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On a differential inequality for equations of a viscous compressible heat conducting fluid bounded by a free surface

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Abstract. We derive a global differential inequality for solutions of a free boundary problem for a viscous compressible heat conducting fluid. The inequality is essential in proving the global existence of solutions.

1. Introduction. The aim of this paper is to derive a global differential inequality for the following free boundary problem for a viscous compressible heat conducting fluid (see [2], Chs. 2 and 5):

$$
\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] + \nabla p - \mu\Delta v - \nu\nabla\operatorname{div} v = \varrho f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho c_v(\theta_t + v\cdot\nabla\theta) + \theta p_\theta \operatorname{div} v - \kappa\Delta\theta\\
&\qquad -\tfrac{1}{2}\mu\sum_{i,j=1}^{3}(v_{i,x_j}+v_{j,x_i})^2 - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\phi_t/|\nabla\phi| && \text{on } \widetilde{S}^T,\\
&\partial\theta/\partial n = \theta_1 && \text{on } \widetilde{S}^T,\\
&v|_{t=0} = v_0,\quad \varrho|_{t=0} = \varrho_0,\quad \theta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\widetilde{\Omega}^T = \bigcup_{t\in(0,T)} \Omega_t \times \{t\}$, Ω_t is a bounded domain of the drop at time t and $\Omega_0 = \Omega$ is its initial domain, $\widetilde{S}^T = \bigcup_{t\in(0,T)} S_t \times \{t\}$, $S_t = \partial\Omega_t$, $\phi(x,t) = 0$ describes S_t, and $\overline{n}$ is the unit outward vector normal to the boundary (i.e. $\overline{n} = \nabla\phi/|\nabla\phi|$).

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Moreover, $v = v(x,t)$ is the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $\theta = \theta(x,t)$ the temperature, $f = f(x,t)$ the external force field per unit mass, $r = r(x,t)$ the heat sources per unit mass, $\theta_1 = \theta_1(x,t)$ the heat flow per unit surface, $p = p(\varrho,\theta)$ the pressure, μ and ν the viscosity coefficients, κ the coefficient of heat conductivity, $c_v = c_v(\varrho,\theta)$ the specific heat at constant volume, and p_0 the external (constant) pressure.

We assume that $c_v > 0$, the coefficients μ, ν, κ are constants, and $\kappa > 0$, $\nu \ge \mu > 0$.

Finally, $\mathbb{T} = \mathbb{T}(v,p)$ denotes the stress tensor of the form

$$\begin{aligned}\mathbb{T} = \{T_{ij}\} &= \{-p\delta_{ij} + \mu(v_{i,x_j} + v_{j,x_i}) + (\nu-\mu)\delta_{ij}\operatorname{div} v\}\\ &\equiv \{-p\delta_{ij} + D_{ij}(v)\},\end{aligned}$$

where $i,j = 1,2,3$, and $\mathbb{D} = \mathbb{D}(v) = \{D_{ij}\}$ is the deformation tensor. Let the domain Ω be given. Then by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi,t),\ \xi \in \Omega\}$, where $x = x(\xi,t)$ is the solution of the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x,t), \qquad x|_{t=0} = \xi \in \Omega,\ \xi = (\xi_1,\xi_2,\xi_3).$$

Therefore, the transformation $x = x(\xi,t)$ connects the Eulerian x and the Langrangian ξ coordinates of the same fluid particle. Hence

$$x = \xi + \int_0^t u(\xi,s)\,ds \equiv X_u(\xi,t), \tag{1.2}$$

where $u(\xi,t) = v(X_u(\xi,t),t)$. Moreover, the kinematic boundary condition $(1.1)_5$ implies that the boundary S_t is a material surface. Thus, if $\xi \in S = S_0$ then $X_u(\xi,t) \in S_t$ and $S_t = \{x : x = X_u(\xi,t),\ \xi \in S\}$.

By the continuity equation $(1.1)_2$ and $(1.1)_5$ the total mass of the drop is conserved and the following relation holds between ϱ and Ω_t:

$$\int_{\Omega_t} \varrho(x,t)\,dx = M.$$

This paper is divided into three sections. In Section 2 we introduce some notation. In Section 3 we derive the main result of the paper, i.e. the differential inequality (3.160) (see Theorem 3.13) which is essential in proving the global existence of a solution of problem (1.1) (see [19]). In order to obtain inequality (3.160) we impose the following assumptions:

a) there exists a sufficiently smooth local solution;

b) the transformation (1.2) together with its inverse exist;

c) the volume and the shape of the domain do not change much in time.

Papers concerning problem (1.1) include [15]–[17] and [20]. In [15] the local-in-time existence and uniqueness of solution to problem (1.1) in the

Sobolev–Slobodetskiĭ spaces is proved. In [17] we prove that under an appropriate choice of ϱ_0, v_0, θ_0, θ_1, p_0, κ and the form of the internal energy per unit mass $\varepsilon = \varepsilon(\varrho, \theta)$, $\operatorname{var}_t |\Omega_t|$ is as small as we need. Paper [20] contains the global existence theorem for problem (1.1). In [15], [18], [19], [21] we consider the motion of a viscous compressible heat conducting fluid bounded by a free surface governed by surface tension. Such a motion is described by equations $(1.1)_1$–$(1.1)_3$ with conditions $(1.1)_5$–$(1.1)_7$ and with the condition

$$\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} \tag{1.3}$$

replacing $(1.1)_4$. In (1.3), σ is the constant coefficient of surface tension, and H is the double mean curvature of S_t.

Similarly to the case $\sigma = 0$, in [15] the local motion of a capillary fluid (the case $\sigma \neq 0$) is considered, while [18], [19] and [21] give, in that case, analogous to those of [17], the present paper and [20], respectively. In [18] conservation laws and global estimates for equations $(1.1)_1$–$(1.1)_3$ with conditions (1.3) and $(1.1)_5$–$(1.1)_7$ are presented. Moreover, we prove in [18] that we can choose ϱ_0, v_0, θ_0, θ_1, p_0, κ, σ and the form of the internal energy per unit mass $\varepsilon = \varepsilon(\varrho, \theta)$ such that $\operatorname{var}_t |\Omega_t|$ is as small as we need. This result is used in [21] to prove the global-in-time existence of solutions to problem $(1.1)_1$–$(1.1)_3$, (1.3), $(1.1)_5$–$(1.1)_7$. Paper [19] is devoted to a differential inequality for problem $(1.1)_1$–$(1.1)_3$, (1.3), $(1.1)_5$–$(1.1)_7$ which is analogous to inequality (3.160). In [21] the global existence theorem for problem $(1.1)_1$–$(1.1)_3$, (1.3), $(1.1)_5$–$(1.1)_7$ is proved. Finally, [16] contains the review of all results from [17]–[21] including the main result proved in this paper.

The motion of a viscous compressible heat conducting fluid in a fixed domain was considered by A. Matsumura and T. Nishida [3]–[7], A.Valli [13], and A. Valli and W. M. Zajączkowski [14]. Papers [3] and [4] are concerned with the initial value problem for equations $(1.1)_1$–$(1.1)_3$ considered in $\mathbb{R}^3 \times (0, \infty)$. In [4] the existence and uniqueness of a global-in-time classical solution of system $(1.1)_1$–$(1.1)_3$ is proved for the initial conditions

$$v|_{t=0} = v_0, \quad \varrho|_{t=0} = \varrho_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

The solution is obtained in a neighbourhood of a constant state $(v, \varrho, \theta) = (0, \overline{\varrho}, \overline{\theta})$, where $\overline{\varrho}$ and $\overline{\theta}$ are positive constants. In [3] the same type of result is obtained for a polytropic gas, i.e. under the assumption that $\varepsilon = c_v\theta$, where ε is the internal energy. In [7] the global existence theorem is proved for system $(1.1)_1$–$(1.1)_3$ considered in $\Omega \times (0, \infty)$ (where Ω is a halfspace or an exterior domain of any bounded region with smooth boundary) with initial conditions (1.4) and with the boundary conditions of Dirichlet or Neumann type. Papers [5], [6], [13] and [14] are concerned with the global motion of a viscous compressible heat conducting fluid in a bounded domain $\Omega \subset \mathbb{R}^3$.

