ANNALES
POLONICI MATHEMATICI
LXI.3 (1995)

A free boundary stationary magnetohydrodynamic problem in connection with the electromagnetic casting process

by Tomasz Roliński (Warszawa)

Abstract. We investigate the behaviour of the meniscus of a drop of liquid aluminium in the neighbourhood of a state of equilibrium under the influence of weak electromagnetic forces. The mathematical model comprises both Maxwell and Navier–Stokes equations in 2D. The meniscus is governed by the Young–Laplace equation, the data being the jump of the normal stress. To show the existence and uniqueness of the solution we use the classical implicit function theorem. Moreover, the differentiability of the operator solving this problem is established.

1. Introduction. At the outset let us describe briefly the 2D mathematical model presented in detail in [3].

Imagine three infinitely long cylindrical conductors with generating lines parallel to the x_3-axis in $\mathbb{R}^3$. The cross sections of the conductors with the Ox_1x_2 plane will be denoted by Ω_0, Ω_1, Ω_2. Let Ω_0 correspond to liquid aluminium, and let Ω_1, Ω_2 correspond to solid conductors. From the point of view of the industrial device Ω_0 is related to the metal ingot, whereas Ω_1, Ω_2 are related to the inductor. The region Ω_0 is assumed to be bounded and simply-connected with sufficiently smooth boundary (cf. Fig. 1).

An electric alternating sinusoidal current travels through the inductor, the total intensity of the current being equal to J in Ω_1 and $-J$ in Ω_2. The inductor creates an electromagnetic field which is responsible for magnetohydrostatic and magnetohydrodynamic effects in the ingot, which in turn influence the shape of the meniscus.

The above is a simplified description of the electromagnetic casting process. The simplification concerns the negligence of other physical phenomena as the natural convection in the ingot resulting from the temperature gra-

1991 *Mathematics Subject Classification*: 76W05, 76D05, 35Q30.

Key words and phrases: free boundary, local existence and uniqueness, implicit function theorem, steady plane magnetohydrodynamics, electromagnetic casting.

This research was financed by a grant from Ecole Polytechnique Fédérale de Lausanne.

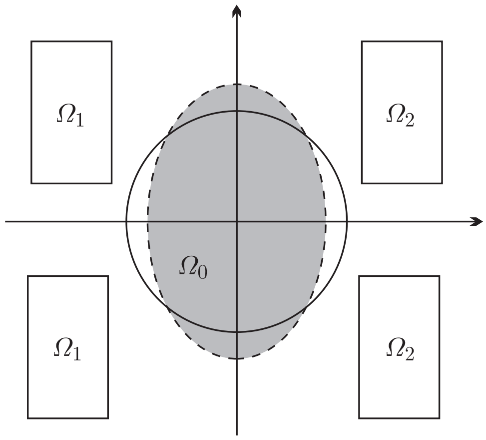

Fig. 1

dient, solidification of the ingot as well as the thermal effects due to the solidification.

The electromagnetic potential $\phi : \mathbb{R}^2 \to \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers) is governed by the Helmholtz equation in the plane, derived from the Maxwell equations (cf. [3]):

$$-\Delta\phi + \alpha\mathbf{u}.\nabla\phi + i\beta(\phi - I(\phi)) = \begin{cases} \mu_0 J/|\Omega_1| & \text{in } \Omega_1, \\ -\mu_0 J/|\Omega_2| & \text{in } \Omega_2, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where $\mathbf{u} : \Omega_0 \to \mathbb{R}^2$ ($\mathbf{u} = (u_1, u_2)$) is the velocity field of the liquid metal contained in Ω_0 and $J \in \mathbb{R}$ is the given current intensity. Moreover, $\alpha = \mu_0\sigma$, $\beta = \omega\mu_0\sigma$, where μ_0 is the magnetic permeability of the vacuum, ω is the angular velocity associated with the frequency of the alternating current, σ is the electric conductivity of the media:

$$\sigma = \begin{cases} \sigma_k & \text{in } \Omega_k,\ k = 0, 1, 2, \\ 0 & \text{otherwise;} \end{cases}$$

and $I(\phi) : \mathbb{R}^2 \to \mathbb{C}$ is the function

$$I(\phi) = \begin{cases} |\Omega_k|^{-1} \int_{\Omega_k} \phi\, dx & \text{in } \Omega_k,\ k = 0, 1, 2, \\ 0 & \text{in } \mathbb{R}^2 \setminus (\Omega_0 \cup \Omega_1 \cup \Omega_2). \end{cases}$$

The behaviour of the liquid metal in the interior of the ingot Ω_0 is described by the velocity field $\mathbf{u}$ and the pressure field $p : \Omega_0 \to \mathbb{R}$ governed by the Navier–Stokes equation, where the data is the Lorentz force (cf. [3]):

$$-2\operatorname{div}\mathbf{D}(\mathbf{u}) + \rho(\mathbf{u}.\nabla)\mathbf{u} + \nabla p = \mathbf{F}(\phi, \mathbf{u}), \tag{1.2}$$

where $\mathbf{D}(\mathbf{u}) = ((\eta/2)(\partial_j u_i + \partial_i u_j))_{i,j=1}^2$ is the symmetric deformation tensor; η, ϱ are the kinematic viscosity and the density of the liquid, respectively.

The Lorentz body force $\mathbf{F}$ results from interaction between the magnetic induction and the current density. Since we seek stationary flows we must average $\mathbf{F}$ over the period $2\pi/\omega$. After the averaging process this force reads (cf. [3], [13])

$$\mathbf{F}(\phi, \mathbf{u}) = \frac{\sigma\omega}{2}(\phi_I \nabla\phi_R - \phi_R \nabla\phi_I) - \frac{\sigma}{2}((\mathbf{u}.\nabla\phi_R)\nabla\phi_R + (\mathbf{u}.\nabla\phi_I)\nabla\phi_I),$$

where ϕ_R and ϕ_I denote the real and imaginary parts of the potential $\phi : \mathbb{R}^2 \to \mathbb{C}$. As we look for a divergence-free velocity field we assume additionally

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_0. \tag{1.3}$$

It follows from physical considerations that we must impose two conditions describing the behaviour of the velocity field at the free boundary $\Gamma_0 = \partial\Omega_0$, i.e. at the meniscus of the ingot. The first one states that $\mathbf{u}$ shall satisfy the slip condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_0, \tag{1.4}$$

where $\mathbf{n} = (n_1, n_2)$ is the exterior unit vector normal to Γ_0. This means that the velocity of the particles at Γ_0 is tangent. The second condition expresses the fact that the fluid cannot resist any tangential stresses:

$$\mathbf{s}(\mathbf{u}, p) \cdot \mathbf{t} = 0 \quad \text{on } \Gamma_0, \tag{1.5}$$

where $\mathbf{t} = (t_1, t_2)$ is the unit vector tangent to Γ_0, and

$$\mathbf{s}(\mathbf{u}, p) = \eta(\partial_j u_i + \partial_i u_j)n_j - \mathbf{n}p \quad \text{on } \Gamma_0 \tag{1.6}$$

is the Cauchy stress tensor (we use the summation convention over repeated indices).

Since we assume the presence of surface tension we shall give the Young–Laplace condition governing the free boundary Γ_0. It says that the change in the curvature of the boundary is proportional to the sum of the jump of the normal stress at the boundary and a constant. In our problem the jump is equal to the normal component of the Cauchy stress tensor (1.6). The constant is unknown.

In the absence of the Lorentz force the liquid assumes the shape of a cylinder with cross section denoted by Ω_{00}, $\Gamma_{00} = \partial\Omega_{00}$. In our analysis we allow for small departures from this state assuming that the perturbed boundary $\Gamma_0(f)$ of Γ_{00} has a polar representation $I \ni \theta \to ((f(\theta) + r_0)\cos\theta, (f(\theta) + r_0)\sin\theta)$, where $I = (-2\pi, 2\pi)$, $f : I \to \mathbb{R}$, $f(\theta) = f(\theta + 2\pi)$, r_0 is the radius of Ω_{00}. The function f can naturally be viewed as one defined on $\mathbb{R}$ and of period 2π. Here we limit the domain to the interval I for purely technical reasons. The angle θ can be defined as the angle between the x_1-axis and the radius of a point at $\Gamma_0(f)$ (cf. Fig. 1). Obviously $\Gamma_0(0) = \Gamma_{00}$. We denote by $\Omega_0(f)$ the star-shaped perturbed liquid region with boundary $\Gamma_0(f)$. Obviously we have $\Omega_0(0) = \Omega_{00}$.

The announced Young–Laplace condition for $f : I \to \mathbb{R}$, together with the side condition expressing the fact that the volume of $\Omega_0(f)$ does not change, read as follows:

$$V(f, \lambda, J) = 0 \quad \text{on } I \tag{1.7}$$

for the given current $J \in \mathbb{R}$, where $V = (\kappa + S_n + \Lambda, \mathrm{vol})$. The operator $f \to \kappa(f)$ describes the curvature of $\Gamma_0(f)$ in polar coordinates:

$$\kappa(f) = \tau \frac{(f(\theta) + r_0)^2 + 2(f'(\theta))^2 - (f(\theta) + r_0) f''(\theta)}{((f(\theta) + r_0)^2 + (f'(\theta))^2)^{3/2}}, \qquad \theta \in I,$$

where the constant $\tau \in \mathbb{R}_+$ is the surface tension. Moreover, S_n denotes the normal component of the Cauchy stress tensor,

$$S_n(J, f) = \{\mathbf{s}(\mathbf{u}(J, f), p(J, f))|_{\Gamma_0(f)} \cdot \mathbf{n}\} \circ \tau_f,$$

where $\mathbf{s}(\mathbf{u}, p)$ is defined in (1.6) and τ_f denotes the polar parametrization of $\Gamma_0(f)$. We assume here that $(\mathbf{u}, p)$ corresponds uniquely to J and the fixed boundary $\Gamma_0(f)$. By [13] this is true, at least for sufficiently regular f and small J. Finally,

$$\Lambda(\lambda) = \lambda - \frac{\tau}{r_0}, \qquad \mathrm{vol}(f) = \frac{1}{2} \int_0^{2\pi} (r_0 + f)^2 \, d\theta - \pi r_0^2,$$

where λ is the constant in the Young–Laplace condition and $\mathrm{vol}(f)$ is the perturbation of the volume of $\Omega_0(f)$.

In what follows we assume the symmetric setup for the inductor and the ingot, which means that $\Omega_1 \cup \Omega_2 \cup \Omega_{00}$ is symmetric w.r.t. the x_1- and x_2-axes (cf. Fig. 1). In the absence of the velocity field $\mathbf{u} : \Omega_0 \to \mathbb{R}^2$ the symmetry of the system implies that for the fixed open disk Ω_{00} and some current $J \in \mathbb{R}$ the electromagnetic potential ϕ is antisymmetric w.r.t. the x_2-axis and symmetric w.r.t. the x_1-axis (for short, x_2-antisymmetric and x_1-symmetric). Thus the Lorentz force $\mathbf{F} = (F_1, F_2)$ satisfies the following condition: F_1 is x_2-antisymmetric and x_1-symmetric, F_2 is x_2-symmetric and x_1-antisymmetric. Hence we can expect that, at least for small currents, the following symmetry conditions on the potential, velocity field, pressure and polar representation of the boundary perturbation for the full free boundary problem are satisfied:

(1.8a) ϕ is x_2-antisymmetric and x_1-symmetric,

(1.8b) u_1 is x_2-antisymmetric and x_1-symmetric, u_2 is x_2-symmetric and x_1-antisymmetric,

(1.8c) p is symmetric w.r.t. both axes,

(1.8d) $\Omega_0(f)$ is symmetric w.r.t. both axes, which means that

a) $f(\theta) = f(-\theta)$,
b) $f(\theta + \pi/2) = f(-\theta + \pi/2), \quad \theta, \theta + \pi/2 \in I$.

Obviously, the assumed symmetries (1.8a–d) imply the symmetries for the fields contained in the images of the operators involved. If we denote by M, A, $\mathbf{N}$, Sl, S_t, respectively, the Helmholtz operator on the left-hand side of (1.1), the data on the right-hand side of (1.1), the Navier–Stokes operator in (1.2), the normal component of the velocity (cf. (1.4)) and the tangent component of the Cauchy stress tensor (cf. (1.6)), then we have the following conditions:

(1.9a) the values of M and A are x_2-antisymmetric and x_1-symmetric,

(1.9b) the values of N_1, F_1 are x_2-antisymmetric and x_1-symmetric, the values of N_2, F_2 are x_2-symmetric and x_1-antisymmetric ($\mathbf{N} = (N_1, N_2)$),

(1.9c) the values of div, Sl, κ, S_n are symmetric w.r.t. both axes,

(1.9d) the values of S_t are antisymmetric w.r.t. both axes.

