Theorem 1 are satisfied.

Hence, by Corollary 3, there exist $g \in H^\infty(E, \mathbb{C}^n)$ and $\varrho \in L^\infty(\partial E)$, $\varrho > 0$, such that

$$Q(\zeta)\varrho(\zeta)F_j^*(\zeta)\frac{|\varphi_j^*(\zeta)|^{2p_j}}{\varphi_j^*(\zeta)} = g_j^*(\zeta) \qquad \text{for a.a. } \zeta \in \partial E,\ j = 1, \ldots, n,$$

where $Q(\zeta) = \prod_{k=1}^m (\zeta - \sigma_k)$ is a polynomial witnessing the m-type. This is equivalent to

$$Q(\zeta)\varrho(\zeta)|F_j^*(\zeta)|^{2p_j} = B_j^*(\zeta)S_j^*(\zeta)g_j^*(\zeta) \qquad \text{for a.a. } \zeta \in \partial E,\ j = 1, \ldots, n.$$

By Lemma 18 there exist $r_j > 0$ and $\alpha_{kj} \in \overline{E}$ such that

$$B_j^*(\zeta)S_j^*(\zeta)g_j^*(\zeta) = r_j \frac{\prod_{k=1}^m (\zeta - \alpha_{kj})(1 - \overline{\alpha}_{kj}\zeta)}{\prod_{k=1}^m (1 - \overline{\sigma}_k\zeta)} \tag{3}$$

and there exist $r_0 > 0$ and $\alpha_{k0} \in \overline{E}$ such that

$$(4) \qquad Q(\zeta)\varrho(\zeta) = \sum_{j=1}^{n} B_j^*(\zeta)S_j^*(\zeta)g_j^*(\zeta) = r_0 \frac{\prod_{k=1}^m (\zeta - \alpha_{k0})(1 - \overline{\alpha}_{k0}\zeta)}{\prod_{k=1}^m (1 - \overline{\sigma}_k \zeta)}.$$

We have

$$(5) \qquad r_0 \prod_{k=1}^{m} (\zeta - \alpha_{k0})(1 - \overline{\alpha}_{k0}\zeta)|F_j(\zeta)|^{2p_j} = r_j \prod_{k=1}^{m} (\zeta - \alpha_{kj})(1 - \overline{\alpha}_{kj}\zeta).$$

Hence

$$(6) \qquad F_j(\zeta) = a_j \prod_{k=1}^{m} \left(\frac{1 - \overline{\alpha}_{kj}\zeta}{1 - \overline{\alpha}_{k0}\zeta} \right)^{1/p_j},$$

where $a_j \in \mathbb{C} \setminus \{0\}$. From (6) it follows that

$$B_j(\zeta) = \prod_{k=1}^{m} \left(\frac{\zeta - \alpha_{kj}}{1 - \overline{\alpha}_{kj}\zeta} \right)^{r_{kj}}, \qquad \text{where } r_{kj} \in \{0, 1\}.$$

Hence

$$S_j(\zeta)g_j(\zeta) = r_j \frac{\prod_{k=1}^m (\zeta - \alpha_{kj})^{1-r_{kj}} (1 - \overline{\alpha}_{kj}\zeta)^{1+r_{kj}}}{\prod_{k=1}^m (1 - \overline{\sigma}_k \zeta)}.$$

Since the right-hand side is an outer function, from Lemma 19 we conclude that $S_j \equiv 1$, $j = 1, \ldots, n$.

From (5) and (6) we see that $|a_j|^{2p_j} = r_j/r_0$ and from (3) and (4) it follows that

$$\sum_{j=1}^{n} |a_j|^{2p_j} \prod_{k=1}^{m} (\zeta - \alpha_{kj})(1 - \overline{\alpha}_{kj}\zeta) = \prod_{k=1}^{m} (\zeta - \alpha_{k0})(1 - \overline{\alpha}_{k0}\zeta), \qquad \zeta \in E.$$

So, we get the required result. ∎

4. The case of complex geodesics

Lemma 20. *Any $\varkappa_D$- and $\widetilde{k}_D$-geodesic is extremal for an appropriate problem $(\mathcal{P})$ of 1-type.*

Proof. *The case of a $\varkappa_D$-geodesic.* Consider problem $(\mathcal{P})$ with linear functionals such that:

- $N = 4n$,
- $w_j := (0, \ldots, 1, \ldots, 0)$ and $a_j := \operatorname{Re} z_j$ for $j = 1, \ldots, n$,
- $w_j := (0, \ldots, -i, \ldots, 0)$ and $a_j := \operatorname{Im} z_j$ for $j = n+1, \ldots, 2n$,
- $w_j := (0, \ldots, 1/\zeta, \ldots, 0)$ and $a_j := \operatorname{Re} X_j$ for $j = 2n+1, \ldots, 3n$,
- $w_j := (0, \ldots, -i/\zeta, \ldots, 0)$ and $a_j := \operatorname{Im} X_j$ for $j = 3n+1, \ldots, 4n$,

where $z \in D$ and $X \in \mathbb{C}^n \setminus \{0\}$.

It is easy to see that the corresponding linear functionals are linearly independent and problem $(\mathcal{P})$ is of 1-type.

Let us show that any $\varkappa_D$-geodesic f for (z, X) is extremal for this problem $(\mathcal{P})$. Suppose that there exists a mapping $g \in \mathcal{O}(E, D)$ such that $g(0) = z$, $g'(0) = X$, and $g(E) \Subset D$. Write $\widetilde{g}(\zeta) := g(\zeta) + \zeta t X$, where $t > 0$ will be defined later. Then $\widetilde{g}(0) = g(0) = z$ and $\widetilde{g}'(0) = g'(0) + tX = (1+t)X$. If we take t such that $\widetilde{g}(E) \subset D$ (that is possible, because $g(E) \Subset D$), then we have a contradiction with f being a $\varkappa_D$-geodesic.

The case of a $\widetilde{k}_D$-geodesic. Consider problem $(\mathcal{P})$ with linear functionals such that $f \in \mathcal{O}(E, D)$ is extremal iff $f(0) = z$, $f(\sigma) = w$, where $\sigma > 0$, and there is no mapping $g \in \mathcal{O}(E, D)$ such that

(1) $g(0) = z$, $g(\sigma) = w$,
(2) $g(E) \Subset D$.

(The functions w_j in this case can be constructed similarly to the case of a $\varkappa_D$-geodesic. It is enough to replace $1/\zeta$ by $1/(\zeta - \sigma)$ and $-i/\zeta$ by $-i/(\zeta - \sigma)$.) It is easy to see that the relevant linear functionals are linearly independent and that the problem $(\mathcal{P})$ is of 1-type.

Let us show that any $\widetilde{k}_D$-geodesic f is extremal for this problem. Suppose that there exists a mapping $g \in \mathcal{O}(E, D)$ such that $g(0) = z$, $g(\sigma) = w$, and $g(E) \Subset D$. Define

$$\widetilde{g}(\zeta) := g(\zeta) + \frac{\zeta}{t\sigma}(g(\sigma) - g(t\sigma)),$$

where $0 < t < 1$ will be defined later. Then $\widetilde{g}(0) = g(0) = z$ and $\widetilde{g}(t\sigma) = g(\sigma) = w$. If we take t such that $\widetilde{g}(E) \subset D$ (use $g(E) \Subset D$), then we have a contradiction, because f is a $\widetilde{k}_D$-geodesic. ■

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