For a compressible barotropic fluid (i.e. when the temperature of the fluid is constant) the problem corresponding to (1.1) has been examined by W. M. Zajączkowski [22]–[25] and V. A. Solonnikov and A. Tani [12]. In [23]–[24] the local motion of a compressible barotropic fluid bounded by a free surface is considered, while [22], [25] and [12] are devoted to the global motion of such a fluid.

In [8] K. Pileckas and W. M. Zajączkowski proved the existence of a stationary motion of a viscous compressible barotropic fluid bounded by a free surface governed by surface tension.

Finally, papers of V. A. Solonnikov [9]–[11] concern free boundary problems for viscous incompressible fluids. In the case of an incompressible fluid $\varrho = \text{const}$, so the continuity equation $(1.1)_2$ reduces to

$$\operatorname{div} v = 0. \tag{1.5}$$

Therefore, the problem examined by V. A. Solonnikov [9]–[11] is described by the Navier–Stokes equations $(1.1)_1$ (where $p = p(x,t)$) and by (1.5) with the initial condition $v|_{t=0} = v_0$ and with the boundary condition being either $(1.1)_4$ or (1.3).

2. Notation. Let $Q = \Omega_t$ or $Q = S_t$ $(t \ge 0)$. By $\|\cdot\|_{l,Q}$ $(l \ge 0)$ and $|\cdot|_{p,Q}$ $(1 \le p \le \infty)$ we denote the norms in the usual Sobolev spaces $W_2^l(Q)$ and in the $L_p(Q)$ spaces, respectively.

Next, we introduce the space $\Gamma_k^l(\Omega)$ of functions u with the norm

$$\|u\|_{\Gamma_k^l(\Omega)} = \sum_{i \le l-k} \|\partial_t^i u\|_{l-i,\Omega} \equiv |u|_{l,k,\Omega}, \qquad \text{where } l > 0,\ k \ge 0.$$

In the sequel we shall use the following notation for derivatives of u. If u is a scalar-valued function we denote by $D_{x,t}^k u$ or $u_{\underbrace{x\ldots x\,t\ldots t}_{k \text{ times}}}$ the vector $(D_x^\alpha \partial_t^i u)_{|\alpha|+i=k}$.

Similarly, if $u = (u_1, u_2, u_3)$ we denote by $D_{x,t}^k u$ or $u_{\underbrace{x\ldots x\,t\ldots t}_{k \text{ times}}}$ the vector $(D_x^\alpha \partial_t^i u_j)_{|\alpha|+i=k, j=1,2,3}$. Hence $|D_{x,t}^k u| = \sum_{|\alpha|+i=k} |D_x^\alpha \partial_t^i u|$.

We use the following lemma.

LEMMA 2.1. *The following imbedding holds*: $W_r^l(Q) \subset L_p^\alpha(Q)$ $(Q \subset \mathbb{R}^3)$, *where* $|\alpha| + 3/r - 3/p \le l$, $l \in \mathbb{Z}$, $1 \le p, r \le \infty$; $L_p^\alpha(\Omega)$ *is the space of functions* u *such that* $|D_x^\alpha u|_{p,\Omega} < \infty$, *and* $W_r^l(Q)$ *is the Sobolev space.*

Moreover, the following interpolation inequalities hold:

$$|D_x^\alpha u|_{p,Q} \le c\varepsilon^{1-\kappa} |D_x^l u|_{r,Q} + c\varepsilon^{-\kappa} |u|_{r,Q},$$

where $\kappa = |\alpha|/l + 3/(lr) - 3/(lp) < 1$, ε *is a parameter*, *and* $c > 0$ *is a constant independent of* u *and* ε; *and*

$$|D_x^\alpha u|_{q,\partial Q} \le c\varepsilon^{1-\kappa} |D_x^l u|_{r,Q} + c\varepsilon^{-\kappa} |u|_{r,Q},$$

where $\kappa = |\alpha|/l + 3/(lr) - 2/(lq) < 1$, ε *is a parameter*, *and* $c > 0$ *is a constant independent of* u *and* ε.

Lemma 2.1 follows from Theorem 10.2 of [1].

3. Global differential inequality. Assume that the existence of a sufficiently smooth local solution of problem (1.1) has been proved. To show the differential inequality we consider the motion near the constant state $v_e = 0$, $p_e = p_0$, $\theta_e = \overline{\theta}_0 = \frac{1}{|\Omega|}\int_\Omega \theta_0\,d\xi$ and ϱ_e, where ϱ_e is a solution of the equation

$$p(\varrho_e, \theta_e) = p_0. \tag{3.1}$$

Let

$$p_\sigma = p - p_0, \quad \varrho_\sigma = \varrho - \varrho_0, \quad \vartheta_0 = \theta - \theta_e, \quad \vartheta = \theta - \theta_{\Omega_t}, \tag{3.2}$$

where

$$\theta_{\Omega_t} = \frac{1}{|\Omega_t|}\int_{\Omega_t} \theta\,dx.$$

Then problem (1.1) takes the form

$$\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p_\sigma) = \varrho f && \text{in } \Omega_t,\ t\in[0,T],\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \Omega_t,\ t\in[0,T],\\
&\varrho c_v(\varrho,\theta)(\vartheta_{0t} + v\cdot\nabla\vartheta_0) + \theta p_\theta(\varrho,\theta)\operatorname{div} v \\
&\qquad - \kappa\Delta\vartheta_0 - \frac{1}{2}\mu\sum_{i,j}(\partial_{x_i}v_j + \partial_{x_j}v_i)^2 \\
&\qquad - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \Omega_t,\ t\in[0,T],\\
&\mathbb{T}(v,p_\sigma)\overline{n} = 0 && \text{on } S_t,\ t\in[0,T],\\
&\partial\vartheta_0/\partial n = \theta_1 && \text{on } S_t,\ t\in[0,T],
\end{aligned} \tag{3.3}$$

where $\mathbb{T}(v,p_\sigma) = \{\mu(\partial_{x_i}v_j + \partial_{x_j}v_i) + (\nu-\mu)\delta_{ij}\operatorname{div} v - p_\sigma\delta_{ij}\}$ and T is the time of local existence.

In the sequel we shall use the following Taylor formula for p_σ:

$$\begin{aligned}
p_\sigma &= p(\varrho,\theta) - p(\varrho_e,\theta_e) = p(\varrho,\theta) - p(\varrho_e,\theta) + p(\varrho_e,\theta) - p(\varrho_e,\theta_e)\\
&= (\varrho-\varrho_e)\int_0^1 p_\varrho(\varrho_e + s(\varrho-\varrho_e),\theta)\,ds\\
&\quad + (\theta-\theta_e)\int_0^1 p_\theta(\varrho_e,\theta_e + s(\theta-\theta_e))\,ds \equiv p_1\varrho_\sigma + p_2\vartheta_0.
\end{aligned} \tag{3.4}$$

We shall also use the formula

$$
\begin{aligned}
(3.5)\quad p_\sigma &= p(\varrho,\theta) - p(\varrho_{\Omega_t},\theta_{\Omega_t})\\
&= (\varrho - \varrho_{\Omega_t})\int\limits_0^1 p_\varrho(\varrho_{\Omega_t} + s(\varrho-\varrho_{\Omega_t}),\theta)\,ds\\
&\quad + (\theta-\theta_{\Omega_t})\int\limits_0^1 p_\theta(\varrho_{\Omega_t},\theta_{\Omega_t}+s(\theta-\theta_{\Omega_t}))\,ds \equiv p_3\overline{\varrho}_{\Omega_t} + p_4\vartheta,
\end{aligned}
$$

where the function $\varrho_{\Omega_t} = \varrho_{\Omega_t}(t)$ is a solution of the problem

$$
(3.6)\qquad p(\varrho_{\Omega_t},\theta_{\Omega_t}) = p_0,\qquad \varrho_{\Omega_t}|_{t=0} = \varrho_e
$$

and

$$
(3.7)\qquad \overline{\varrho}_{\Omega_t} = \varrho - \varrho_{\Omega_t}.
$$

The functions p_i $(i = 1,2,3,4)$ in (3.4) and (3.5) are positive and $p_1 = p_1(\varrho,\theta)$, $p_2 = p_2(\varrho_e,\theta)$, $p_3 = p_3(\varrho_{\Omega_t},\varrho,\theta)$, $p_4 = p_4(\varrho_{\Omega_t},\theta_{\Omega_t},\theta)$.