In this paper we shall consider the case where the domains and images of operators are sets of functions from suitable Sobolev spaces (cf. Sec. 2), satisfying additionally the above symmetry conditions. The condition (1.8d) implies that the center of gravity of the cross-section of the ingot $\Omega_0(f)$ does not change, which is a typical condition for this kind of problem (cf. [2]). We want to show that for small currents J in the inductor the shape of the ingot adjusts itself uniquely in a symmetric way to the change of the normal stress coming from the Lorentz forces. Thus the main result of the paper is the following theorem:

THEOREM 1.1. *There exist a neighbourhood $\mathcal{U}_V$ of* 0 *in the domain of the operator V and a function $J \to (f, \lambda)$ such that $(f, \lambda, J) \in \mathcal{U}_V$ and* $V(f, \lambda, J) = 0$ (cf. (1.7)). *This function is unique and continuously Fréchet differentiable.*

To prove Theorem 1.1 we study the differential properties of the operator V (cf. (1.7)). The crucial step here is to prove the differentiability of the operator $(J, f) \to S_n(J, f)$. This can be done by considering an auxiliary problem in which the domain of the operator consists of the deformed velocity fields and the deformed pressures that are defined on the same reference open disk Ω_{00}. The introduction of such a problem is useful since we want to compare different velocity fields and pressures for different regions.

The definition of the auxiliary problem is based on a family of invertible transformations $T_f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_f(\Omega_0(f)) = \Omega_{00}$. These transformations are different from the identity in the vicinity of Γ_{00} only (cf. Sec. 2, (2.1)). The relation to be satisfied for the deformed potential field

$\overline{\phi} : \mathbb{R}^2 \to \mathbb{C}$, $\overline{\mathbf{u}} : \Omega_{00} \to \mathbb{R}^2$ ($\overline{\mathbf{u}} = (\overline{u}_1, \overline{u}_2)$) and $\overline{p} : \Omega_{00} \to \mathbb{R}$ reads

$$\mathbb{L}(\overline{\phi}, \overline{\mathbf{u}}, \overline{p}, J, f) = 0, \tag{1.10}$$

where $J \in \mathbb{R}$ is the given current, $f : I \to \mathbb{R}$ is the given boundary perturbation, and $\mathbb{L} = (\mathbb{M} - \mathbb{A},\ \mathbb{N} - \mathbb{F},\ \mathbb{D}\mathrm{iv},\ \mathbb{S}l,\ \mathbb{S}_t)$, with

$$\begin{aligned}
\mathbb{M}(\overline{\phi}, \overline{\mathbf{u}}, f) &= \{M(\overline{\phi} \circ T_f,\ \overline{\mathbf{u}} \circ T_f)\} \circ T_f^{-1} \qquad (\overline{\mathbf{u}} \circ T_f = (\overline{u}_1 \circ T_f, \overline{u}_2 \circ T_f)),\\
\mathbb{A}(J) &= A(J),\\
\mathbb{N}(\overline{\mathbf{u}}, \overline{p}, f) &= \{\mathbf{N}(\overline{\mathbf{u}} \circ T_f,\ \overline{p} \circ T_f)\} \circ T_f^{-1},\\
\mathbb{F}(\overline{\phi}, \overline{\mathbf{u}}, f) &= \{\mathbf{F}(\overline{\phi} \circ T_f,\ \overline{\mathbf{u}} \circ T_f)\} \circ T_f^{-1},\\
\mathbb{D}\mathrm{iv}(\overline{\mathbf{u}}, f) &= \{\partial_i(\overline{u}_i \circ T_f)|J(T_f^{-1})|\} \circ T_f^{-1},\\
\mathbb{S}l(\overline{\mathbf{u}}, f) &= \{(\overline{u}_i \circ T_f)|_{\Gamma_0(f)} \cdot n_i \cdot |J(\tau_f)|\} \circ \tau_f,\\
\mathbb{S}_t(\overline{\mathbf{u}}, f) &= \eta\{(\partial_j(\overline{u}_i \circ T_f) + \partial_i(\overline{u}_j \circ T_f))|_{\Gamma_0(f)}\ n_j t_i\ |J(\tau_f)|^2\} \circ \tau_f,
\end{aligned}$$

where $|J(T_f^{-1})|$ and $|J(\tau_f)|$ are the Jacobians of T_f^{-1} and of the polar parametrization τ_f of the boundary, respectively.

Notice that the domain and the image of the operator $\mathbb{L}$ consist of functions defined on the fixed region Ω_{00}, the plane $\mathbb{R}^2$ and the interval I. Moreover, if $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p})$ is a solution of problem (1.10) for some J, f sufficiently small then $\phi = \overline{\phi} \circ T_f$, $\mathbf{u} = \overline{\mathbf{u}} \circ T_f$, $p = \overline{p} \circ T_f$ satisfy (1.1)–(1.5) for the same current J and the regions Ω_1, Ω_2, $\Omega_0(f)$ (cf. Remark 2.2, Sec. 2).

We shall show that the operator $\mathbb{L}$ is differentiable and the partial derivative of $\mathbb{L}$ w.r.t. $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p})$ at 0 is an isomorphism in suitable Sobolev spaces. Consequently, the classical implicit function theorem yields the local existence, uniqueness and differentiability of the function $(J, f) \to (\overline{\phi}, \overline{\mathbf{u}}, \overline{p})$ such that (1.10) is satisfied. This means that if we run a small current through the inductor and put the liquid metal into a container of the shape close to a cylinder, symmetric w.r.t. both axes (cf. Fig. 1), we obtain a unique electromagnetic potential, velocity field and pressure satisfying the symmetry conditions (1.8abc). Moreover, these quantities change smoothly with the change of the current and the shape of the cylinder. Then we establish the differentiability of the function $(J, f) \to S_n(J, f)$ which is the normal stress function from (1.7), modifying the shape of the free boundary $\Gamma_0(f)$. Subsequently, we show that the operator V from (1.7) is differentiable and the partial derivative of V w.r.t. (f, λ) at 0 is an isomorphism in suitable Sobolev spaces. Finally, the local existence, uniqueness and differentiability of the function $J \to (f, \lambda)$ are verified.

At this moment we stress that to prove that the linearization of $\mathbb{L}$ and V yields isomorphisms between suitable spaces (cf. (1.10), (1.7) and Sec. 4) we use the symmetry properties (1.8abcd) of the functions from the domains of these operators. In the case of $\mathbb{L}$ the linearization process gives the Stokes operator together with the boundary operators Sl and S_t. We know that the solutions of the linear problem for such operators are unique up to rigid rotations of the liquid (cf. [15]). The latter can be rejected by assuming (1.8b). Similarly, the linearization of V gives a Fredholm operator. The solutions of the linear problem for this operator are unique up to the functions $\sin\theta$, $\cos\theta$, $\theta \in I$ (cf. Sec. 3). The latter can be rejected by assuming condition (1.8d) since it allows for functions of period π only.

The model described here was given treatment in [3], [13]. In [3] this model was derived from the Maxwell and Navier–Stokes equations, and a numerical iterative procedure based on the finite element technique and the Newton method was proposed. Some references concerning a more detailed description of the electromagnetic casting phenomena and suitable numerical procedures were given there as well.

In [13] the authors deal with the fixed boundary model and prove the existence of a solution for strong magnetic fields via the Leray–Schauder homotopy lemma. A uniqueness result is also given for weak magnetic fields via the contraction principle.

There exists a review paper [14] concerning free boundary problems for the Navier–Stokes equations in the presence of surface tension. In this paper the results concerning non-stationary and stationary cases are cited. For non-stationary problems the introduction of Lagrangian coordinates was a major step in obtaining the local existence and uniqueness theorems.

For stationary problems the main tool was the coercive Schauder estimates for the linearized problem and the contraction principle applied to the free boundary condition to obtain the local existence and uniqueness theorems. For example in [2] a sequence of successive approximations was constructed by updating the free boundary via the free boundary conditions, where the solution of the Navier–Stokes equations in the previous domain was used. Then it was proved that this sequence converges to the solution.

In this paper we reduce the whole problem to a problem posed on fixed reference domains. Then we use the classical implicit function theorem directly to the reduced problem without constructing a sequence of approximate solutions. Thus we obtain the desired result in a straightforward manner. The analysis is performed in Sobolev spaces as opposed to the usual analysis in Hölder spaces (cf. [14], [2]) and, consequently, we obtain the uniqueness of the free boundary in a wider class of functions.

2. The supplementary problem. Existence and uniqueness of solution for small currents and deformations. In what follows we use the Sobolev spaces of scalar or 2-vector functions defined on a region $\mathcal{O} \subset \mathbb{R}^2$: $W^{m,\alpha}(\mathcal{O})^n$, $m = 0, 1, 2$, $n = 1, 2$, $\alpha > 2$, with the standard notation for their seminorms: $|\cdot|_{m,\alpha,\mathcal{O},n}$, and norms: $\|\cdot\|_{m,\alpha,\mathcal{O},n}$. The case $m = 0$ corresponds to the spaces of functions integrable with exponent α, which we denote by $L^\alpha(\mathcal{O})^n$. We also use the Sobolev spaces $H^m(\mathcal{O})^n$ of scalar or 2-vector functions which are square integrable together with their distributional derivatives, with the standard notation for their seminorms: $|\cdot|_{m,\mathcal{O},n}$, and norms: $\|\cdot\|_{m,\mathcal{O},n}$.

To deal efficiently with the external problem for electromagnetic potentials we use the weighted Sobolev spaces $W^m_l(\mathbb{R}^2)$, $(m,l)=(1,0),(0,1),(2,1)$, of complex-valued functions defined as follows:

$$\begin{aligned}
W^1_0(\mathbb{R}^2) &= \{\phi\in\mathcal{D}'(\mathbb{R}^2):\phi\cdot(1+\overline{r}^2)^{-1/2}(1+\log(1+\overline{r}^2))^{-1/2}\in L^2(\mathbb{R}^2),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nabla\phi\in L^2(\mathbb{R}^2)^2\},\\
W^0_1(\mathbb{R}^2) &= \{\phi\in\mathcal{D}'(\mathbb{R}^2):(1+\overline{r}^2)^{1/2}\phi\in L^2(\mathbb{R}^2)\},\\
W^2_1(\mathbb{R}^2) &= \{\phi\in W^1_0(\mathbb{R}^2):(1+\overline{r}^2)^{1/2}D^\gamma\phi\in L^2(\mathbb{R}^2),\ |\gamma|=2\},
\end{aligned}$$

where $\gamma=(\gamma_1,\gamma_2)$, $\overline{r}^2=x_1^2+x_2^2$, $(x_1,x_2)\in\mathbb{R}^2$. The standard notation for the seminorms and norms in these spaces is $|\cdot|_{m,l,\mathbb{R}^2}$ and $\|\cdot\|_{m,l,\mathbb{R}^2}$ (for details see [12]). The weighted Sobolev spaces were used by many authors (cf. e.g. [12], [8], [9], [10]) to analyse external elliptic problems. Here we use them for the potentials ϕ.

We also need spaces of functions defined on the sufficiently smooth boundary $\partial\mathcal{O}$ of the region $\mathcal{O}$: $W^{m-1/\alpha,\alpha}(\partial\mathcal{O})$, $m=1,2$, with the standard notation for the seminorms: $|\cdot|_{m-1/\alpha,\alpha,\partial\mathcal{O}}$, and the norms: $\|\cdot\|_{m-1/\alpha,\alpha,\partial\mathcal{O}}$, as well as the spaces $H^{m-1/2}(\partial\mathcal{O})$, the seminorms and norms being denoted by $|\cdot|_{m-1/2,\partial\mathcal{O}}$ and $\|\cdot\|_{m-1/2,\partial\mathcal{O}}$. The latter spaces consist of the traces of functions from $W^{m,\alpha}(\mathcal{O})$ or $H^m(\mathcal{O})$ (for detailed description see [11]). Moreover, we use some spaces defined on the interval $I=(-2\pi,2\pi)$: $W^{m,\alpha}(I)$, $H^m(I)$ for $m=0,1,2$, $W^{m-1/\alpha,\alpha}(I)$, $H^{m-1/2}(I)$ for $m=1,2,3$. The symbols $n,\mathcal{O},\mathbb{R}^2,\partial\mathcal{O},I$ in the notation of spaces, norms and seminorms are often dropped in unambiguous situations.

In what follows we are concerned with the following regularities of the functions introduced in Section 1: $\overline{\phi}\in W^2_1(\mathbb{R}^2)$, $\overline{\mathbf{u}}\in W^{2,\alpha}(\Omega_{00})^2$, $\overline{p}\in W^{1,\alpha}(\Omega_{00})$, $f\in W^{3-1/\alpha,\alpha}(I)$. Functions from these spaces will also be denoted $\psi,\mathbf{v},q,g$, respectively ($\mathbf{v}=(v_1,v_2)$). We stress that if we consider these functions as elements of wider or narrower spaces it will be stated explicitly.

Next, to complement the definition of $\mathbb{L}$ (cf. (1.10)) we must define the transformations T_f. In polar coordinates they read

$$\begin{cases}\overline{r}=r-f(\theta)\mu(r),\\ \overline{\theta}=\theta,\end{cases}\tag{2.1}$$

where $r^2=y_1^2+y_2^2$, $\overline{r}^2=x_1^2+x_2^2$, $\mu:\mathbb{R}_+\to\langle 0,1\rangle$, $\mu\in C^\infty(\mathbb{R}_+)$, $\mu(r)=1$ for $r_0-\delta_1\le r\le r_0+\delta_1$ and $\operatorname{supp}\mu\subset\{r_0-\delta_2<r<r_0+\delta_2\}$, $0<\delta_1<\delta_2<r_0$.

