ANNALES
POLONICI MATHEMATICI
LXII.1 (1995)

Sufficient conditions for multivalent starlikeness

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Abstract. Let $\mathbb{S}^*(p)$ be the class of functions $f(z)$ which are p-valently starlike in the open unit disk $\mathbb{U}$. Two sufficient conditions for a function $f(z)$ to be in the class $\mathbb{S}^*(p)$ are shown.

1. Introduction. Let $\mathbb{A}(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. A function $f(z)$ belonging to $\mathbb{A}(p)$ is said to be *p-valently starlike* in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.2}$$

We denote by $\mathbb{S}^*(p)$ the subclass of $\mathbb{A}(p)$ consisting of functions $f(z)$ which are p-valently starlike in $\mathbb{U}$. Also, we write $\mathbb{S}^*(1) \equiv \mathbb{S}^*$.

Let $\mathbb{Q}$ denote the class of all analytic functions $q(z)$ in $\mathbb{U}$ which are normalized by $q(0)=1$. Using Jack's lemma (see [1], also [2]), Nunokawa [3] has shown that

Lemma 1. *Let $q(z) \in \mathbb{Q}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that* $\operatorname{Re}(q(z)) > 0$ $(|z| < |z_0|)$, $\operatorname{Re}(q(z_0)) = 0$ *and* $q(z_0) \neq 0$. *Then*

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik, \tag{1.3}$$

where k is real and $|k| \geq 1$.

1991 *Mathematics Subject Classification*: Primary 30C45.
Key words and phrases: analytic, open unit disk, p-valently starlike, Jack's lemma.
Research of the first author was supported, in part, by the Japanese Ministry of Education, Science and Culture under Grant-in-Aid for General Scientific Research (No. 046204).

Lemma 1 yields

LEMMA 2. *Let* $q(z) \in \mathbb{Q}$ *and suppose that there exists a point* $z_0 \in \mathbb{U}$ *such that* $\operatorname{Re}(q(z)) > 0$ $(|z| < |z_0|)$, $\operatorname{Re}(q(z_0)) = 0$ *and* $q(z_0) \neq 0$. *Then*

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{k}{2}\left(a + \frac{1}{a}\right)i, \tag{1.4}$$

where $q(z_0) = ia$, k *is real and* $k \geq 1$.

More recently, Owa, Nunokawa and Fukui [4] have given

THEOREM A. *If* $f(z) \in \mathbb{A}(p)$ *satisfies* $f(z) \neq 0$ $(0 < |z| < 1)$ *and*

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \left(1 + \frac{1}{4p}\right)\right\}\right| > 0 \quad (z \in \mathbb{U}), \tag{1.5}$$

then $f(z) \in \mathbb{S}^*(p)$ *and*

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p \quad (z \in \mathbb{U}). \tag{1.6}$$

In the present paper, we give an improvement of Theorem A.

2. Main results. An application of Lemma 2 gives us the following condition for $f(z) \in \mathbb{S}^*(p)$.

THEOREM 1. *If* $f(z) \in \mathbb{A}(p)$ *satisfies* $f(z) \neq 0$ $(0 < |z| < 1)$ *and*

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \left(1 + \frac{1}{2p}\right)\right\}\right| > 0 \quad (z \in \mathbb{U}), \tag{2.1}$$

then $f(z) \in \mathbb{S}^*(p)$.

Proof. For $f(z) \in \mathbb{A}(p)$ satisfying the condition of the theorem, we define the function $q(z)$ by

$$q(z) = \frac{zf'(z)}{pf(z)}. \tag{2.2}$$

Then, since $q(z)$ is analytic in $\mathbb{U}$ and $q(0) = 1$, we have $q(z) \in \mathbb{Q}$. Note that

$$1 + \frac{zf''(z)}{f'(z)} = pq(z) + \frac{zq'(z)}{q(z)}. \tag{2.3}$$

Therefore, our condition (2.1) implies that

$$\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \frac{zq'(z)}{pq(z)^2} \neq \alpha \quad (z \in \mathbb{U}), \tag{2.4}$$

where $\alpha \geq 1 + 1/(2p)$.

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\mathrm{Re}(q(z)) > 0$ $(|z| < |z_0|)$, $\mathrm{Re}(q(z_0)) = 0$ and $q(z_0) \neq 0$. Then, applying Lemma 2, we see that

$$\tag{2.5} \begin{aligned} \frac{f(z_0)}{z_0 f'(z_0)}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= 1 + \frac{z_0 q'(z_0)}{pq(z_0)^2} \\ &= 1 + \frac{k}{2ap}\left(a + \frac{1}{a}\right) \\ &= 1 + \frac{k}{2p}\left(1 + \frac{1}{a^2}\right) \\ &\geq 1 + \frac{k}{2p} \geq 1 + \frac{1}{2p}, \end{aligned}$$

which contradicts (2.4). Thus $\mathrm{Re}(q(z)) > 0$ $(z \in \mathbb{U})$, that is, $f(z) \in \mathbb{S}^*(p)$. This proves the assertion of our theorem.

Remark. The condition for $f(z)$ to be in the class $\mathbb{S}^*(p)$ in Theorem 1 is an improvement of Theorem A due to Owa, Nunokawa and Fukui [4].

Letting $p = 1$ in Theorem 1, we have

COROLLARY 1. *If* $f(z) \in \mathbb{A}(1)$ *satisfies* $f(z) \neq 0$ $(0 < |z| < 1)$ *and*

$$\tag{2.6} \left|\arg\left\{\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \frac{3}{2}\right\}\right| > 0 \qquad (z \in \mathbb{U}),$$

then $f(z) \in \mathbb{S}^*$.

Next, we derive

THEOREM 2. *If* $f(z) \in \mathbb{A}(p)$ *satisfies* $f(z) \neq 0$ $(0 < |z| < 1)$ *and*

$$\tag{2.7} \left|\arg\left\{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{p}{2}\right\}\right| < \pi \qquad (z \in \mathbb{U}),$$

then $f(z) \in \mathbb{S}^*(p)$.

Proof. Define the function $q(z)$ by (2.2). Then $q(z) \in \mathbb{Q}$ and

$$\tag{2.8} \frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) = p^2 q(z)^2 + pzq'(z) \neq \alpha \qquad (z \in \mathbb{U}),$$

where $\alpha \leq -p/2$. If there exists a point $z_0 \in \mathbb{U}$ such that $\mathrm{Re}(q(z)) > 0$ $(|z| < |z_0|)$, $\mathrm{Re}(q(z_0)) = 0$ and $q(z_0) \neq 0$, then Lemma 2 leads us to

$$\tag{2.9} \begin{aligned} \frac{z_0 f'(z_0)}{f(z_0)}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= p^2 q(z_0)^2 + pz_0 q'(z_0) \\ &= -p^2 a^2 - \frac{pk}{2}(1 + a^2) \leq -\frac{pk}{2} \leq -\frac{p}{2}, \end{aligned}$$

which contradicts (2.8). Consequently, $f(z) \in \mathbb{S}^*(p)$.

Setting $p = 1$ in Theorem 2, we have

COROLLARY 2. *If* $f(z) \in \mathbb{A}(1)$ *satisfies* $f(z) \neq 0$ $(0 < |z| < 1)$ *and*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)+\frac{1}{2}\right\}\right| < \pi \qquad (z \in \mathbb{U}), \tag{2.10}$$

then $f(z) \in \mathbb{S}^*$.

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Reçu par la Rédaction le 10.9.1994