Now we point out the following facts concerning the estimates in Lemmas 3.1–3.12 and Theorem 3.13:

• By ε we denote small constants and for simplicity we do not distinguish them.

• By C_1 and C_2 we denote constants which depend on ϱ_*, ϱ^*, θ_*, θ^*, T, $\int_0^T \|v\|^2_{3,\Omega_{t'}}\,dt'$, $\|S\|_{4-1/2}$, on the parameters which guarantee the existence of the inverse transformation to $x = x(\xi,t)$ and also the constants of the imbedding theorems and the Korn inequalities. C_1 is always the coefficient of a linear term, while C_2 is the coefficient of a nonlinear term. For simplicity we do not distinguish different C_1's and C_2's.

• By c we denote absolute constants which may depend on μ, ν, κ, and by $c_0 < 1$ we denote positive constants which may depend on μ, ν, κ, ϱ_*, ϱ^*, θ_*, θ^*. For simplicity we do not distinguish different c's and c_0's.

• We underline that all the estimates are obtained under the assumption that there exists a local-in-time solution of (1.1), so all the quantities ϱ_*, ϱ^*, θ_*, θ^*, T, $\int_0^T \|v\|^2_{3,\Omega_{t'}}\,dt'$, $\|S\|_{4-1/2}$ are estimated by the data functions. Moreover, the existence of the inverse transformation to $x = x(\xi,t)$ is guaranteed by the estimates for the local solution (see [14]).

Lemma 3.1. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of* (3.3). *Then*

$$
\begin{aligned}
(3.8)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v^2 + \frac{p_1}{\varrho}\varrho_\sigma^2 + \overline{\varrho}^2_{\Omega_t} + \frac{p_2\varrho c_v}{p_\theta\theta}\vartheta_0^2\right)dx\\
&\quad + c_0\|v\|^2_{1,\Omega_t} + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_t} + c_0\|\vartheta_{0x}\|^2_{0,\Omega_t}
\end{aligned}
$$

$$\le \varepsilon(\|p_\sigma\|^2_{0,\Omega_t} + \|\vartheta_{0tx}\|^2_{0,\Omega_t}) + C_1(\|v\|^2_{0,\Omega_t} + \|r\|^2_{0,\Omega_t} + \|r\|_{0,\Omega_t} + \|\theta_1\|^2_{1,\Omega_t} + \|\theta_1\|_{1,\Omega_t} + \|f\|^2_{0,\Omega_t}) + C_2(\|\varrho_\sigma\|^4_{2,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^4_{2,\Omega_t} + \|v\|^4_{2,\Omega_t} + \|\vartheta_0\|^4_{2,\Omega_t}),$$

where $\varepsilon > 0$ *is sufficiently small.*

Proof. Multiplying $(3.3)_1$ by v, integrating over Ω_t and using the continuity equation $(3.3)_2$ and (3.4) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \varrho v^2\,dx + \frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_t} - \int_{\Omega_t} p_1\varrho_\sigma \operatorname{div} v\,dx - \int_{\Omega_t} p_2\vartheta_0 \operatorname{div} v\,dx = \int_{\Omega_t} \varrho f v\,dx, \tag{3.9}$$

where $E_{\Omega_t}(v) = \int_{\Omega_t}\sum_{i,j=1}^3(\partial_{x_i}v_j + \partial_{x_j}v_i)^2\,dx$.

By the continuity equation $(3.3)_2$, energy equation $(3.3)_3$ and condition $(3.3)_5$ we have

$$-\int_{\Omega_t} p_1\varrho_\sigma \operatorname{div} v\,dx = \int_{\Omega_t}\frac{p_1}{\varrho}\varrho_\sigma(\varrho_{\sigma t} + v\cdot\nabla\varrho_\sigma)\,dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\frac{p_1\varrho_\sigma^2}{\varrho}\,dx + I_1, \tag{3.10}$$

where

$$|I_1| \le \varepsilon(\|v_x\|^2_{0,\Omega_t} + \|\vartheta_{0x}\|^2_{0,\Omega_t}) + C_1(\|r\|^2_{0,\Omega_t} + \|\theta_1\|^2_{1,\Omega_t}) + C_2(\|\varrho_\sigma\|^4_{1,\Omega_t} + \|v\|^2_{1,\Omega_t}\|\varrho_\sigma\|^2_{2,\Omega_t} + \|v\|^2_{1,\Omega_t}\|\vartheta_0\|^2_{2,\Omega_t} + \|v\|^2_{2,\Omega_t}\|\varrho_\sigma\|^2_{1,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t}\|\varrho_\sigma\|^2_{1,\Omega_t}). \tag{3.11}$$

Next, dividing equation $(3.3)_3$ by $\theta\varrho_\theta$, multiplying the result by $p_2\vartheta_0$ and integrating over Ω_t we get

$$\int_{\Omega_t}\frac{p_2\varrho c_v}{\theta p_\theta}\left(\partial_t\frac{\vartheta_0^2}{2} + v\cdot\nabla\frac{\vartheta_0^2}{2}\right)dx + \int_{\Omega_t} p_2\vartheta_0\operatorname{div} v\,dx - \int_{\Omega_t}\frac{p_2\kappa\Delta\vartheta_0}{\theta p_\theta}\vartheta_0\,dx - \int_{\Omega_t}\frac{p_2\mu}{2\theta p_\theta}\sum_{i,j}(\partial_{x_i}v_j + \partial_{x_j}v_i)^2\vartheta_0\,dx - \int_{\Omega_t}\frac{p_2(\nu-\mu)}{\theta p_\theta}(\operatorname{div} v)^2\vartheta_0\,dx = \int_{\Omega_t}\frac{p_2\varrho r}{\theta p_\theta}\vartheta_0\,dx. \tag{3.12}$$

Hence applying the boundary condition $(3.3)_5$ we have

$$\int\limits_{\Omega_t}\frac{p_2\varrho c_v}{\theta p_\theta}\left(\partial_t\frac{\vartheta_0^2}{2}+v\cdot\nabla\frac{\vartheta_0^2}{2}\right)dx+\int\limits_{\Omega_t}p_2\vartheta_0\operatorname{div}v\,dx+\int\limits_{\Omega_t}\frac{p_2\kappa}{\theta p_\theta}|\vartheta_{0x}|^2\,dx = I_2+\int\limits_{\Omega_t}\frac{p_2\varrho r}{\theta p_\theta}\vartheta_0\,dx+\int\limits_{S_t}\frac{p_2\kappa}{\theta p_\theta}\theta_1\vartheta_0\,dx, \tag{3.13}$$

where

$$|I_2|\le\varepsilon(\|v_x\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{0,\Omega_t}^2) + C_2\|\vartheta_0\|_{1,\Omega_t}^2(\|v\|_{2,\Omega_t}^2+\|\varrho_\sigma\|_{2,\Omega_t}^2+\|\vartheta_0\|_{2,\Omega_t}^2). \tag{3.14}$$