The mapping T_f is of class C^2, which is a consequence of $f \in W^{3-1/\alpha,\alpha}(I)$, $\alpha > 2$, and the embedding $W^{1-1/\alpha,\alpha}(I) \hookrightarrow C^{0,\beta}(I)$, $0 \le \beta < 1 - 2/\alpha$ (cf. [11]). The Jacobian $|J(T_f)|$ of T_f in polar coordinates is equal to

$$\frac{d\overline{r}}{dr} = 1 - f(\theta)\frac{d\mu}{dr}$$

and thus it is positive for f sufficiently small. Consequently, T_f is a C^2-diffeomorphism (cf. [4], Cor. 4.2.2, Th. 5.4.4, Ch. 1).

LEMMA 2.1. *The operator* $\mathbb{L}$ *maps*

$$W_1^2(\mathbb{R}^2) \times W^{2,\alpha}(\Omega_{00})^2 \times W^{1,\alpha}(\Omega_{00}) \times \mathbb{R} \times W^{3-1/\alpha,\alpha}(I)$$

into

$$W_1^0(\mathbb{R}^2) \times L^{\alpha}(\Omega_{00})^2 \times W^{1,\alpha}(\Omega_{00}) \times W^{2-1/\alpha,\alpha}(I) \times W^{1-1/\alpha,\alpha}(I).$$

Proof. We begin by the statement of some facts which we shall often need in the further parts of the proof:

(2.2a) Since T_f is a C^2-diffeomorphism, for any bounded region $\mathcal{O} \subset \mathbb{R}^2$ it induces (via superposition) an isomorphism between the spaces $W^{m,\alpha}(\mathcal{O})$ $(H^m(\mathcal{O}))$ and $W^{m,\alpha}(T_f(\mathcal{O}))$ $(H^m(T_f(\mathcal{O})))$, $m = 0, 1, 2$ (cf. [11], Lemma 3.4, Ch. 2).

(2.2b) For any bounded region $\mathcal{O}$ with sufficiently smooth boundary there exists a trace operator from $W^{m,\alpha}(\mathcal{O})$ onto $W^{m-1/\alpha,\alpha}(\partial\mathcal{O})$, $m = 1, 2$ (cf. [11], Th. 5.5, Ch. 2).

(2.2c) T_f is the identity beyond the annulus $r_0 - \delta_2 \le r \le r_0 + \delta_2$, $r^2 = y_1^2 + y_2^2$ (cf. (2.1)).

The image of $\mathbb{M}$ is in $W_1^0(\mathbb{R}^2)$ by the definition of the space $W_1^0(\mathbb{R}^2)$ and the properties (2.2ac).

The image of $\mathbb{A}$ is in $W_1^0(\mathbb{R}^2)$, which is obvious (cf. (1.1)).

The image of $\mathbb{N}$ is in $L^{\alpha}(\Omega_{00})^2$ by the property (2.2a) and the fact that $\overline{\mathbf{u}} \in C^0(\Omega_{00})^2$ by the embedding $W^{1,\alpha}(\Omega_{00}) \hookrightarrow C^{0,\beta}(\Omega_{00})$, $\beta < 1 - 2/\alpha$ (cf. [11]).

The image of $\mathbb{F}$ is in $L^{\alpha}(\Omega_{00})^2$ since

$$\overline{\phi}_I, \quad \overline{\phi}_R, \quad \left(\frac{\partial\overline{\phi}_I \circ T_f}{\partial y_i}\right) \circ T_f^{-1}, \quad \left(\frac{\partial\overline{\phi}_R \circ T_f}{\partial y_i}\right) \circ T_f^{-1}$$

restricted to Ω_{00} are in $L^{\delta}(\Omega_{00})$ for any $\delta \ge 1$ in view of (2.2a) and the embedding $H^1(\Omega_{00}) \hookrightarrow L^{\delta}(\Omega_{00})$ (cf. [11]).

The image of $\mathbb{D}\mathrm{iv}$ is in $W^{1,\alpha}(\Omega_{00})$ by (2.2a) and the fact that $|J(T_f^{-1})| \in C^2(\Omega_{00})$ since $f \in C^2(I)$ in view of the embedding $W^{1-1/\alpha,\alpha}(I) \hookrightarrow C^{0,\beta}(I)$ (cf. (2.1) and the formula for the Jacobian below).

In order to show that the image of $\mathbb{S}l$ is in $W^{2-1/\alpha,\alpha}(I)$ observe that the following formulae hold:

$$n_1 = t_2 = \frac{f'_\theta \sin\theta + (f+r_0)\cos\theta}{(f'^2_\theta + (f+r_0)^2)^{1/2}}, \tag{2.3a}$$

$$t_1 = -n_2 = \frac{f'_\theta \cos\theta - (f+r_0)\sin\theta}{(f'^2_\theta + (f+r_0)^2)^{1/2}}, \tag{2.3b}$$

$$|J(\tau_f)| = (f'^2_\theta + (f+r_0)^2)^{1/2}. \tag{2.3c}$$

Hence it is clear that $n_i|J(\tau_f)|\in W^{2-1/\alpha,\alpha}(I)$, $i=1,2$. On the other hand, $\{\mathrm{tr}\,|_{\Gamma_0(f)}(\overline{u}_i \circ T_f)\}\circ\tau_f \in W^{2-1/\alpha,\alpha}(I)$, which is a consequence of (2.2ab). Now since the product of two functions from $W^{2-1/\alpha,\alpha}(I)$ is in $W^{2-1/\alpha,\alpha}(I)$ by the embedding $W^{1-1/\alpha,\alpha}(I)\hookrightarrow C^{0,\beta}(I)$, $\beta<1-2/\alpha$, we see that $\mathbb{S}l(\mathbf{u})\in W^{2-1/\alpha,\alpha}(I)$.

Finally, the image of $\mathbb{S}_t$ is in $W^{1-1/\alpha,\alpha}(I)$ since $\{\mathrm{tr}\,|_{\Gamma_0(f)}\partial_j(\overline{u}_i\circ T_f)\}\circ\tau_f\in W^{1-1/\alpha,\alpha}(I)$ by (2.2ab) and the fact that $n_j|J(\tau_f)|,\ t_i|J(\tau_f)|\in W^{2-1/\alpha,\alpha}(I)$ by the formulae (2.3abc). The product of these functions is in $W^{1-1/\alpha,\alpha}(I)$ in view of the embedding $W^{1-1/\alpha,\alpha}(I)\hookrightarrow C^{0,\beta}(I)$. ∎

Remark 2.1. What needs some explanation here is the choice of the potential spaces $W^2_1(\mathbb{R}^2)$ for the deformed electromagnetic potentials. First, observe that by (2.2ac) the potentials $\phi=\overline{\phi}\circ T_f$ are in $W^2_1(\mathbb{R}^2)$ as well. In our problem (cf. (1.1)) the solution is a potential ϕ which is regular at infinity, and the Biot–Savart formula for electromagnetic induction yields $\phi(x)=O(\log|x|)$ as $|x|\to\infty$ (cf. [13]). Then from potential theory together with the condition $\int_{\Omega_0}\phi\,dx=0$ (this condition is satisfied naturally in view of the symmetry condition (1.8a)) we obtain that (cf. [13]) $\phi(x)=c+O(r^{-1})$, $\nabla\phi(x)=O(r^{-2})$, $r\to\infty$, which implies $\phi\in W^1_0(\mathbb{R}^2)$ since $\phi\in H^1_{\mathrm{loc}}(\mathbb{R}^2)$. The theory of potentials yields $D^\gamma\phi(x)=O(r^{-3})$, $\gamma=(\gamma_1,\gamma_2)$, $|\gamma|=2$, $r\to\infty$, as well, which implies $\phi\in W^2_1(\mathbb{R}^2)$ since $\phi\in H^2_{\mathrm{loc}}(\mathbb{R}^2)$.

Remark 2.2. By the property (2.2a) the velocity field $\mathbf{u}=\overline{\mathbf{u}}\circ T_f$ and the pressure field $p=\overline{p}\circ T_f$ are in $W^{2,\alpha}(\Omega_0(f))^2$ and $W^{1,\alpha}(\Omega_0(f))$, respectively. Moreover, we have already noticed in Remark 2.1 that the electromagnetic potential $\phi=\overline{\phi}\circ T_f$ is in $W^2_1(\mathbb{R}^2)$. Thus if we assume that f is small enough so that the Jacobians of T_f^{-1} and τ_f are positive, then $(\overline{\phi},\overline{\mathbf{u}},\overline{p})$ is a solution of problem (1.10) iff ϕ, $\mathbf{u}$, p satisfy (1.1)–(1.5). ∎

Our aim is to prove the following

Theorem 2.1. *There exists a neighbourhood* $\mathcal{U}_{\mathbb{L}}$ *of zero in the domain of* $\mathbb{L}$ *and a function* $(J,f)\to(\overline{\phi},\overline{\mathbf{u}},\overline{p})$ *such that* $(\overline{\phi},\overline{\mathbf{u}},\overline{p},J,f)\in\mathcal{U}_{\mathbb{L}}$ *and* $\mathbb{L}(\overline{\phi},\overline{\mathbf{u}},\overline{p},J,f)=0$ (cf. (1.10)). *This function is unique and of class* C^1. ∎

First, we formulate and prove some lemmas concerning the regularity of $\mathbb{L}$.

LEMMA 2.2. *The Fréchet partial derivative of* $\mathbb{L}$ *w.r.t.* $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p}, J)$, *which we denote by* $D_1\mathbb{L}$, *exists and is continuous w.r.t.* $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p}, J)$.

Proof. The nonlinear operators in the definition of problem (1.10) are sums of terms that are linear, bilinear or trilinear w.r.t. $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p}, J)$. The Gateaux derivatives of these terms w.r.t. $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p}, J)$ are, respectively, constant, linear and bilinear functions. Due to the well-known embeddings we get

$$\begin{aligned} |\mathbf{u}|_{0,\delta} &\le C\|\mathbf{u}\|_{1,\alpha}, \\ |\nabla u_i|_{0,\delta} &\le C\|\mathbf{u}\|_{2,\alpha}, \qquad i = 1,2, \\ |\phi|_{0,\delta,\Omega_0(f)} &\le C\|\phi\|_{1,0,\mathbb{R}^2}, \\ |\nabla\phi|_{0,\delta,\Omega_0(f)} &\le C\|\phi\|_{2,1,\mathbb{R}^2}, \end{aligned}$$

for any $\delta \ge 1$. Thus the Hölder inequality implies that the Gateaux derivatives are Fréchet derivatives that are continuous w.r.t. $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p}, J)$. ∎

LEMMA 2.3. *The Fréchet partial derivative of* $\mathbb{L}$ *w.r.t.* f, *which we denote by* $D_2\mathbb{L}$, *exists and is continuous.* ∎

Before we prove Lemma 2.3 we show some additional lemmas.

From the definition of $\mathbb{L}$ it follows that if we show the existence and continuity of the Fréchet derivative of the following functions in suitable Sobolev spaces:

$$\begin{aligned} &\text{a) } f \to \{D^\gamma(\overline{\phi} \circ T_f)\} \circ T_f^{-1},\ 1 \le |\gamma| \le 2, \\ &\text{b) } f \to \{D^\gamma(\overline{u}_i \circ T_f)\} \circ T_f^{-1},\ 1 \le |\gamma| \le 2,\ i = 1,2, \\ &\text{c) } f \to \{D^\gamma(\overline{p} \circ T_f)\} \circ T_f^{-1},\ |\gamma| = 1, \\ &\text{d) } f \to \{n_i|_{\Gamma_0(f)} \cdot |J(\tau_f)|\} \circ \tau_f,\ i = 1,2, \\ &\text{e) } f \to \{t_i|_{\Gamma_0(f)} \cdot |J(\tau_f)|\} \circ \tau_f,\ i = 1,2, \\ &\text{f) } f \to \{|J(T_f^{-1})|\} \circ T_f^{-1}, \end{aligned} \tag{2.4}$$

then the existence and continuity of $D_2\mathbb{L}$ can be obtained easily.

Next, let $H : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $H(r, \overline{r}, \overline{f}) = r - \mu(r)\overline{f} - \overline{r}$, $r, \overline{r} \in \mathbb{R}_+$, $\overline{f} \in (-\overline{f}_0, \overline{f}_0)$, $\overline{f}_0 \in \mathbb{R}_+$. For sufficiently small $\overline{f}_0$ we have $\partial H/\partial r > 0$. The implicit function theorem (cf. [4], Th. 4.7.1, Cor. 5.4.5, Ch. 1) yields the local existence and regularity of the function $(\overline{r}, \overline{f}) \to r = \nu(\overline{r}, \overline{f})$ such that $H(r, \overline{r}, \overline{f}) = 0$. The monotonicity of H with respect to r implies that ν is defined in the band $\mathbb{R}_+ \times (-\overline{f}_0, \overline{f}_0)$. Obviously, the inverse of T_f can be expressed in polar coordinates as follows:

$$\begin{cases} r = \nu(\overline{r}, f(\overline{\theta})), \\ \theta = \overline{\theta}. \end{cases} \tag{2.5}$$

Let $\varrho: \mathbb{R}^2 \to \mathbb{R}$ be in $C^1_B(\mathbb{R}^2)$, the space of continuous functions that are bounded on the whole plane together with their continuous first derivatives. Let $\overline{\varrho}$ be the polar representation of ϱ. Define $\widetilde{\varrho}: \mathbb{R}_+ \times (-\overline{f}_0, \overline{f}_0) \times I \to \mathbb{R}$ by the following formula: $\widetilde{\varrho}(\overline{r}, \overline{f}, \overline{\theta}) = \overline{\varrho}(\nu(\overline{r}, \overline{f}), \overline{\theta})$. Moreover, define

$$d_\varrho(f)[g](\overline{r}, \overline{\theta}) = \frac{\partial \widetilde{\varrho}}{\partial \overline{f}}(\overline{r}, f(\overline{\theta}), \overline{\theta}) \cdot g(\overline{\theta}), \qquad \forall (\overline{r}, \overline{\theta}) \in \mathbb{R}_+ \times I \tag{2.6}$$

for any functions $f, g \in C^0(I)$, f having its graph in the band $I \times (-\overline{f}_0, \overline{f}_0)$.