Moreover,

$$\left|\int\limits_{\Omega_t}\frac{p_2\varrho r}{\theta p_\theta}\vartheta_0\,dx\right|\le\left|\int\limits_{\Omega_t}\frac{p_2\varrho r}{\theta p_\theta}\vartheta\,dx\right|+\left|\int\limits_{\Omega_t}\frac{p_2\varrho r}{\theta p_\theta}(\theta_{\Omega_t}-\theta_e)\,dx\right| \le\varepsilon\|\vartheta\|_{0,\Omega_t}^2+C_1(\|r\|_{0,\Omega_t}^2+\|r\|_{0,\Omega_t}) \tag{3.15}$$

and

$$\left|\int\limits_{S_t}\frac{p_2\kappa}{\theta p_\theta}\theta_1\vartheta_0\,ds\right|\le\varepsilon(\|\vartheta\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{0,\Omega_t}^2)+C_1(\|\theta_1\|_{1,\Omega_t}^2+\|\theta_1\|_{1,\Omega_t}). \tag{3.16}$$

Next, using equations $(3.3)_2$, $(3.3)_3$ and condition $(3.3)_5$ yields

$$\int\limits_{\Omega_t}\frac{p_2\varrho c_v}{\theta p_\theta}\left(\partial_t\frac{\vartheta_0^2}{2}+v\cdot\nabla\frac{\vartheta_0^2}{2}\right)dx=\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\frac{p_2\varrho c_v}{\theta p_\theta}\vartheta_0^2\,dx+I_3, \tag{3.17}$$

where

$$|I_3|\le\varepsilon(\|v_x\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{0,\Omega_t}^2)+C_1(\|r\|_{0,\Omega_t}^2+\|\theta_1\|_{1,\Omega_t}^2) + C_2\|\vartheta_0\|_{1,\Omega_t}^2(\|\vartheta_0\|_{2,\Omega_t}^2+\|v\|_{2,\Omega_t}^2+\|\varrho_\sigma\|_{2,\Omega_t}^2). \tag{3.18}$$

Taking into account (3.9)–(3.11), (3.13)–(3.18), using Lemma 5.2 of [21] and the Poincaré inequality

$$\|\vartheta\|_{0,\Omega_t}\le C_1\|\vartheta_{0x}\|_{0,\Omega_t} \tag{3.19}$$

we obtain for sufficiently small ε,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}&\left(\varrho v^2+\frac{p_1\varrho_\sigma^2}{\varrho}+\frac{p_2\varrho c_v}{\theta p_\theta}\vartheta_0^2\right)dx+c_0\|v\|_{1,\Omega_t}^2\\ &+(\nu-\mu)\|\operatorname{div}v\|_{0,\Omega_t}^2+c_0\|\vartheta_{0x}\|_{0,\Omega_t}^2\\ \le{}& C_1(\|v\|_{0,\Omega_t}^2+\|r\|_{0,\Omega_t}^2+\|r\|_{0,\Omega_t}+\|\theta_1\|_{1,\Omega_t}^2+\|\theta\|_{1,\Omega_t}+\|f\|_{0,\Omega_t}^2)\\ &+C_2[\|\varrho_\sigma\|_{1,\Omega_t}^2(\|\varrho_\sigma\|_{1,\Omega_t}^2+\|v\|_{2,\Omega_t}^2)\\ &+\|\varrho_\sigma\|_{2,\Omega_t}^2(\|v\|_{1,\Omega_t}^2+\|\vartheta_0\|_{2,\Omega_t}^2)+\|\vartheta_0\|_{1,\Omega_t}^2(\|\vartheta_0\|_{2,\Omega_t}^2+\|v\|_{2,\Omega_t}^2)].\end{aligned} \tag{3.20}$$

Finally, by $(3.3)_2$ and (3.7) we have

$$\partial_t \overline{\varrho}_{\Omega_t} + v \cdot \nabla \overline{\varrho}_{\Omega_t} + \varrho \operatorname{div} v + \partial_t \varrho_{\Omega_t} = 0, \tag{3.21}$$

where in view of (3.6),

$$\partial_t \varrho_{\Omega_t} = -\frac{p_{\theta_{\Omega_t}}}{p_{\varrho_{\Omega_t}}} \partial_t \theta_{\Omega_t}. \tag{3.22}$$

Using the definition of θ_{Ω_t} we calculate

$$\begin{aligned} \partial_t \theta_{\Omega_t} = {} & \frac{1}{|\Omega_t|} \int_{\Omega_t} \vartheta_{0t}\, dx + \frac{1}{|\Omega_t|} \int_{\Omega_t} \theta \operatorname{div} v\, dx \\ & - \frac{1}{|\Omega_t|^2} \Big(\int_{\Omega_t} \theta\, dx \Big) \Big(\int_{\Omega_t} \operatorname{div} v\, dx \Big). \end{aligned} \tag{3.23}$$

Consider now

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} \overline{\varrho}_{\Omega_t}^2\, dx = {} & - \int_{\Omega_t} \overline{\varrho}_{\Omega_t}^2 \varrho \operatorname{div} v\, dx - \int_{\Omega_t} \overline{\varrho}_{\Omega_t} \partial_t \varrho_{\Omega_t}\, dx \\ & + \frac{1}{2} \int_{\Omega_t} \overline{\varrho}_{\Omega_t}^2 \operatorname{div} v\, dx, \end{aligned} \tag{3.24}$$

where we have used equation (3.21). Since by $(3.3)_3$,

$$\begin{aligned} & \|\vartheta_{0t}\|_{0,\Omega_t}^2 \\ & \le \varepsilon \|\vartheta_{0xt}\|_{0,\Omega_t}^2 + C_1 (\|r\|_{0,\Omega_t}^2 + \|\theta_1\|_{1,\Omega_t}^2 + \|v_x\|_{0,\Omega_t}^2 + \|\vartheta_{0x}\|_{0,\Omega_t}^2) \\ & \quad + C_2 (\|v\|_{1,\Omega_t}^2 \|\vartheta_0\|_{2,\Omega_t}^2 + \|v\|_{1,\Omega_t}^4 + \|\varrho_\sigma\|_{2,\Omega_t}^2 \|\vartheta_0\|_{2,\Omega_t}^2 + \|\vartheta_0\|_{2,\Omega_t}^4), \end{aligned} \tag{3.25}$$

relations (3.22)–(3.24) give the estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} \overline{\varrho}_{\Omega_t}^2\, dx \le {} & \varepsilon (\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \|\vartheta_{0tx}\|_{0,\Omega_t}^2) \\ & + C_1 (\|r\|_{0,\Omega_t}^2 + \|\theta_1\|_{1,\Omega_t}^2 + \|v_x\|_{0,\Omega_t}^2 + \|\vartheta_{0x}\|_{0,\Omega_t}^2) \\ & + C_2 (\|v\|_{1,\Omega_t}^2 \|\vartheta_0\|_{2,\Omega_t}^2 + \|v\|_{2,\Omega_t}^4 + \|\overline{\varrho}_{\Omega_t}\|_{1,\Omega_t}^4 \\ & + \|\varrho_\sigma\|_{2,\Omega_t}^2 \|\vartheta_0\|_{2,\Omega_t}^2 + \|\vartheta_0\|_{2,\Omega_t}^4). \end{aligned} \tag{3.26}$$

By (3.5) and the Poincaré inequality (3.19) we have

$$\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t} \le C_1 (\|\vartheta_{0x}\|_{0,\Omega_t} + \|\varrho_\sigma\|_{0,\Omega_t}). \tag{3.27}$$

The estimates (3.20), (3.26) and (3.27) imply (3.8). ■

LEMMA 3.2. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of* (3.3). *Then*

$$
\begin{aligned}
(3.28)\quad & \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_t+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma t}^2+\frac{\varrho c_v}{\theta}\vartheta_{0t}^2\right)dx\\
&\quad+c_0\|v_t\|_{1,\Omega_t}^2+(\nu-\mu)\|\operatorname{div}v_t\|_{0,\Omega_t}^2+c_0\|\vartheta_{0t}\|_{1,\Omega_t}^2\\
&\le\varepsilon(\|v\|_{1,\Omega_t}^2+\|\vartheta_{0x}\|_{0,\Omega_t}^2)\\
&\quad+C_1(|f|_{1,0,\Omega_t}^2+|r|_{1,0,\Omega_t}^2+|\theta_1|_{2,1,\Omega_t}^2)+C_2X_1^2(1+X_1),
\end{aligned}
$$

where $X_1=|v|_{2,1,\Omega_t}^2+|\varrho_\sigma|_{2,1,\Omega_t}^2+|\vartheta_0|_{2,1,\Omega_t}^2$.

Proof. Differentiating $(3.3)_1$ with respect to t, multiplying by v_t and integrating over Ω_t we obtain

$$
\begin{aligned}
(3.29)\quad & \int_{\Omega_t}\left(\varrho\partial_t\frac{v_t^2}{2}+\varrho v\cdot\nabla\frac{v_t^2}{2}\right)dx+\frac{\mu}{2}E_{\Omega_t}(v_t)+(\nu-\mu)\|\operatorname{div}v_t\|_{0,\Omega_t}^2\\
&-\int_{\Omega_t}p_\varrho\varrho_{\sigma t}\operatorname{div}v_t\,dx-\int_{\Omega_t}p_\theta\vartheta_{0t}\operatorname{div}v_t\,dx\\
&+\int_{\Omega_t}(\varrho_tv_t^2+\varrho_tv_t\cdot(v\nabla v)+\varrho v_t\cdot(v_t\nabla v))\,dx+\int_{S_t}(\mathbb{T}(v,p_\sigma)n_t)\cdot v_t\,ds\\
&=\int_{\Omega_t}\partial_t(\varrho f)\cdot v_t\,dx,
\end{aligned}
$$

where we have used the boundary condition $(3.3)_4$.

The continuity equation $(3.3)_2$ yields

$$
\begin{aligned}
(3.30)\quad & -\int_{\Omega_t}p_\varrho\varrho_{\sigma t}\operatorname{div}v_t\,dx\\
&=\int_{\Omega_t}\left(\frac{p_\varrho}{\varrho}\varrho_{\sigma t}\varrho_{\sigma tt}+\frac{p_\varrho}{\varrho}\varrho_{\sigma t}^2\operatorname{div}v+\frac{p_\varrho}{\varrho}\varrho_{\sigma t}v_t\nabla\varrho_\sigma+\frac{p_\varrho}{\varrho}\varrho_{\sigma t}v\nabla\varrho_{\sigma t}\right)dx\\
&=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\frac{p_\varrho}{\varrho}\varrho_{\sigma t}^2\,dx+I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
(3.31)\quad |I_4|&\le\varepsilon(\|v_t\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2)\\
&\quad+C_2[\|\varrho_{\sigma t}\|_{1,\Omega_t}^2(\|\varrho_\sigma\|_{2,\Omega_t}^2+\|\vartheta_{0t}\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{1,\Omega_t}^2+\|\vartheta_0\|_{2,\Omega_t}^2)].
\end{aligned}
$$

Using (3.30), (3.31) and Lemma 5.3 of [21] we obtain from (3.29) the inequality

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_t^2+\frac{p_\varrho}{\varrho}\varrho_{\sigma t}^2\right)dx \tag{3.32}$$
$$+c_0\|v_t\|_{1,\Omega_t}^2+(\nu-\mu)\|\operatorname{div} v_t\|_{0,\Omega_t}^2-\int\limits_{\Omega_t}p_\theta\vartheta_{0t}\operatorname{div} v_t\,dx$$
$$\le\varepsilon\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+C_1|f|_{1,0,\Omega_t}^2+C_2X_1^2(1+X_1),$$

Dividing now $(3.3)_3$ by θ, differentiating with respect to t, multiplying by ϑ_{0t}, integrating over Ω_t and next applying the Hölder and Young inequalities and the Sobolev lemma gives

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\frac{\varrho c_v}{\theta}\vartheta_{0t}^2\,dx+\int\limits_{\Omega_t}p_\theta\vartheta_{0t}\operatorname{div} v_t\,dx+\frac{\kappa}{\theta^*}\int\limits_{\Omega_t}|\vartheta_{0tx}|^2\,dx \tag{3.33}$$
$$\le\varepsilon(\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|v_t\|_{0,\Omega_t}^2+\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|\vartheta_{0tx}\|_{0,\Omega_t}^2)$$
$$+C_1(\|r\|_{0,\Omega_t}^2+\|r_t\|_{0,\Omega_t}^2+|\theta_1|_{2,1,\Omega_t}^2)+C_2X_1^2.$$

From the continuity equation $(3.3)_2$ it follows that

$$\|\varrho_{\sigma t}\|_{0,\Omega_t}^2\le C_1\|v\|_{1,\Omega_t}^2+C_2\|v\|_{1,\Omega_t}^2\|\varrho_\sigma\|_{2,\Omega_t}^2. \tag{3.34}$$

Finally, adding inequalities (3.32)–(3.33) and using (3.25) and (3.34) we obtain (3.28). ∎

Lemmas 3.1 and 3.2 imply

LEMMA 3.3. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of* (3.3). *Then*

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\varrho(v^2+v_t^2)\,dx+\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\frac{1}{\varrho}(p_1\varrho_\sigma^2+p_\varrho\varrho_{\sigma t}^2)\,dx \tag{3.35}$$
$$+\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\overline{\varrho}_{\Omega_t}^2\,dx+\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\frac{\varrho c_v}{\theta}\left(\frac{p_2}{p_\theta}\vartheta_0^2+\vartheta_{0t}^2\right)dx+c_0(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2)$$
$$+(\nu-\mu)(\|\operatorname{div} v\|_{0,\Omega_t}^2+\|\operatorname{div} v_t\|_{0,\Omega_t}^2)$$
$$+c_0(\|\vartheta_{0x}\|_{0,\Omega_t}^2+\|\vartheta_{0t}\|_{1,\Omega_t}^2)$$
$$\le\varepsilon\|p_\sigma\|_{0,\Omega_t}^2+C_1(\|v\|_{0,\Omega_t}^2+|r|_{1,0,\Omega_t}^2+\|r\|_{0,\Omega_t}+|\theta_1|_{2,1,\Omega_t}^2+\|\theta_1\|_{0,\Omega_t}+|f|_{1,0,\Omega_t}^2)$$
$$+C_2[\|\overline{\varrho}_{\Omega_t}\|_{2,\Omega_t}^4+X_1^2(1+X_1)],$$

where X_1 is defined in Lemma 3.2.

In order to obtain an inequality for derivatives with respect to x we rewrite problem (3.3) in the Lagrangian coordinates and next we introduce a partition of unity in the fixed domain Ω. Thus we have

$$
\begin{aligned}
&\eta u_{it} - \nabla_{u_j} T_u^{ij}(u, p_\sigma) = \eta g_i, \quad i = 1,2,3,\\
&\eta_{\sigma t} + \eta \nabla_u \cdot u = 0,\\
&\eta c_v(\eta, \Gamma)\gamma_{0t} - \kappa \nabla_u^2 \gamma_0\\
&\quad = \eta k - \Gamma p_\Gamma(\eta, \Gamma) \nabla_u \cdot u\\
&\qquad + \frac{\mu}{2} \sum_{i,j=1}^{3} (\xi_{kx_i} \partial_{\xi_k} u_j + \xi_{kx_j} \partial_{\xi_k} u_i)^2 + (\nu - \mu)(\nabla_u \cdot u)^2,\\
&\mathbb{T}_u(u, p_\sigma)\overline{n}(\xi, t) = 0,\\
&\overline{n} \cdot \nabla_u \Gamma = \Gamma_1,
\end{aligned}
\tag{3.36}
$$

where

$$
\begin{aligned}
&\eta(\xi,t) = \varrho(x(\xi,t),t), \quad u(\xi,t) = v(x(\xi,t),t), \quad g(\xi,t) = f(x(\xi,t),t),\\
&\Gamma(\xi,t) = \theta(x(\xi,t),t), \quad \gamma_0(\xi,t) = \vartheta_0(x(\xi,t),t), \quad \Gamma_1 = \theta_1(x(\xi,t),t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{T}_u(u, p_\sigma) = \{T_u^{ij}(u, p_\sigma)\}\\
&\qquad = \{-p_\sigma \delta_{ij} + \mu(\nabla_{u_j} u_i + \nabla_{u_i} u_j) + (\nu - \mu)\delta_{ij} \nabla_u \cdot u\},\\
&\nabla_u = \xi_x \partial_\xi \equiv (\xi_{ix_k} \partial_{\xi_i})_{k=1,2,3}, \quad \nabla_{u_i} = \xi_{kx_i} \partial_{\xi_k},\\
&\operatorname{div}_u \mathbb{T}_u(u, p_\sigma) = \nabla_u \mathbb{T}_u(u, p_\sigma).
\end{aligned}
\tag{3.37}
$$