Remark 2.3. Observe that $\partial\widetilde{\varrho}/\partial\overline{f} = 0$ in a neighbourhood of 0 in view of the definition of the function μ. ∎

In what follows we often use the spaces of linear operators from a space $\mathcal{X}$ into a space $\mathcal{Y}$, which we denote by $[\mathcal{X} \to \mathcal{Y}]$. Now we are ready to formulate:

Lemma 2.4. *Let $\varrho \in C^{m+2}_B(\mathbb{R}^2)$ and let d_ϱ be the function $C^m(I) \to [C^m(I) \to C^m_B(\mathbb{R}^2)]$ defined by* (2.6) *for $f, g \in C^m(I)$, $m = 0, 1$. Then d_ϱ is the continuous Fréchet derivative of the function $f \to \varrho \circ T_f^{-1}$.*

Proof. By (2.5) we have $(\varrho \circ T_f^{-1})(\overline{r}, \overline{\theta}) = \widetilde{\varrho}(\overline{r}, f(\overline{\theta}), \overline{\theta})$. Denote

$$\mathcal{J}(\overline{r}, \overline{\theta}) = \widetilde{\varrho}(\overline{r}, f(\overline{\theta}) + g(\overline{\theta}), \overline{\theta}) - \widetilde{\varrho}(\overline{r}, f(\overline{\theta}), \overline{\theta}) - \frac{\partial \widetilde{\varrho}}{\partial \overline{f}}(\overline{r}, f(\overline{\theta}), \overline{\theta}) \cdot g(\overline{\theta}).$$

In view of $\varrho \in C^2_B(\mathbb{R}^2)$ we get $|\mathcal{J}|_{0,\infty} \le C(\varrho, \nu)|g|^2_{0,\infty}$ and Lemma 2.4 is proved for $m = 0$.

By differentiating $\mathcal{J}$ w.r.t. $\overline{r}$ and $\overline{\theta}$ and assuming $\varrho \in C^3_B(\mathbb{R}^2)$ we get $|\partial\mathcal{J}/\partial\overline{r}|_{0,\infty} \le C(\varrho, \nu)|g|^2_{0,\infty}$ and $|\partial\mathcal{J}/\partial\theta|_{0,\infty} \le C(\varrho, \nu)\|g\|^2_{1,\infty}(1 + |f|_{1,\infty})$. Thus in view of the formulae

$$\begin{aligned} &\text{a)}\ \frac{\partial \mathcal{J}}{\partial x_1} = \cos\overline{\theta}\frac{\partial \mathcal{J}}{\partial \overline{r}} - \sin\overline{\theta}\frac{1}{\overline{r}}\frac{\partial \mathcal{J}}{\partial \overline{\theta}}, \\ &\text{b)}\ \frac{\partial \mathcal{J}}{\partial x_2} = \sin\overline{\theta}\frac{\partial \mathcal{J}}{\partial \overline{r}} + \cos\overline{\theta}\frac{1}{\overline{r}}\frac{\partial \mathcal{J}}{\partial \overline{\theta}} \end{aligned} \tag{2.7}$$

and Remark 2.3 we obtain Lemma 2.4 for $m = 1$ (the continuity of the derivative is obvious in view of the formula (2.6)). ∎

Using Lemma 2.4 we prove the following

Lemma 2.5. *The functions in* (2.4abc) *are continuously Fréchet differentiable, the derivatives being understood as $C^2(I) \to [C^2(I) \to \mathcal{X}]$, where*

$\mathcal{X} = W^0_1(\mathbb{R}^2)$ *for* (2.4a) *with* $|\gamma| = 2$,

$\mathcal{X} = W^1_0(\mathbb{R}^2)$ *for* (2.4a) *with* $|\gamma| = 1$,

$\mathcal{X} = L^\alpha(\Omega_{00})$ *for* (2.4b) *with* $|\gamma| = 2$ *and for* (2.4c),

$\mathcal{X} = W^{1,\alpha}(\Omega_{00})$ *for* (2.4b) *with* $|\gamma| = 1$.

P r o o f. We concentrate on the calculation of the derivative of the functions (2.4a), the cases (2.4bc) being analogous.

The chain rule yields

$$\text{(2.8a)}\qquad \frac{\partial(\overline{\phi}\circ T_f)}{\partial y_i}=\left\{\frac{\partial\overline{\phi}}{\partial x_1}\right\}\circ T_f\cdot\frac{\partial x_1}{\partial y_i}+\left\{\frac{\partial\overline{\phi}}{\partial x_2}\right\}\circ T_f\cdot\frac{\partial x_2}{\partial y_i},\qquad i=1,2,$$

$$\text{(2.8b)}\qquad \frac{\partial^2(\overline{\phi}\circ T_f)}{\partial y_i\partial y_j}=\sum_{\substack{k,l=0\\k+l=2}}^{2}C(k,l)\left\{\frac{\partial^2\overline{\phi}}{\partial x_1^k\partial x_2^l}\right\}\circ T_f\cdot\left(\frac{\partial x_1}{\partial y_i}\right)^k\left(\frac{\partial x_2}{\partial y_j}\right)^l$$
$$+\sum_{\substack{(k,l)=(0,1)\\(k,l)=(1,0)}}\left\{\frac{\partial\overline{\phi}}{\partial x_1^k\partial x_2^l}\right\}\circ T_f\cdot\left(\frac{\partial^2 x_1}{\partial y_i\partial y_j}\right)^k\left(\frac{\partial^2 x_2}{\partial y_i\partial y_j}\right)^l,\qquad i,j=1,2,$$

where the constant $C(k,l)=2$ if $k=l=1$, and $C(k,l)=1$ otherwise. Using the formulae analogous to (2.7ab) to express the derivatives of T_f in polar coordinates we arrive at

$$\text{(2.9a)}\qquad \frac{\partial x_i}{\partial y_j}=\sum_{\substack{0\le|\gamma|\le2\\0\le\gamma_k\le1\\k=1,2,3}}P_{i,j,\gamma}(\sin\theta,\cos\theta)\frac{d^{\gamma_1}\mu}{dr^{\gamma_1}}\cdot\frac{1}{r^{\gamma_2}}\cdot\frac{d^{\gamma_3}f}{d\theta^{\gamma_3}},\qquad i,j=1,2,$$

where $P_{i,j,\gamma}$ is a form of two variables of degree 2 (we assume that for $\gamma=0$ the corresponding term in (2.9a) is 1), and

$$\text{(2.9b)}\qquad \frac{\partial^2 x_i}{\partial y_j\partial y_k}=\sum_{\substack{2\le|\gamma|\le4\\0\le\gamma_l\le2\\l=1,2,3}}P_{i,j,k,\gamma}(\sin\theta,\cos\theta)\frac{d^{\gamma_1}\mu}{dr^{\gamma_1}}\cdot\frac{1}{r^{\gamma_2}}\cdot\frac{d^{\gamma_3}f}{d\theta^{\gamma_3}},\qquad i,j,k=1,2,$$

where $P_{i,j,k,\gamma}$ is a form of two variables of degree 3.

Now by the formulae (2.8a), (2.9a) the function

$$f\to\left\{\frac{\partial\overline{\phi}\circ T_f}{\partial y_j}\right\}\circ T_f^{-1}$$

can be viewed as the sum of the following products:

$$\text{(2.10)}\qquad \frac{\partial\overline{\phi}}{\partial x_i}P_{i,j,\gamma}(\sin\overline{\theta},\cos\overline{\theta})\left\{\frac{d^{\gamma_1}\mu}{dr^{\gamma_1}}\cdot\frac{1}{r^{\gamma_2}}\right\}\circ T_f^{-1}\cdot\frac{d^{\gamma_3}f}{d\overline{\theta}^{\gamma_3}}$$

for $i=1,2$, $0\le|\gamma|\le2$, $0\le\gamma_k\le1$, $k=1,2,3$.

The application of Lemma 2.4 for $m = 1$ gives the existence and continuity of the derivative of

$$f \to \left\{ \frac{d^{\gamma_1}\mu}{dr^{\gamma_1}} \cdot \frac{1}{r^{\gamma_2}} \right\} \circ T_f^{-1}.$$

The assumptions of Lemma 2.4 hold since μ has support in the annulus $r_0 - \delta_2 \le r \le r_0 + \delta_2$, $r^2 = y_1^2 + y_2^2$ so that the function in braces is in $C_B^3(\mathbb{R}^2)$. Moreover, the function $f \to d^{\gamma_3} f / d\overline{\theta}^{\gamma_3}$ is linear.

In this way the function (2.10) is continuously differentiable as the product of differentiable functions and the case of functions of type (2.4a) for $|\gamma| = 1$ is proved.

The case (2.4a) for $|\gamma| = 2$ is treated in a similar way by making use of (2.8b), (2.9b) and Lemma 2.4 for $m = 0$. As mentioned above the cases (2.4bc) can be treated analogously. Hence the lemma is proved. ∎

Now we deal with the functions defined in (2.4def).

LEMMA 2.6. *The functions* (2.4de) *are continuously Fréchet differentiable*, *the derivative being understood as a function* $W^{3-1/\alpha,\alpha}(I) \to [W^{3-1/\alpha,\alpha}(I) \to W^{2-1/\alpha,\alpha}(I)]$. *The same statement is true for the function* (2.4f), *the derivative being understood as a function* $C^1(I) \to [C^1(I) \to C_B^1(\mathbb{R}^2)]$.

Proof. The first statement is obvious since by the formulae (2.3abc) the functions involved are affine.

For the second, notice that by the definition of the mapping T_f the Jacobian of the inverse in Cartesian coordinates reads as follows:

$$|J(T_f^{-1})| = \frac{r}{\overline{r}} \left(\frac{d\overline{r}}{dr} \right)^{-1} = J_1 \cdot J_2, \tag{2.11}$$

where

$$J_1(f) = \frac{1}{1 - \left(f \dfrac{\mu}{r} \right) \circ T_f^{-1}}, \qquad J_2(f) = \frac{1}{1 - \left(f \dfrac{d\mu}{dr} \right) \circ T_f^{-1}}.$$

Let us deal with the function $f \to J_1(f)$. Define $\sigma : (-1/2, 1/2) \to \mathbb{R}$ by $\sigma(\lambda) = 1/(1-\lambda)$ and let $\widehat{\varrho} \in C_B^1(\mathbb{R}^2)$ have values in $(-1/2, 1/2)$. Then the function $\widehat{\varrho} \to \sigma(\widehat{\varrho})$ is continuously differentiable as a function $C_B^1(\mathbb{R}^2) \to C_B^1(\mathbb{R}^2)$, the derivative being

$$d_\sigma(\widehat{\varrho})[\check{\varrho}] = \frac{d\sigma}{d\lambda}(\widehat{\varrho}) \cdot \check{\varrho}.$$

Moreover, $f \to \left(f \frac{\mu}{r} \right) \circ T_f^{-1}$ is continuously differentiable as a function $C^1(I) \to C_B^1(\mathbb{R}^2)$ (cf. the end of the proof of Lemma 2.5). Thus $f \to J_1(f)$

as the superposition of two continuously differentiable functions is also continuously differentiable. The continuous differentiability of $f \to J_2(f)$ can be established in exactly the same way. Hence (2.11) is differentiable as well and the lemma is proved. ■

Proof of Lemma 2.3. Denote

$$D_2\mathbb{L} = (D_2\mathbb{M} - D_2\mathbb{A},\ D_2\mathbb{N} - D_2\mathbb{F},\ D_2\,\mathbb{D}\mathrm{iv},\ D_2\mathbb{S}l,\ D_2\mathbb{S}_t).$$

In view of Lemmas 2.5 and 2.6 we have:

- $D_2\mathbb{M}$ is the sum of
  - — two derivatives of functions of type (2.4a) for $|\gamma| = 2$,
  - — two derivatives of functions of type (2.4a) for $|\gamma| = 1$ multiplied by $\alpha\overline{u}_i$ (cf. (1.1));
- $D_2\mathbb{A} = 0$;
- $D_2\mathbb{N}$ is the sum of
  - — two derivatives of two vector-valued functions with components of type (2.4b) for $|\gamma| = 2$, multiplied by η,
  - — two derivatives of two vector-valued functions with components of type (2.4b) for $|\gamma| = 1$, multiplied by $\varrho\overline{u}_i$,
  - — the derivative of a vector-valued function with components of type (2.4c);
- $D_2\mathbb{F}$ is the sum of
  - — two derivatives of two vector-valued functions whose components are the real or imaginary parts of functions of type (2.4a) for $|\gamma| = 1$, multiplied by $\frac{1}{2}\sigma\omega\overline{\phi}_I$ or $\frac{1}{2}\sigma\omega\overline{\phi}_R$,
  - — four derivatives of four vector-valued functions whose components are the real or imaginary parts of functions of type (2.4a) for $|\gamma| = 1$, multiplied by $\frac{1}{2}\sigma\overline{u}_i\{\partial_j\overline{\phi}_R \circ T_f\} \circ T_f^{-1}$ or $\frac{1}{2}\sigma\overline{u}_i\{\partial_j\overline{\phi}_I \circ T_f\} \circ T_f^{-1}$;
- $D_2\,\mathbb{D}\mathrm{iv}$ is the sum of
  - — two derivatives of two functions of type (2.4b) for $|\gamma| = 1$, multiplied by the function (2.4f),
  - — the derivative of the function (2.4f) multiplied by the sum of two functions of type (2.4b) for $|\gamma| = 1$;
- $D_2\mathbb{S}l$ is the derivative of an affine function in view of (2.3abc);
- $D_2\mathbb{S}_t$ is the sum of
  - — eight derivatives of eight functions of type (2.4b) for $|\gamma| = 1$ composed with the trace operator on $\Gamma_0(f)$ and multiplied by the product of two functions of type (2.4d) or (2.4e) and the square of the function (2.4f),

— four derivatives of four bi-affine functions (comp. (2.3abc) and the definition of $\mathbb{S}_t$ in (1.10)), multiplied by the traces of functions of type (2.4b).