By (3.4), (3.5) we have respectively

$$
p_\sigma = p_1 \eta_\sigma + p_2 \gamma_0
\tag{3.38}
$$

and

$$
p_\sigma = p_3 \overline{\eta}_{\Omega_t} + p_4 \gamma,
\tag{3.39}
$$

where

$$
\begin{aligned}
&\eta_\sigma = \eta - \varrho_e, \quad \gamma_0 = \Gamma - \theta_e, \quad \overline{\eta}_{\Omega_t} = \eta - \varrho_{\Omega_t}, \quad \gamma = \Gamma - \theta_{\Omega_t},\\
&p_1 = p_1(\eta, \Gamma), \quad p_2 = p_2(\eta, \Gamma), \quad p_3 = p_3(\varrho_{\Omega_t}, \eta, \Gamma),\\
&p_4 = p_4(\varrho_{\Omega_t}, \theta_{\Omega_t}, \Gamma), \quad p_i > 0 \quad (i = 1,2,3).
\end{aligned}
$$

Let us introduce a partition of unity $(\{\widetilde{\Omega}_i\}, \{\zeta_i\})$, $\Omega = \bigcup_i \widetilde{\Omega}_i$. Let $\widetilde{\Omega}$ be one of the $\widetilde{\Omega}_i$'s and $\zeta(\xi) = \zeta_i(\xi)$ be the corresponding function. If $\widetilde{\Omega}$ is an interior subdomain then let $\widetilde{\omega}$ be a set such that $\widetilde{\omega} \subset \widetilde{\Omega}$ and $\zeta(\xi) = 1$ for $\xi \in \widetilde{\omega}$. Otherwise we assume that $\overline{\widetilde{\Omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \subset \overline{\widetilde{\Omega}}$. Take any $\beta \in \overline{\widetilde{\omega}} \cap S \subset \overline{\widetilde{\Omega}} \cap S = \overline{\widetilde{S}}$ and introduce local coordinates $\{y\}$ associated with $\{\xi\}$ by

$$
y_k = \alpha_{kl}(\xi_l - \beta_l), \quad \alpha_{3k} = n_k(\beta), \quad k = 1,2,3,
\tag{3.40}
$$

where α_{kl} is a constant orthogonal matrix such that $\widetilde{S}$ is determined by the equation $y_3 = F(y_1, y_2)$, $F \in W_2^{4-1/2}$ and

$$\widetilde{\Omega} = \{y : |y_i| < d,\ i = 1,2,\ F(y') < y_3 < F(y') + d,\ y' = (y_1, y_2)\}.$$

Next, we introduce functions u', η', Γ', γ_0', γ', Γ_1' by

$$\begin{aligned} u_i'(y) &= \alpha_{ij} u_j(\xi)|_{\xi=\xi(y)}, & \eta'(y) &= \eta(\xi)|_{\xi=\xi(y)}, \\ \Gamma'(y) &= \Gamma(\xi)|_{\xi=\xi(y)}, & \gamma_0'(y) &= \gamma_0(\xi)|_{\xi=\xi(y)}, \\ \gamma'(y) &= \gamma(\xi)|_{\xi=\xi(y)}, & \Gamma_1'(y) &= \Gamma_1(\xi)|_{\xi=\xi(y)}, \end{aligned} \tag{3.41}$$

where $\xi = \xi(y)$ is the inverse transformation to (3.40). Further, we introduce new variables by

$$z_i = y_i \quad (i = 1,2), \quad z_3 = y_3 - \widetilde{F}(y), \quad y \in \widetilde{\Omega}, \tag{3.42}$$

which will be denoted by $z = \Phi(y)$ (where $\widetilde{F}$ is an extension of F with $\widetilde{F} \in W_2^4$).

Let $\widehat{\Omega} = \Phi(\widetilde{\Omega}) = \{z : |z_i| < d,\ i = 1,2,\ 0 < z_3 < d\}$ and $\widehat{S} = \Phi(\widetilde{S})$. Define

$$\begin{aligned} \widehat{u}(z) &= u'(y)|_{y=\Phi^{-1}(z)}, & \widehat{\eta}(z) &= \eta'(y)|_{y=\Phi^{-1}(z)}, \\ \widehat{\Gamma}(z) &= \Gamma'(y)|_{y=\Phi^{-1}(z)}, & \widehat{\gamma}_0(z) &= \gamma_0'(y)|_{y=\Phi^{-1}(z)}, \\ \widehat{\gamma}(z) &= \gamma'(y)|_{y=\Phi^{-1}(z)}, & \widehat{\Gamma}_1(z) &= \Gamma_1'(y)|_{y=\Phi^{-1}(z)}. \end{aligned} \tag{3.43}$$

Set $\widehat{\nabla}_k = \xi_{lx_k}(\xi) z_{i\xi_l} \nabla_{z_i}|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi) = \Phi(\psi(\xi))$ and $y = \psi(\xi)$ is described by (3.40). We also introduce the following notation:

$$\begin{aligned} \widetilde{u}(\xi) &= u(\xi)\zeta(\xi), & \widetilde{\eta}(\xi) &= \eta(\xi)\zeta(\xi), \\ \widetilde{\Gamma}(\xi) &= \Gamma(\xi)\zeta(\xi), & \widetilde{\gamma}_0(\xi) &= \gamma_0(\xi)\zeta(\xi), \\ \widetilde{\gamma}(\xi) &= \gamma(\xi)\zeta(\xi), & \widetilde{\Gamma}_1(\xi) &= \Gamma_1(\xi)\zeta(\xi) \end{aligned} \tag{3.44}$$

for $\xi \in \widetilde{\Omega}$, $\widetilde{\Omega} \cap S = \emptyset$, and

$$\begin{aligned} \widetilde{u}(z) &= \widehat{u}(z)\widehat{\zeta}(z), & \widetilde{\eta}(z) &= \widehat{\eta}(z)\widehat{\zeta}(z), \\ \widetilde{\Gamma}(z) &= \widehat{\Gamma}(z)\widehat{\zeta}(z), & \widetilde{\gamma}_0(z) &= \widehat{\gamma}_0(z)\widehat{\zeta}(z), \\ \widetilde{\gamma}(z) &= \widehat{\gamma}(z)\widehat{\zeta}(z), & \widetilde{\Gamma}_1(z) &= \widehat{\Gamma}_1(z)\widehat{\zeta}(z) \end{aligned} \tag{3.45}$$

for $z \in \widehat{\Omega} = \Phi(\widetilde{\Omega})$, $\overline{\widetilde{\Omega}} \cap S \neq \emptyset$, where $\widehat{\zeta}(z) = \zeta(\xi)|_{\xi=\chi^{-1}(z)}$.

Using the above notation and (3.2) we can rewrite problem (3.36) in the following form in an interior subdomain:

$$
\begin{aligned}
&\eta\widetilde{u}_{it}-\nabla_{u_j}T_u^{ij}(\widetilde{u},\widetilde{p}_\sigma)=\eta\widetilde{g}_i-\nabla_{u_j}B_u^{ij}(u,\zeta)-T_u^{ij}(u,p_\sigma)\nabla_{u_j}\zeta\\
&\qquad\qquad\equiv\eta\widetilde{g}_i+k_1,\qquad i=1,2,3,\\
&\widetilde{\eta}_{\sigma t}+\eta\nabla_u\cdot\widetilde{u}=\eta u\cdot\nabla_u\zeta\equiv k_2,\\
&\eta c_v(\eta,\Gamma)\widetilde{\gamma}_t-\kappa\nabla_u^2\widetilde{\gamma}+\Gamma p_\Gamma(\eta,\Gamma)\nabla_u\cdot\widetilde{u}\\
&\quad=\eta\widetilde{k}+\bigg[\frac{1}{2}\mu\sum_{i,j=1}^3(\xi_{kx_i}\partial_{\xi_k}u_j+\xi_{kx_j}\partial_{\xi_k}u_i)^2+(\nu-\mu)(\nabla_u\cdot u)^2\bigg]\zeta\\
&\qquad+\Gamma p_\Gamma(\eta,\Gamma)u\cdot\nabla_u\zeta-\kappa(\nabla_u^2\zeta\gamma+2\nabla_u\zeta\cdot\nabla_u\gamma)\\
&\qquad-\eta c_v(\eta,\Gamma)\zeta\partial_t\theta_{\Omega_t}\equiv\eta\widetilde{k}+k_3,
\end{aligned}
\tag{3.46}
$$

where $p_\sigma=p_\sigma\zeta$ and

$$\mathbb{B}_u(u,\zeta)=\{B_u^{ij}(u,\zeta)\}=\{\mu(u_i\nabla_{u_j}\zeta+u_j\nabla_{u_i}\zeta)+(\nu-\mu)\partial_{ij}u\cdot\nabla_u\zeta\}.$$

In boundary subdomains we have

$$
\begin{aligned}
&\widehat{\eta}\widetilde{u}_{it}-\widehat{\nabla}_j\widehat{T}^{ij}(\widetilde{u},\widetilde{p}_\sigma)=\widehat{\eta}\widetilde{g}_i-\widehat{\nabla}_j\widehat{B}^{ij}(\widehat{u},\widehat{\zeta})-\widehat{T}^{ij}(\widehat{u},p_\sigma)\widehat{\nabla}_j\widehat{\zeta}\\
&\qquad\qquad\equiv\widehat{\eta}\widetilde{g}_i+k_4^i,\\
&\widetilde{\eta}_{\sigma t}+\widehat{\eta}\widehat{\nabla}\cdot\widetilde{u}=\widehat{\eta}\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta}\equiv k_5,\\
&\widehat{\eta}c_v(\widehat{\eta},\widehat{\Gamma})\widetilde{\gamma}_t-\kappa\widehat{\nabla}^2\widetilde{\gamma}+\widehat{\Gamma}p_\Gamma(\widehat{\eta},\widehat{\Gamma})\widehat{\nabla}\cdot\widetilde{u}\\
&\quad=\widehat{\eta}\widetilde{k}+\bigg[\frac{1}{2}\mu\sum_{i,j=1}^3(\widehat{\nabla}_i\widehat{u}_j+\widehat{\nabla}_j\widehat{u}_i)^2+(\nu-\mu)(\widehat{\nabla}\cdot\widehat{u})^2\bigg]\widehat{\zeta}\\
&\qquad+\widehat{\Gamma}p_\Gamma(\widehat{\eta},\widehat{\Gamma})\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta}-\kappa(\widehat{\nabla}^2\widehat{\zeta}\cdot\widehat{\gamma}+2\widehat{\nabla}\widehat{\zeta}\cdot\widehat{\nabla}\widehat{\gamma})\\
&\qquad-\widehat{\eta}c_v(\widehat{\eta},\widehat{\Gamma})\partial_t\theta_{\Omega_t}\widehat{\zeta}\equiv\widehat{\eta}\widetilde{k}+k_6,\\
&\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma)\widehat{n}=k_7,\\
&\widehat{n}\cdot\widehat{\nabla}\widetilde{\gamma}=\widetilde{\Gamma}_1+k_8,
\end{aligned}
\tag{3.47}
$$

where $k_7^i=\widehat{B}^{ij}(\widehat{u},\widehat{\zeta})\widehat{n}_j$, $k_8=\widehat{n}\cdot\widehat{\nabla}\widehat{\zeta}\widehat{\gamma}$, $\widehat{\nabla}=(\widehat{\nabla}_j)_{j=1,2,3}$, and $\widehat{T}$ and $\widehat{B}$ indicate that the operator ∇_u is replaced by $\widehat{\nabla}$.

In the considerations below we denote z_1, z_2 by τ and z_3 by n.

Lemma 3.4. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of problem* (3.3). *Then*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\bigg(\varrho v_x^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma x}^2+\frac{\varrho c_v}{\theta}\vartheta_{0x}^2\bigg)\,dx\\
&\quad+c_0(\|v_x\|_{1,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\varrho_{\sigma x}\|_{0,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{1,\Omega_t}^2)
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
&\le \varepsilon(\|v_{xt}\|^2_{0,\Omega_t} + \|\vartheta_{0xt}\|^2_{0,\Omega_t})\\
&\quad + C_1(|v|^2_{1,0,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t} + \|\vartheta\|^2_{0,\Omega_t} + \|\vartheta_{0x}\|^2_{0,\Omega_t}\\
&\quad + \|\vartheta_{0t}\|^2_{0,\Omega_t} + \|f\|^2_{1,\Omega_t} + \|r\|^2_{1,\Omega_t} + \|\theta_1\|^2_{2,\Omega_t})\\
&\quad + C_2\Big(X_2 + \int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt'\Big)Y_2,
\end{aligned}
$$

where

$$
v_x^2 = \sum_{i,j=1}^3 v_{ix_j}^2, \quad \varrho_{\sigma x}^2 = \sum_{i=1}^3 \varrho_{\sigma x_i}^2, \quad \vartheta_{0x}^2 = \sum_{i=1}^3 \vartheta_{0x_i}^2,
$$

$$
X_2 = |v|^2_{2,1,\Omega_t} + |\varrho_\sigma|^2_{2,1,\Omega_t} + |\vartheta_0|^2_{2,1,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t},
$$

$$
Y_2 = X_2 + \|v\|^2_{3,\Omega_t} + \|\vartheta_{0x}\|^2_{2,\Omega_t} + \|\vartheta\|^2_{0,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}.
$$

Proof. First we obtain the estimate in interior subdomains. Differentiating $(3.46)_1$ with respect to ξ, multiplying the result by $\widetilde{u}_\xi A$ (where A is the Jacobian of the transformation $x = x(\xi)$) and integrating over $\widetilde{\Omega}$ we get

$$
\begin{aligned}
(3.49)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\tilde\Omega} \eta\widetilde{u}_\xi^2 A\,d\xi + \frac{1}{2}\mu\int_{\tilde\Omega} (\nabla_{u_i}\widetilde{u}_{j\xi} + \nabla_{u_j}\widetilde{u}_{i\xi})^2 A\,d\xi\\
&\quad + (\nu-\mu)\|\nabla_u\cdot\widetilde{u}_\xi\|^2_{0,\tilde\Omega} - \int_{\tilde\Omega} \widetilde{p}_{\sigma\xi}\cdot(\nabla_u\cdot\widetilde{u}_\xi)A\,d\xi\\
&\le \varepsilon(\|u_{\xi\xi}\|^2_{0,\tilde\Omega} + \|\eta_{\sigma\xi}\|^2_{0,\tilde\Omega} + \|\gamma_{0\xi}\|^2_{0,\tilde\Omega})\\
&\quad + C_1(\|u\|^2_{1,\tilde\Omega} + \|\gamma_{0\xi}\|^2_{0,\tilde\Omega} + \|\overline{\eta}_{\Omega_t}\|^2_{0,\tilde\Omega} + \|\gamma\|^2_{0,\tilde\Omega} + \|\widetilde{g}\|^2_{0,\tilde\Omega})\\
&\quad + C_2\Big(X_2(\widetilde{\Omega}) + \int_0^t \|u\|^2_{3,\tilde\Omega}\,dt'\Big)Y_2(\widetilde{\Omega}),
\end{aligned}
$$