To end the proof notice that $D_2\mathbb{L}$ is continuous since the functions defined in (2.4) are continuously differentiable. ∎

Proof of Theorem 2.1. By Lemma 2.2 the derivative $D_1\mathbb{L}$ is continuous w.r.t. $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p}, J)$. The continuity of $D_1\mathbb{L}$ w.r.t. f comes easily from the continuity of the functions defined in (2.4), which in turn is a consequence of the differentiability of these functions (cf. Lemmas 2.5, 2.6). Thus $D_1\mathbb{L}$ is continuous. The existence and continuity of $D_1\mathbb{L}$ and, by Lemma 2.3, of $D_2\mathbb{L}$ yield the continuous Fréchet differentiability of $\mathbb{L}$ for sufficiently small f.

The linearization process for the nonlinear problem (1.10) at 0 yields two decoupled linear problems.

One of them is the Stokes problem on the disk Ω_{00} for the velocity field $\mathbf{v}$ and the pressure q with the boundary conditions on the normal component of the velocity and the tangent component of the Cauchy stress tensor (cf. (4.1)). By Theorem 4.1 the linear Stokes operator together with the divergence and boundary operators is an isomorphism between the space $W^{2,\alpha}(\Omega_{00})^2 \times W^{1,\alpha}(\Omega_{00})/P_0$ with the symmetry conditions(1.8bc) and the space $L^\alpha(\Omega_{00})^2 \times W^{1,\alpha}(\Omega_{00}) \times W^{2-1/\alpha,\alpha}(I) \times W^{1-1/\alpha,\alpha}(I)$ with the symmetry conditions (1.9bcd) and the compatibility condition (4.2). Notice that by the definition of $\mathbb{L}$ (cf. (1.10)) the compatibility condition (4.2) is also satisfied for the functions from its image.

The second linear problem is defined in (4.16). This is an elliptic problem for the electromagnetic potential ψ. By Theorem 4.2 this problem defines an isomorphism between the space $W^2_1(\mathbb{R}^2)$ with the symmetry condition (1.8a) and the space $\mathcal{Z} = W^0_1(\mathbb{R}^2) \cap (W^1_0(\mathbb{R}^2))'$ with the symmetry condition (1.9a). Notice that the operator $\mathbb{L}$ remains continuously differentiable if we replace the space $W^1_0(\mathbb{R}^2)$ with the symmetry condition (1.9a) by the space $\mathcal{Z}$ with the same symmetry condition. This is a consequence of the fact that the derivative of the function

$$f \to \left\{\frac{\partial^2(\overline{\phi} \circ T_f)}{\partial y_i^2}\right\} \circ T_f^{-1}$$

has its values in the space of functions with support in the annulus $r_0 - \delta_2 \le \overline{r} \le r_0 + \delta_2$, $\overline{r}^2 = x_1^2 + x_2^2$, since T_f^{-1} is the identity beyond it (cf. Lemmas 2.4, 2.5 and (2.1)). In other words, the change of the function f does not affect the behaviour of $\{\partial^2(\overline{\phi} \circ T_f)/\partial y_i^2\} \circ T_f^{-1}$ at infinity.

Now a straightforward application of the implicit function theorem (cf. [4]) yields the theorem. ∎

3. The free boundary problem. Existence and uniqueness of solution for small currents. Problem (1.7) is studied in this section in a similar way to the supplementary problem (1.10). This means that first we establish the existence and continuity of the derivative of V. Next, we show that the linearized problem defines an isomorphism between the spaces involved. Finally, Theorem 1.1 is proved by the implicit function theorem.

LEMMA 3.1. *The operator V defined by* (1.7) *has a continuous Fréchet derivative as an operator from $W^{3-1/\alpha,\alpha}(I)\times\mathbb{R}\times\mathbb{R}$ into $W^{1-1/\alpha,\alpha}(I)\times\mathbb{R}$.*

Proof. First, we establish the differentiability of the curvature operator κ (cf. (1.7)) which can be written in the following way:

$$\kappa(f)=\sigma_1(f,f'_\theta)+f''_\theta\sigma_2(f,f'_\theta), \tag{3.1}$$

where

$$\sigma_1(\overline{f},\overline{f}_\theta)=\frac{(\overline{f}+r_0)^2+2\overline{f}_\theta^2}{((\overline{f}+r_0)^2+\overline{f}_\theta^2)^{3/2}}, \tag{3.2a}$$

$$\sigma_2(\overline{f},\overline{f}_\theta)=-\frac{\overline{f}+r_0}{((\overline{f}+r_0)^2+\overline{f}_\theta^2)^{3/2}}, \tag{3.2b}$$

for $\overline{f}\in\left(-\frac{1}{2}r_0,\frac{1}{2}r_0\right)$, $\overline{f}_\theta\in\mathbb{R}$. Notice that $\sigma_i(f,f'_\theta)\in C^1(I)$ since $f\in C^2(I)$ in view of the embedding $W^{3-1/\alpha,\alpha}(I)\hookrightarrow C^{2,\beta}(I)$, $\beta<1-2/\alpha$, and $f''_\theta\in W^{1-1/\alpha,\alpha}(I)$. Hence $\kappa(f)\in W^{1-1/\alpha,\alpha}(I)$. Now we can verify easily that the derivative of $f\to\sigma_i(f,f'_\theta)$ reads

$$d_{\sigma_i}(f)[g]=\frac{\partial\sigma_i}{\partial\overline{f}}(f,f'_\theta)\cdot g+\frac{\partial\sigma_i}{\partial\overline{f}_\theta}(f,f'_\theta)\cdot g'_\theta,\qquad i=1,2. \tag{3.3}$$

The above derivative can be understood as a function $C^2(I)\to[C^2(I)\to C^1(I)]$. Thus in view of the formulae (3.1), (3.2), (3.3) together with the embedding $W^{3-1/\alpha,\alpha}(I)\hookrightarrow C^{2,\beta}(I)$ the operator $f\to\kappa(f)$ has a continuous derivative as a function $W^{3-1/\alpha,\alpha}(I)\to[W^{3-1/\alpha,\alpha}(I)\to W^{1-1/\alpha,\alpha}(I)]$.

Now we deal with the operator S_n (cf. (1.7)):

$$\begin{aligned}(J,f)\to S_n(J,f)=\eta\{&\mathrm{tr}\,|_{\Gamma_0(f)}(\partial_j(\overline{u}_i(J,f)\circ T_f)\\&+\partial_i(\overline{u}_j(J,f)\circ T_f))\cdot n_j|_{\Gamma_0(f)}n_i|_{\Gamma_0(f)}\\&-\mathrm{tr}\,|_{\Gamma_0(f)}(\overline{p}\circ T_f)\}\circ\tau_f\end{aligned}$$

(τ_f stands for the polar parametrization of $\Gamma_0(f)$). First, we establish the differentiability of the function

$$(J,f)\to\{\partial_j(\overline{u}_i(J,f)\circ T_f)\}\circ T_f^{-1}. \tag{3.4}$$

This function can be viewed as the superposition of $(J,f)\to(J,f,f)$ with

$$(J,f,g)\to\{\partial_j(\overline{u}_i(J,f)\circ T_g)\}\circ T_g^{-1}. \tag{3.5}$$

The partial derivative of (3.5) w.r.t. (J,f) is equal to the derivative of $(J,f)\to\overline{u}_i$ composed with the linear function $\overline{u}_i\to\{\partial_j(\overline{u}_i\circ T_g)\}\circ T_g^{-1}$. The former is continuously differentiable by Theorem 2.1. Hence the partial derivative of (3.5) w.r.t. (J,f) exists and is continuous as a function $\mathbb{R}\times W^{3-1/\alpha,\alpha}(I)\to[\mathbb{R}\times W^{3-1/\alpha,\alpha}(I)\to W^{1,\alpha}(\Omega_{00})]$. The partial derivative of (3.5) w.r.t. g exists and is continuous, being the derivative of a function of type (2.4b) for $|\gamma|=1$ (cf. Lemma 2.5). Thus (3.5) is continuously differentiable and, consequently, the differentiability of (3.4) follows.

Next, we establish the differentiability of the function $f\to n_i|_{\Gamma_0(f)}$. By the formulae (2.3ab) this function can be written as $f\to\sin\theta\cdot\sigma_{i1}(f,f'_\theta)+\cos\theta\cdot\sigma_{i2}(f,f'_\theta)$, where $(\overline{f},\overline{f}_\theta)\to\sigma_{ij}(\overline{f},\overline{f}_\theta)\in\mathbb{R}$, $i,j=1,2$, are regular functions of the parameters $\overline{f}_\theta\in\mathbb{R}$ and $\overline{f}\in(-\frac{1}{2}r_0,\frac{1}{2}r_0)$. Now the derivative of $f\to\sigma_{ij}(f,f'_\theta)$ can be expressed by the following formula:

$$d_{\sigma_{ij}}(f)[g]=\frac{\partial\sigma_{ij}}{\partial\overline{f}}(f,f'_\theta)\cdot g+\frac{\partial\sigma_{ij}}{\partial\overline{f}_\theta}(f,f'_\theta)\cdot g'_\theta,\qquad i,j=1,2,$$

and this derivative can be understood as a function $C^2(I)\to[C^2(I)\to C^1(I)]$. Hence $f\to n_i|_{\Gamma_0(f)}$ is continuously differentiable.

Moreover, the function $(J,f)\to\{\operatorname{tr}|_{\Gamma_0(f)}(\overline{p}(J,f)\circ T_f)\}\circ\tau_f=\{\operatorname{tr}|_{\Gamma_{00}}\overline{p}\}\circ\tau_0$ is differentiable in suitable spaces by Theorem 2.1.

Now the continuous differentiability of $(J,f)\to S_n(J,f)$ can be established easily since this function is a polynomial in differentiable functions, the derivative being understood as a function $\mathbb{R}\times W^{3-1/\alpha,\alpha}(I)\to[\mathbb{R}\times W^{3-1/\alpha,\alpha}(I)\to W^{1-1/\alpha,\alpha}(I)]$.

Next, let us establish the differentiability of vol (cf. (1.7) and the explanations below). Define a function $L^2(I)\to[L^2(I)\to\mathbb{R}]$ by

$$d_{\mathrm{vol}}(f)[g]=\int_0^{2\pi}(f+r_0)g\,d\theta. \tag{3.6}$$

One can easily verify that this is the continuous derivative of $f\to\mathrm{vol}(f)$.

Next, the differentiability of Λ is obvious since it is an affine function (cf. (1.7) and the explanations below).

In view of the differentiability of $\kappa,S_n,\Lambda,\mathrm{vol}$ the differentiability of V is obvious and the lemma is proved. ■

Presently, let us calculate the partial derivative D_1V of V w.r.t. (f,λ) at 0. Denote $D_1V=(D_1\kappa+D_1S_n+D_1\Lambda,\ D_1\,\mathrm{vol})$. We obtain:

$$D_1\kappa(0)[g]=-\frac{\tau}{r_0^2}(g''_\theta+g), \tag{3.7a}$$

$$D_1S_n(0,0)[g]=0, \tag{3.7b}$$

$$D_1\Lambda(0)[\mu]=\mu, \tag{3.7c}$$

$$D_1 \operatorname{vol}(0)[g] = r_0 \int_0^{2\pi} g(\theta)\, d\theta. \tag{3.7d}$$

The partial derivative $D_1 S_n$ is zero since for $J = 0$ the solution $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p})$ of problem (1.10) is zero so that $S_n(0, f) = 0$.

We want to show the following

LEMMA 3.2. *The partial derivative $D_1 V$ of V w.r.t. (f, λ) at 0 is an isomorphism.*

Proof. In view of the formulae (3.7a–d) consider the following linear problem: find $g \in W^{3-1/\alpha,\alpha}(I)$, $\mu \in \mathbb{R}$ such that g satisfies the symmetry conditions (1.8d) and

$$\begin{aligned} &\text{a) } -\frac{\tau}{r_0^2}(g'' + g) + \mu = h, \\ &\text{b) } r_0 \int_0^{2\pi} g(\theta)\, d\theta = \nu, \end{aligned} \tag{3.8}$$

where $h \in W^{1-1/\alpha,\alpha}(I)$, $\nu \in \mathbb{R}$ and h satisfies the symmetry conditions (1.9c).

First, consider the following supplementary problem: find $g \in W^{3-1/\alpha,\alpha}(I)$ satisfying (1.8d) such that

$$-g'' + g = h, \tag{3.9}$$

where $h \in W^{1-1/\alpha,\alpha}(I)$ and h satisfies (1.9c).

Problem (3.9) has the following variational formulation:

$$\int_{-2\pi}^{2\pi} g'\chi'\, d\theta + \int_{-2\pi}^{2\pi} g\chi\, d\theta = \int_{-2\pi}^{2\pi} h\chi\, d\theta, \qquad \forall \chi \in C^\infty(I). \tag{3.10}$$

The variational problem (3.10) can be obtained from (3.9) by multiplication of (3.9) by $\chi \in C^\infty(I)$ and integration over the interval I. Then we decompose χ into its symmetric and antisymmetric parts: $\chi = \chi_1 + \chi_2$, $\chi_1(\theta) = \frac{1}{2}(\chi(\theta) + \chi(-\theta))$, $\chi_2(\theta) = \frac{1}{2}(\chi(\theta) - \chi(-\theta))$. Obviously, the integrals containing χ_2 are zero since g'', g, h are even. Then we integrate by parts to obtain (3.10) for χ_1 (the boundary terms disappear since χ_1 is even and g' is π-periodic since g is (cf. (1.8d)). The last step is to replace χ_1 by χ, which is possible since the corresponding integrals containing χ_2 are zero.

The form on the left-hand side of (3.10) is $H^1(I)$-elliptic. Thus by the Lax–Milgram theorem we obtain a unique solution g of (3.10) such that $g \in H^1(I)$. Obviously this solution satisfies condition (1.8d).

Moreover, the definition of distributional derivatives yields

$$\langle g'', \chi \rangle = \int\limits_{-2\pi}^{2\pi} (g-h)\chi\, dx, \qquad \forall \chi \in C_0^\infty(I), \tag{3.11}$$

which means $g'' \in H^{0.5}(I)$ since $W^{1-1/\alpha,\alpha}(I) \hookrightarrow H^{0.5}(I)$, which in turn means $g \in H^{2.5}(I)$. Next, by the embedding $H^{2.5}(I) \hookrightarrow W^{1,\alpha}(I)$ and by referring once again to (3.11) we obtain $g'' \in W^{1-1/\alpha,\alpha}(I)$. Hence $g \in W^{3-1/\alpha,\alpha}(I)$ and g is a unique solution of problem (3.9). Thus the operator $g \to h$ defined by (3.9) is an isomorphism between the spaces $W^{3-1/\alpha,\alpha}(I)$ and $W^{1-1/\alpha,\alpha}(I)$ with the symmetry conditions (1.8d), (1.9c). ∎

Now let us go back to the full problem (3.8). The above result concerning problem (3.9), the fact that the operators in (3.7cd) are one-dimensional and the operator $g \to -2(\tau/r_0^2)g$ is compact from $W^{3-1/\alpha,\alpha}(I)$ into $W^{1-1/\alpha,\alpha}(I)$ (since the embedding $W^{3-1/\alpha,\alpha}(I) \hookrightarrow W^{1-1/\alpha,\alpha}(I)$ is compact) we see that the operator $(g,\mu) \to (h,\nu)$ defined by (3.8) is a Fredholm operator.

We want to show that this operator is injective. Assume $(h,\nu) = 0$ and integrate (3.8a) over the interval $(0, 2\pi)$. We have $\int_0^{2\pi} g''\, d\theta = 0$ since g is π-periodic by the symmetry conditions (1.8d), and $\int_0^{2\pi} g\, d\theta = 0$ since $\nu = 0$ in (3.8b). Thus $\mu = 0$. The condition $-g'' - g = 0$ yields two linearly independent functions: $\sin\theta, \cos\theta$. On the other hand, we have already noticed that g is π-periodic. Hence $g = 0$ and the injectivity follows.

Now the operator $(g,\mu) \to (h,\nu)$ is an injective Fredholm operator. Hence it is an isomorphism and the lemma is proved. ∎

The main result of this section and the main result of this paper is Theorem 1.1 announced in Section 1. We are now ready to prove it.

Proof of Theorem 1.1. The operator V of the free boundary problem (1.7) is continuously differentiable by Lemma 3.1. Moreover, the partial derivative of V w.r.t. (f, λ) at 0 is an isomorphism by Lemma 3.2. Thus a straightforward application of the implicit function theorem (cf. [4]) yields the existence and uniqueness of the function $J \to (f,\lambda)$ for J and (f,λ) sufficiently small. This function is continuously differentiable in the spaces involved. ∎

4. The linear problems. The calculation of the partial derivative of $\mathbb{L}$ w.r.t. $(\overline{\phi}, \overline{\mathbf{u}}, \overline{p})$ at zero leads to the Helmholtz operator and the Stokes operator together with the boundary operators Sl and S_t (cf. (1.4), (1.5) and (1.9)). First, let us deal with the Stokes problem: find $(\mathbf{v}, q) \in W^{2,\alpha}(\Omega_{00})^2 \times W^{1,\alpha}(\Omega_{00})/P_0$ satisfying the respective symmetry conditions (1.8bc) such that

$$(4.1)\qquad \begin{aligned} &\text{a) } -2\,\mathrm{div}\,\mathbf{D}(\mathbf{v})+\nabla q=\mathbf{h} && \text{in } \Omega_{00},\\ &\text{b) } \mathrm{div}\,\mathbf{v}=d && \text{in } \Omega_{00},\\ &\text{c) } \mathbf{v}.\mathbf{n}=s_1 && \text{on } \Gamma_{00},\\ &\text{d) } \mathbf{s}(\mathbf{v},q).\mathbf{t}=s_2 && \text{on } \Gamma_{00}, \end{aligned}$$

where $\mathbf{h} \in L^\alpha(\Omega_{00})^2$, $d \in W^{1,\alpha}(\Omega_{00})$, $s_1 \in W^{2-1/\alpha,\alpha}(\Gamma_{00})$, $s_2 \in W^{1-1/\alpha,\alpha}(\Gamma_{00})$, and $\mathbf{h}$, d, s_1, s_2 satisfy the respective symmetry conditions (1.9bcd); the symmetric deformation tensor $\mathbf{D}$ is defined in (1.2), the Cauchy stress tensor $\mathbf{s}$ is defined in (1.6), P_0 is the space of constant functions. We recall that Ω_{00} is an open disk of radius r_0 and Γ_{00} is its boundary. Moreover, the following compatibility condition is satisfied:

$$(4.2)\qquad \int_{\Omega_{00}} d\,dx=\int_{\Gamma_{00}} s_1\,d\sigma.$$

THEOREM 4.1. *The operator* $(\mathbf{v},q)\to(\mathbf{h},d,s_1,s_2)$ *defined by problem* (4.1) *is an isomorphism.* ■

Before we prove Theorem 4.1 we need some auxiliary results. Let us consider the following problem:

(4.3) find $(\mathbf{v},q)\in H^2(\Omega_{00})^2\times H^1(\Omega_{00})/P_0$ such that (4.1) is satisfied for $\mathbf{h}\in L^2(\Omega_{00})^2$, $d\in H^1(\Omega_{00})$, $s_1\in H^{1.5}(\Gamma_{00})$, $s_2\in H^{0.5}(\Gamma_{00})$.

Next, define a partially homogeneous problem: find $(\overline{\mathbf{v}},\overline{q})\in H^2(\Omega_{00})^2\times H^1(\Omega_{00})/P_0$ such that

$$(4.4)\qquad \begin{aligned} &\text{a) } -2\,\mathrm{div}\,\mathbf{D}(\overline{\mathbf{v}})+\nabla\overline{q}=\overline{\mathbf{h}} && \text{in } \Omega_{00},\\ &\text{b) } \mathrm{div}\,\overline{\mathbf{v}}=0 && \text{in } \Omega_{00},\\ &\text{c) } \overline{\mathbf{v}}.\mathbf{n}=0 && \text{on } \Gamma_{00},\\ &\text{d) } \mathbf{s}(\overline{\mathbf{v}},\overline{q}).\mathbf{t}=\overline{s}_2 && \text{on } \Gamma_{00}, \end{aligned}$$

where $\overline{\mathbf{h}}\in L^2(\Omega_{00})^2$, $\overline{s}_2\in H^{0.5}(\Gamma_{00})$. We have the following

LEMMA 4.1. *The pair* $(\overline{\mathbf{v}},\overline{q})$ *is a solution of problem* (4.4) *for* $\overline{\mathbf{h}}=\mathbf{h}+2\,\mathrm{div}\,\mathbf{D}(\mathbf{grad}\,\overline{w})$, $\overline{s}_2=s_2-\mathbf{s}(\mathbf{grad}\,\overline{w},q).\mathbf{t}$ *iff* $(\mathbf{v}=\overline{\mathbf{v}}+\mathbf{grad}\,\overline{w},\ q=\overline{q})$ *is a solution of problem* (4.3), *where* $\overline{w}$ *is a solution of the following Neumann problem*: *find* $\overline{w}\in H^3(\Omega_{00})$ *such that*

$$(4.5)\qquad \begin{aligned} &\Delta\overline{w}=d && \textit{in } \Omega_{00},\\ &\frac{\partial\overline{w}}{\partial\mathbf{n}}=s_1 && \textit{on } \Gamma_{00}. \end{aligned}$$

Proof. If we substitute $\overline{\mathbf{v}}+\mathbf{grad}\,\overline{w}$ for $\mathbf{v}$ in (4.3) then we obtain that $(\overline{\mathbf{v}},q)$ satisfies (4.4). Conversely, if we substitute $\mathbf{v}-\mathbf{grad}\,\overline{w}$ for $\overline{\mathbf{v}}$ in (4.4) then we obtain that $(\mathbf{v}, q)$ satisfies (4.3). The existence of the solution $\overline{w} \in H^3(\Omega_{00})$ follows from the compatibility condition (4.2) (cf. [16], [7]). ■

Since problem (4.4) is considered in dimension 2 we can represent its solution $\overline{\mathbf{v}}$ as

$$\overline{\mathbf{v}} = \mathbf{curl}\, w = \left(\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1}\right),$$

where w is a scalar function (cf. [16], Proposition 2.3, Ch. I). Condition (4.4b) is then satisfied automatically, $\overline{\mathbf{v}}.\mathbf{n} = \partial w/\partial \mathbf{t}$ and condition (4.4c) implies that w is constant along Γ_{00}. Let then $w = 0$ on Γ_{00}. By acting with the curl operator ($\operatorname{curl} \overline{\mathbf{h}} = \partial \overline{h}_2/\partial x_1 - \partial \overline{h}_1/\partial x_2$) on (4.4a) we arrive at the following problem: find $w \in H^3(\Omega_{00})$ such that

$$(4.6)\qquad \begin{aligned} &\text{a)}\ -2\operatorname{curl}\operatorname{div}\mathbf{D}(\mathbf{curl}\, w) = \widehat{h} && \text{in } \Omega_{00},\\ &\text{b)}\ w = 0 && \text{on } \Gamma_{00},\\ &\text{c)}\ \mathbf{s}(\mathbf{curl}\, w, q).\mathbf{t} = \widehat{s}_2 && \text{on } \Gamma_{00}, \end{aligned}$$

where $\widehat{h} \in H^{-1}(\Omega_{00})$, $\widehat{s}_2 \in H^{0.5}(\Gamma_{00})$.

We have the following:

LEMMA 4.2. *Let* $\widehat{h} = \operatorname{curl} \overline{\mathbf{h}}$ *and* $\widehat{s}_2 = \overline{s}_2$. *If* $\overline{\mathbf{v}}$ *is the first component of a solution of problem* (4.4), *then there exists* $w \in H^3(\Omega_{00})$ *such that* $\mathbf{curl}\, w = \overline{\mathbf{v}}$ *and* w *is a solution of* (4.6). *Conversely*, *if* $w \in H^3(\Omega_{00})$ *is a solution of problem* (4.6), *then there exists a unique* $\overline{q} \in H^1(\Omega_{00})/P_0$ *such that* $(\overline{\mathbf{v}} = \mathbf{curl}\, w,\ \overline{q})$ *is a solution of* (4.4).

Proof. For the first part of the lemma it remains to notice that if $\overline{\mathbf{v}} \in H^2(\Omega_{00})^2$ satisfies (4.4b), then there exists a stream function $w \in H^3(\Omega_{00})$ such that $\overline{\mathbf{v}} = \mathbf{curl}\, w$ (cf. [7], Th. 3.1, Ch. I, and the remark just after the proof). Assume now that $w \in H^3(\Omega_{00})$ is a solution of (4.6). Then by Theorem 2.9, Ch. I of [7] we obtain the existence and uniqueness of a pressure field $\overline{q} \in H^1(\Omega_{00})/P_0$ such that the second assertion holds. ■

Now we study problem (4.6). We are interested in obtaining the existence and uniqueness of the stream function $w \in H^3(\Omega_{00})$. To this end we derive a suitable Green formula and, consequently, a generalized (variational) form of problem (4.6). Then we study the coercivity and ellipticity of the bilinear form of the variational problem.

By the definition of distributional derivatives we obtain

$$(4.7)\qquad \begin{aligned} &-\langle 2\operatorname{curl}\operatorname{div}\mathbf{D}(\mathbf{curl}\, w),\ \chi\rangle_{H^{-1}\times H^1}\\ &= -\int_{\Omega_{00}} (2\operatorname{div}\mathbf{D}(\mathbf{curl}\, w)).\mathbf{curl}\,\chi\, dx, \qquad \forall \chi \in C^\infty(\Omega_{00}) \cap H_0^1(\Omega_{00}). \end{aligned}$$

After writing out the integrand on the right-hand side of (4.7) and using the Gauss formula we arrive at

$$(4.8)\qquad -\int\limits_{\Omega_{00}} \{(2\partial_1^2\partial_2 w - \partial_2\partial_1^2 w + \partial_2^3 w)\cdot\partial_2\chi - (-\partial_1^3 w + \partial_1\partial_2^2 w - 2\partial_2^2\partial_1 w)\cdot\partial_1\chi\}\,dx$$
$$= \mathcal{A}(w,\chi) - \int\limits_{\Gamma_{00}} \{2\partial_1\partial_2 w\partial_2\chi\cdot n_1 - \partial_1^2 w\partial_2\chi\cdot n_2 + \partial_2^2 w\partial_2\chi\cdot n_2 + \partial_1^2 w\partial_1\chi\cdot n_1 - \partial_2^2 w\partial_1\chi\cdot n_1 + 2\partial_2\partial_1 w\partial_1\chi\cdot n_2\}\,d\sigma,\qquad \forall\chi\in C^\infty(\Omega_{00})\cap H_0^1(\Omega_{00}),$$

where the linear form $\mathcal{A}$ is defined as follows:

$$(4.9)\qquad \mathcal{A}(w,\chi) = \int\limits_{\Omega_{00}} \{2\partial_1\partial_2 w\partial_1\partial_2\chi - \partial_1^2 w\partial_2^2\chi + \partial_2^2 w\partial_2^2\chi + \partial_1^2 w\partial_1^2\chi - \partial_2^2 w\partial_1^2\chi + 2\partial_2\partial_1 w\partial_2\partial_1\chi\}\,dx.$$

Now we use the following formulae:

$$\partial_2\chi = +\frac{\partial\chi}{\partial\mathbf{t}}n_1 - \frac{\partial\chi}{\partial\mathbf{n}}t_1 \quad \text{on } \Gamma_{00},$$
$$\partial_1\chi = -\frac{\partial\chi}{\partial\mathbf{t}}n_2 + \frac{\partial\chi}{\partial\mathbf{n}}t_2 \quad \text{on } \Gamma_{00},$$

which, in view of $\partial\chi/\partial\mathbf{t} = 0$ and (4.7), (4.8), yield

$$(4.10)\qquad -\langle 2\,\mathrm{curl}\,\mathrm{div}\,\mathbf{D}(\mathbf{curl}\,w),\ \chi\rangle_{H^{-1}\times H^1} = \mathcal{A}(w,\chi) + \int\limits_{\Gamma_{00}} \overline{s}(w)\cdot\frac{\partial\chi}{\partial\mathbf{n}}\,d\sigma,$$

where

$$\overline{s}(w) = 2\partial_1\partial_2 w\cdot n_1 t_1 - 2\partial_2\partial_1 w\cdot n_2 t_2 + (\partial_2^2 w - \partial_1^2 w)\cdot(n_1 t_2 - n_2 t_1).$$

The expression $\overline{s}(w)$ is just the boundary operator on the left-hand side of (4.6c): $\overline{s}(w) = \mathbf{s}(\mathbf{curl}\,w, q).\mathbf{t}$ (we omitted q in the definition of $\overline{s}$ since the tangent component of $\mathbf{s}$ does not depend on q). Formula (4.10) is just the desired Green formula for problem (4.6), which justifies the introduction of the following variational problem: find $w\in H^2(\Omega_{00})$ such that

$$(4.11)\qquad \mathcal{A}(w,\chi) = \langle\widehat{h},\chi\rangle_{H^{-1}\times H^1} - \int\limits_{\Gamma_{00}} \widehat{s}_2(w)\frac{\partial\chi}{\partial\mathbf{n}}\,d\sigma,\qquad \forall\chi\in H^2(\Omega_{00})\cap H_0^1(\Omega_{00}).$$

We introduce the following space:

$$\mathcal{X} = \{\chi\in H^2(\Omega_{00})\cap H_0^1(\Omega_{00}) : \chi(x_1,x_2) = -\chi(-x_1,x_2) = -\chi(x_1,-x_2)\},$$

with the standard norm of $H^2(\Omega_{00})$. Now we are ready to prove the following

LEMMA 4.3. *The form $\mathcal{A}: \mathcal{X}\times\mathcal{X}\to\mathbb{R}$ defined in* (4.9) *is $\mathcal{X}$-elliptic and continuous.*

Proof. We first prove the coercivity of $\mathcal{A}$ in $H^2(\Omega_{00})$, i.e.

$$\mathcal{A}(\chi,\chi)+\|\chi\|^2_{1,\Omega_{00}}\ge C\|\chi\|^2_{2,\Omega_{00}},\quad \forall\chi\in H^2(\Omega_{00}). \tag{4.12}$$

We have

$$\mathcal{A}(\chi,\chi)=\int\limits_{\Omega_{00}}\{4(\partial_1\partial_2\chi)^2+(\partial_2^2\chi-\partial_1^2\chi)^2\}\,dx. \tag{4.13}$$

Define the space

$$\mathcal{Y}=\{\chi\in H^1(\Omega_{00}):\partial_1\partial_2\chi\in L^2(\Omega_{00}),\ \partial_2^2\chi-\partial_1^2\chi\in L^2(\Omega_{00})\},$$

with the norm $(\mathcal{A}(\chi,\chi)+\|\chi\|^2_{1,\Omega_{00}})^{1/2}$. Let $\chi\in\mathcal{Y}$. We have $\partial_2^2\chi\in H^{-1}(\Omega_{00})$, $\partial_1\partial_2^2\chi=\partial_2(\partial_1\partial_2\chi)\in H^{-1}(\Omega_{00})$ and $\partial_2\partial_2^2\chi=\partial_2(\partial_2^2\chi-\partial_1^2\chi)+\partial_1(\partial_1\partial_2\chi)\in H^{-1}(\Omega_{00})$ since $\partial_1\partial_2\chi$ and $\partial_2^2\chi-\partial_1^2\chi$ are in $L^2(\Omega_{00})$.

Now using Theorem 3.2, Ch. 3 of [6] we get $\partial_2^2\chi\in L^2(\Omega_{00})$ and, consequently, $\partial_1^2\chi\in L^2(\Omega_{00})$, which means $\mathcal{Y}=H^2(\Omega_{00})$. The identity operator from $H^2(\Omega_{00})$ onto $\mathcal{Y}$ is obviously continuous, and so is its inverse, which yields (4.12).

To prove the ellipticity we want to show that $\mathcal{A}(\cdot,\cdot)^{1/2}$ is a norm equivalent to $\|\cdot\|_{2,\Omega_{00}}$ in $\mathcal{X}$. First, we check that

$$\mathcal{A}(\chi,\chi)=0\Leftrightarrow\chi=0,\quad\forall\chi\in\mathcal{X}. \tag{4.14}$$

By (4.13) the condition $\mathcal{A}(\chi,\chi)=0$ implies $\partial_1\partial_2\chi=0$, which means $\chi(x_1,x_2)=\chi_1(x_1)+\chi_2(x_2)$. Moreover, $\partial_2^2\chi=\partial_1^2\chi$ (cf. (4.13)), which implies $\chi_1''(x_1)=\chi_2''(x_2)=c$, where c is a constant. Hence $\chi_1=cx_1^2+c_1x_1+c_2$, $\chi_2=cx_2^2+c_3x_2+c_4$ a.e., where c_i, $i=1,\dots,4$, are some constants. On the other hand, we have $\chi\in\mathcal{X}$, which implies $\chi\equiv 0$.

We show that

$$\mathcal{A}(\chi,\chi)\ge C\|\chi\|^2_{2,\Omega_{00}},\quad\forall\chi\in\mathcal{X}. \tag{4.15}$$

Assume the contrary to (4.15), from which it follows that there exists a sequence $\{\chi_n\}_{n=1}^\infty$ such that $\|\chi_n\|_{2,\Omega_{00}}=1$ and $\mathcal{A}(\chi_n,\chi_n)\to 0$. We can derive a subsequence of the above sequence, still denoted by χ_n, that converges weakly in $\mathcal{X}$ to some $\chi\in\mathcal{X}$. Since $H^2(\Omega_{00})$ is compactly embedded in $H^1(\Omega_{00})$, $\chi_n\to\chi$ strongly in $H^1(\Omega_{00})$.

Moreover, since the function $\mathbb{R}^3\ni(x_1,x_2,x_3)\to 4x_1^2+(x_2-x_3)^2\in\mathbb{R}$ is convex the form $\mathcal{A}$ is weakly lower semicontinuous in $H^2(\Omega_{00})$ (cf. [5], Th. 1.1, Ch. 1, and remark (iii) after it), i.e. $\liminf_{n\to\infty}\mathcal{A}(\chi_n,\chi_n)\ge\mathcal{A}(\chi,\chi)$, which implies $\mathcal{A}(\chi,\chi)=0$ and, consequently, by (4.14), $\chi=0$. Now by referring to the coercivity of $\mathcal{A}$ (cf. (4.12)) we see that the sequence $\{\chi_n\}_{n=1}^\infty$ converges strongly to 0 in $H^2(\Omega_{00})$. On the other hand, $\|\chi_n\|_{2,\Omega_{00}}=1$, which is a contradiction.

We have proved that the form $\mathcal{A}$ is $\mathcal{X}$-elliptic. The continuity of $\mathcal{A}$ is obvious. Hence the lemma follows. ∎

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. First, we show the injectivity of the operator. Let $(\mathbf{h}, d, s_1, s_2) = 0$ and let $(\mathbf{v}, q)$ be a solution of (4.1). Consequently, there exists a stream function $w \in H^3(\Omega_{00})$ such that $\mathbf{curl}\, w = \mathbf{v}$ and w is a solution of (4.6) for $(\widehat{h}, \widehat{s}_2) = 0$. Moreover, in view of the Green formula (4.10), w is a solution of the variational problem (4.11) as well. Moreover, we can choose w in such a way that $w \in \mathcal{X}$ since $\mathbf{v}$ satisfies the symmetry condition (1.8b) and by the relation $\overline{\mathbf{v}}.\mathbf{n} = \partial w/\partial \mathbf{t}$ the stream function w is constant along the boundary, so that we can always change it by a constant to obtain $w \in H^1_0(\Omega_{00})$. The Lax–Milgram theorem and the $\mathcal{X}$-ellipticity of the form $\mathcal{A}$ (cf. Lemma 4.3) yield the uniqueness of the solution in $\mathcal{X}$. Hence we get $w = 0$. Consequently, $\mathbf{v} = 0$ and by Lemma 4.2 (uniqueness in the quotient space) $q \in P_0$. This shows the injectivity.

Now we want to check that the operator $(\mathbf{v}, q) \to (\mathbf{h}, d, s_1, s_2)$ is surjective. By Lemmas 4.1, 4.2 and the derivation of the variational problem (4.11) we know that $\mathbf{v}$ is the first component of a solution of problem (4.3) iff $w \in H^3(\Omega_{00})$ is a solution of the variational problem (4.11), where $\mathbf{curl}\, w = \overline{\mathbf{v}}$, $\overline{\mathbf{v}} = \mathbf{v} - \mathbf{grad}\, \overline{w}$, and $\overline{w} \in H^3(\Omega_{00})$ is a solution of the Neumann problem (4.5), $\widehat{h} = \mathrm{curl}(\mathbf{h} + 2\,\mathrm{div}\,\mathbf{D}(\mathbf{grad}\, \overline{w}))$, $\widehat{s}_2 = s_2 - \mathbf{s}(\mathbf{grad}\, \overline{w}, q).\mathbf{t}$ (cf. (4.11)).

Now the form $\mathcal{A}$ is $\mathcal{X}$-elliptic by Lemma 4.3. Thus the Lax-Milgram theorem yields the existence of a solution $w \in \mathcal{X}$ such that the variational equation (4.11) is satisfied for any $\chi \in \mathcal{X}$. The symmetry conditions for $\mathbf{h}$, d, s_1, s_2 (cf. (4.1)) imply that $\widehat{s}_2$ satisfies the same symmetry condition as s_2 and $\overline{\mathbf{h}} = \mathbf{h} + 2\,\mathrm{div}\,\mathbf{D}(\mathbf{grad}\, \overline{w})$ satisfies the same symmetry conditions as $\mathbf{h}$. Moreover, for every χ we have the decomposition $\chi = \chi_1 + \chi_2 + \chi_3$, where χ_1 is antisymmetric w.r.t. both axes, χ_2 is antisymmetric w.r.t. one axis and symmetric w.r.t. to the other and χ_3 is symmetric w.r.t. to one axis. This yields that the variational equation (4.11) is satisfied for all $\chi \in H^2(\Omega_{00}) \cap H^1_0(\Omega_{00})$. The form $\mathcal{A}$ is coercive in $H^2(\Omega_{00})$ (cf. (4.12)). Thus taking into account standard results concerning the regularity of solutions of elliptic problems (cf. e.g. [11], Ch. 4) we conclude that $w \in H^3(\Omega_{00}) \cap \mathcal{X}$. In view of Lemmas 4.1 and 4.2 we see that there exists a pressure field $q \in H^1(\Omega_{00})/P_0$ such that $(\mathbf{v} = \mathbf{curl}\, w + \mathbf{grad}\, \overline{w}, q)$ satisfies problem (4.3). Since $w \in \mathcal{X}$ and $\overline{w}$ is obviously symmetric w.r.t. both axes, $\mathbf{v}$ and q satisfy the appropriate symmetry conditions (cf. (4.1)). Thus if we add the symmetry conditions (1.8bc) and (1.9bcd) to the formulation of problem (4.3), then this problem defines an isomorphism, so that we can estimate $(\mathbf{v}, p)$ in the norm of the space $H^2(\Omega_{00})^2 \times H^1(\Omega_{00})/P_0$ by the data.

Now we show that $\mathbf{v} \in W^{2,\alpha}(\Omega_{00})^2$ and $q \in W^{1,\alpha}(\Omega_{00})/P_0$, $\alpha > 2$. To this end we use the theory presented in [1]. First, observe that there exists a sequence $\{\mathbf{h}_n, d_n, s_{1n}, s_{2n}\}_{n=1}^{\infty}$ satisfying

$$\begin{aligned}
&\mathbf{h}_n \in H^1(\Omega_{00})^2 \text{ and } \mathbf{h}_n \to \mathbf{h} \text{ in } L^{\alpha}(\Omega_{00})^2,\\
&d_n \in H^2(\Omega_{00} \text{ and } d_n \to d \text{ in } W^{1,\alpha}(\Omega_{00}),\\
&s_{1n} \in H^{2.5}(\Gamma_{00}) \text{ and } s_{1n} \to s_1 \text{ in } W^{2-1/\alpha,\alpha}(\Gamma_{00}),\\
&s_{2n} \in H^{1.5}(\Gamma_{00}) \text{ and } s_{2n} \to s_2 \text{ in } W^{1-1/\alpha,\alpha}(\Gamma_{00}).
\end{aligned}$$

Moreover, $\mathbf{h}_n$, d_n, s_{1n}, s_{2n} satisfy the symmetry conditions analogous to those for $\mathbf{h}$, d, s_1, s_2 (cf. (4.1) and Sec. 1). The system of three scalar equations (4.1ab) for the three unknowns v_1, v_2, q is equivalent to the same system with $-2\operatorname{div}\mathbf{D}(\mathbf{v})$ replaced by $-\Delta\mathbf{v}$. After this change it is easy to see that this last system is uniformly elliptic and satisfies the "Supplementary Condition on $\mathbb{L}$" from [1] (cf. also [16], Proposition 2.2, Ch. I). The boundary conditions (4.1cd) satisfy the "Complementing Boundary Condition" from [1] (cf. also [15]). Thus by Theorem 10.5 of [1], bearing in mind that for $\mathbf{h}_n$, d_n, s_{1n}, s_{2n} the solution $(\mathbf{v}_n, q_n) \in H^2(\Omega_{00})^2 \times H^1(\Omega_{00})$ of problem (4.3) exists, we get $\mathbf{v}_n \in H^3(\Omega_{00})^2$ and $q_n \in H^2(\Omega_{00})$. By the appropriate embeddings of Sobolev spaces we obtain $\mathbf{v}_n \in W^{2,\alpha}(\Omega_{00})^2$, $q_n \in W^{1,\alpha}(\Omega_{00})$, $\mathbf{h}_n \in L^{\alpha}(\Omega_{00})^2$, $d_n \in W^{1,\alpha}(\Omega_{00})$, $s_{1n} \in W^{2-1/\alpha,\alpha}(\Gamma_{00})$, $s_{2n} \in W^{1-1/\alpha,\alpha}(\Gamma_{00})$. Then we apply once again Theorem 10.5 of [1] to estimate $(\mathbf{v}_n, q_n)$ in terms of $(\mathbf{h}_n, d_n, s_{1n}, s_{2n})$ in the norms of the above spaces. In this estimation the term on the right-hand side involving the L^{α} norms of $\mathbf{v}_n$ and q_n can be estimated by the data since, as we have already stated, problem (4.3) with the appropriate symmetry conditions defines an isomorphism. Since $\{(\mathbf{h}_n, d_n, s_{1n}, s_{2n})\}_{n=1}^{\infty}$ is a Cauchy sequence the linearity of problem (4.1) and the above estimate imply that $\{(\mathbf{v}_n, q_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^{2,\alpha}(\Omega_{00})^2 \times W^{1,\alpha}(\Omega_{00})/P_0$. On the other hand, the sequence $\{(\mathbf{v}_n, q_n)\}_{n=1}^{\infty}$ converges to the solution $(\mathbf{v}, q)$ of problem (4.3) in $H^2(\Omega_{00})^2 \times H^1(\Omega_{00})/P_0$. Hence $(\mathbf{v}, q) \in W^{2,\alpha}(\Omega_{00})^2 \times W^{1,\alpha}(\Omega_{00})/P_0$ and the surjectivity of the operator $(\mathbf{v}, q) \to (\mathbf{h}, d, s_1, s_2)$ follows. The injectivity and surjectivity imply that the linear operator is an isomorphism. ∎

Presently, we deal with the linearized problem for the potential ψ: find $\psi \in W_1^2(\mathbb{R}^2)$ satisfying the symmetry condition (1.8a) such that

$$-\Delta\psi + i\beta(\psi - I(\psi)) = \xi, \tag{4.16}$$

where $\xi \in \mathcal{Z} = W_1^0(\mathbb{R}^2) \cap (W_0^1(\mathbb{R}^2))'$ and ξ satisfies (1.8a).

In view of the density of the space $\mathcal{D}(\mathbb{R}^2)$ of smooth complex-valued functions with compact support in the space $W_0^1(\mathbb{R}^2)$ (cf. [12], [10]) and the definition of the distributional differentiation we can consider a generalized problem for the potential: find $\psi \in W_0^1(\mathbb{R}^2)$ such that

$$\mathcal{B}(\psi,\overline{\chi}) = \langle \xi,\overline{\chi}\rangle_{(W_0^1)'\times W_0^1}, \qquad \forall \chi \in \mathcal{D}(\mathbb{R}^2), \tag{4.17}$$

where $\mathcal{B}$ is a sesquilinear form defined as follows:

$$\mathcal{B}(\psi,\overline{\chi}) = \sum_{i=1}^{2} \int_{\mathbb{R}^2} \frac{\partial \psi}{\partial x_i} \cdot \frac{\partial \overline{\chi}}{\partial x_i}\, dx + \sum_{k=0}^{2} \Big\{ i\beta \int_{\Omega_k} \psi\overline{\chi}\, dx - \frac{i\beta}{|\Omega_k|} \int_{\Omega_k} \psi\, dx \int_{\Omega_k} \overline{\chi}\, dx \Big\}, \tag{4.18a}$$

where

$$\langle \xi,\chi\rangle_{(W_0^1)'\times W_0^1} = \int_{\mathbb{R}^2} \xi\overline{\chi}\, dx. \tag{4.18b}$$

The integration in the second term on the right-hand side of (4.18a) extends over the bounded regions Ω_k, $k = 0,1,2$ only, since β is zero beyond them. We need the following

THEOREM 4.2. *The operator $\psi \to \xi$ defined by problem* (4.16) *is an isomorphism.*

Proof. The form $\mathcal{B}$ is continuous in the space $W_0^1(\mathbb{R}^2) \times W_0^1(\mathbb{R}^2)$ and $W_0^1(\mathbb{R}^2)/P_0$-elliptic. The continuity comes from the Schwarz inequality applied to both members in the definition (4.18a) of $\mathcal{B}$, and the fact that $W_0^1(\mathbb{R}^2) \hookrightarrow H^1_{\mathrm{loc}}(\mathbb{R}^2)$. The $W_0^1(\mathbb{R}^2)/P_0$-ellipticity comes from the fact that the seminorm $|\cdot|_{1,0,\mathbb{R}^2}$ of $W_0^1(\mathbb{R}^2)$ is a norm in the quotient space $W_0^1(\mathbb{R}^2)/P_0$, equivalent to the standard one (cf. [12], [9] or [10]).

First, we check the injectivity. Assume that $\xi = 0$. If ψ is a solution of (4.16) then it is a solution of the variational problem (4.17). Since the form $\mathcal{B}$ is $W_0^1(\mathbb{R}^2)/P_0$-elliptic the Lax–Milgram theorem yields the uniqueness of solution of (4.17) in the quotient space. Thus $\psi \in P_0$, and the symmetry condition (1.8a) yields $\psi = 0$, which means that the operator $\psi \to \xi$ is injective.

Next, we verify the surjectivity. Assume that $\xi \in \mathcal{Z}$. Since ξ satisfies the symmetry condition (1.8a) as well we infer that $\xi \in W_1^0(\mathbb{R}^2) \cap (W_0^1(\mathbb{R}^2)/P_0)'$. The Lax–Milgram theorem yields the existence of the potential ψ in $W_0^1(\mathbb{R}^2)$ satisfying (4.17). If we substitute $\widehat{\psi}$ for ψ and $\widehat{\chi}$ for χ in (4.17), where $\widehat{\psi}(x_1,x_2) = -\psi(-x_1,x_2)$ and $\widehat{\chi}(x_1,x_2) = -\chi(-x_1,x_2)$, then, in view of the symmetry condition for ξ, $\widehat{\psi}$ again satisfies (4.17). Similarly, $\check{\psi}(x_1,x_2) = \psi(x_1,-x_2)$ satisfies this equation. The uniqueness in the quotient space implies $\psi(x_1,x_2) = -\psi(-x_1,x_2) + c_1 = \psi(x_1,-x_2) + c_2$, where c_1, c_2 are some constants. On the other hand, $\psi \in W_0^1(\mathbb{R}^2)$, which implies that $c_2 = 0$, and thus we have the symmetry conditions for $\psi - c_1/2$. It remains to show that $\psi \in W_1^2(\mathbb{R}^2)$. Equation (4.16) can be written as $-\Delta\psi = \overline{\xi}$, where $\overline{\xi} = \xi - i\beta(\psi - I(\psi))$. Obviously, $\overline{\xi} \in \mathcal{Z}$ since $\beta = 0$ beyond $\Omega_0 \cup \Omega_1 \cup \Omega_2$. Now we can use the regularity result of [10] to obtain $\psi \in W_1^2(\mathbb{R}^2)$, which gives the surjectivity of the operator $\psi \to \xi$. Obviously, this operator is continuous from $W_1^2(\mathbb{R}^2)$ into $\mathcal{Z}$. Thus the operator is injective, surjective and continuous. Hence it is an isomorphism. ■

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math. 17 (1964), 35–92.

[2] J. Bemelmans, *Gleichgewichtsfiguren zäher Flüssigkeiten mit Oberflächenspannung*, Analysis 1 (1981), 241–282.

[3] O. Besson, J. Bourgeois, P. A. Chevalier, J. Rappaz and R. Touzani, *Numerical modelling of electromagnetic casting processes*, J. Comput. Phys. 92 (2) (1991), 482–507.

[4] H. Cartan, *Cours de calcul différentiel*, Hermann, Paris, 1977.

[5] B. Dacorogna, *Weak Continuity and Weak Lower Semicontinuity of Non-Linear Functionals,* Springer, Berlin, 1982.

[6] G. Duvaut et J.-L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

[7] V. Girault and P. A. Raviart, *Finite Elements Methods for Navier–Stokes Equations. Theory and Algorithms*, Springer, Berlin, 1986.

[8] J. Giroire and J. C. Nedelec, *Numerical solution of the exterior Neumann problem using a double layer potential*, Math. Comput. 32 (1987), 973–990.

[9] M. N. Le Roux, *Méthode d'éléments finis pour la résolution numérique de problèmes extérieurs en dimension* 2, RAIRO Anal. Numér. 11 (1977), 27–60.

[10] —, *Résolution numérique du problème du potentiel dans le plan par une méthode variationnelle d'éléments finis*, Thèse, L'Université de Rennes, U.E.R., Mathématiques et Informatique, 1974.

[11] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.

[12] J. C. Nedelec, *Approximation des équations intégrales en mécanique et en physique*, Rapport du Centre de Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, 1977.

[13] J. Rappaz and R. Touzani, *On a two-dimensional magnetohydrodynamic problem I. Modelling and analysis*, RAIRO Modél. Math. Anal. Numér. 26 (2) (1991), 347–364.

[14] V. A. Solonnikov, *Free boundary problems and problems in noncompact domains for the Navier–Stokes equations*, in: Proc. Internat. Congr. Math., Berkeley, California, 1986, Vol. 2, 1987, 1113–1122.

[15] V. A. Solonnikov and V. E. Shchadilov, *On a boundary value problem for a stationary system of Navier–Stokes equations*, Trudy Mat. Inst. Steklov. 125 (1973) (in Russian); English transl.: Proc. Steklov Inst. Math. 125 (1973), 186–199.

[16] R. Temam, *Navier–Stokes Equations. Theory and Numerical Analysis*, Elsevier, Amsterdam, 1985.

INSTITUTE OF PHYSICAL CHEMISTRY
POLISH ACADEMY OF SCIENCES
KASPRZAKA 44/52
01-224 WARSZAWA, POLAND
E-mail: ROLINSKI@ICHF.EDU.PL

Reçu par la Rédaction le 3.3.1993
Révisé le 30.5.1993 et 27.12.1994