where $\|h\|_{0,\tilde\Omega} = (\int_{\tilde\Omega}|h|^2A\,d\xi)^{1/2}$, $\widetilde{u}_\xi^2 = \sum_{i=1}^3 \widetilde{u}_{i\xi}^2$ and

$$
X_2(\widetilde{\Omega}) = |u|^2_{2,1,\tilde\Omega} + |\eta_\sigma|^2_{2,1,\tilde\Omega} + |\gamma_0|^2_{2,1,\tilde\Omega} + \|\overline{\eta}_{\Omega_t}\|^2_{0,\tilde\Omega},
$$

$$
Y_2(\widetilde{\Omega}) = X_2(\widetilde{\Omega}) + \|u\|^2_{3,\tilde\Omega} + \|\gamma\|^2_{3,\tilde\Omega} + \|\overline{\eta}_{\Omega_t}\|^2_{0,\tilde\Omega}.
$$

Next, we have

$$
\begin{aligned}
(3.50)\qquad -\int_{\tilde\Omega} \widetilde{p}_{\sigma\xi}(\nabla_u\cdot\widetilde{u}_\xi)A\,d\xi &= -\int_{\tilde\Omega} p_{\sigma\xi}\widetilde{\gamma}_\xi(\nabla_u\cdot\widetilde{u}_\xi)A\,d\xi\\
&\quad - \int_{\tilde\Omega} p_{\sigma\xi}\widetilde{\overline{\eta}}_{\Omega_t\xi}(\nabla_u\cdot\widetilde{u}_\xi)A\,d\xi + I_5,
\end{aligned}
$$

where

$$|I_5| \le \varepsilon \|\widetilde{u}_{\xi\xi}\|_{0,\tilde{\Omega}}^2 + C_1(\|\gamma\|_{0,\tilde{\Omega}}^2 + \|\overline{\eta}_{\Omega_t}\|_{0,\tilde{\Omega}}^2). \tag{3.51}$$

In order to consider $-\int_{\tilde{\Omega}} p_{\sigma\xi}\widetilde{\overline{\eta}}_{\Omega_t\xi}(\nabla_u\cdot\widetilde{u}_\xi)A\,d\xi$ we rewrite equation (3.21) in the Lagrangian coordinates to obtain

$$\partial_t\widetilde{\overline{\eta}}_{\Omega_t} + \eta\nabla_u\cdot\widetilde{u} = \eta u\cdot\nabla_u\zeta - \zeta\partial_t\varrho_{\Omega_t}. \tag{3.52}$$

Differentiating (3.52) with respect to ξ yields

$$\begin{aligned}\nabla_u\cdot\widetilde{u}_\xi = &-\frac{\partial_t\widetilde{\overline{\eta}}_{\Omega_t\xi}}{\eta} + \frac{\eta_{\sigma\xi}\nabla_u\cdot\widetilde{u}}{\eta} - \xi'_x\int_0^t u_{\xi\xi}\,dt'\,\widetilde{u}_\xi + \frac{\eta_{\sigma\xi}}{\eta}u\cdot\nabla_u\zeta\\ &+ u_\xi\nabla_u\zeta + u\xi'_x\int_0^t u_{\xi\xi}\,dt'\,\zeta_\xi + u\nabla_u\zeta_\xi - \frac{\zeta_\xi}{\eta}\partial_t\varrho_{\Omega_t}.\end{aligned}$$

Hence

$$\begin{aligned}-\int_{\tilde{\Omega}} p_{\sigma\xi}\widetilde{\overline{\eta}}_{\Omega_t\xi}(\nabla_u\cdot\widetilde{u}_\xi)A\,d\xi = \frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}}\frac{p_{\sigma\xi}}{\eta}\widetilde{\overline{\eta}}^2_{\Omega_t\xi}A\,d\xi\\ + \int_{\tilde{\Omega}} p_{\sigma\xi}\widetilde{\overline{\eta}}_{\Omega_t\xi}\frac{\zeta_\xi}{\eta}\partial_t\varrho_{\Omega_t}\,d\xi + I_6,\end{aligned} \tag{3.53}$$

where

$$\begin{aligned}|I_6| \le{}& \varepsilon\|\eta_{\sigma\xi}\|_{0,\tilde{\Omega}}^2 + C_1(\|\overline{\eta}_{\Omega_t}\|_{0,\tilde{\Omega}}^2 + \|u\|_{1,\tilde{\Omega}}^2)\\ &+ C_2\Big[\|\eta_\sigma\|_{2,\tilde{\Omega}}^2\|u\|_{2,\tilde{\Omega}}^2 + \|u\|_{2,\tilde{\Omega}}^2\Big\|\int_0^t u\,dt'\Big\|_{3,\tilde{\Omega}}^2\\ &+ \|\overline{\eta}_{\Omega_t}\|_{2,\tilde{\Omega}}^2(|\eta_\sigma|_{2,1,\tilde{\Omega}}^2 + |\gamma_0|_{2,1,\tilde{\Omega}}^2 + \|u\|_{2,\tilde{\Omega}}^2)\Big]\end{aligned} \tag{3.54}$$

and by (3.22) and (3.23),

$$\begin{aligned}\int_{\tilde{\Omega}}\Big|p_{\sigma\eta}\widetilde{\overline{\eta}}_{\Omega_t\xi}\frac{\zeta_\xi}{\eta}\partial_t\varrho_{\Omega_t}\Big|\,d\xi\\ \le \varepsilon\|\eta_{\sigma\xi}\|_{0,\tilde{\Omega}}^2 + C_1(\|\overline{\eta}_{\Omega_t}\|_{0,\tilde{\Omega}}^2 + \|\vartheta_{0t}\|_{0,\tilde{\Omega}}^2 + \|v\|_{1,\tilde{\Omega}}^2).\end{aligned} \tag{3.55}$$

Next, dividing $(3.46)_3$ by Γ, differentiating the result with respect to ξ, multiplying by $\widetilde{\gamma}_\xi A$ and integrating over $\widetilde{\Omega}$ yields

$$\frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}}\frac{\eta c_v}{\Gamma}\widetilde{\gamma}_\xi^2A\,d\xi + \int_{\tilde{\Omega}} p_{\sigma\Gamma}\nabla_u\cdot\widetilde{u}_\xi\widetilde{\gamma}_\xi A\,d\xi + \int_{\tilde{\Omega}}\frac{\kappa}{\Gamma}|\nabla_u\widetilde{\gamma}_\xi|^2A\,d\xi \tag{3.56}$$

$$
\begin{aligned}
&\le \varepsilon(\|\eta_{\sigma\xi}\|^2_{0,\tilde\Omega} + \|\widetilde\gamma_{\xi\xi}\|^2_{0,\tilde\Omega} + \|\widetilde u_{\xi\xi}\|^2_{0,\tilde\Omega})\\
&\quad + C_1(\|\gamma_{0\xi}\|^2_{0,\tilde\Omega} + \|\gamma\|^2_{0,\tilde\Omega} + \|\vartheta_{0t}\|^2_{0,\Omega_t} + \|u\|^2_{1,\tilde\Omega} + \|v\|^2_{1,\Omega_t} + \|\widetilde k\|^2_{0,\tilde\Omega})\\
&\quad + C_2\Big[\Big(X_2(\widetilde\Omega) + \int_0^t \|u\|^2_{3,\tilde\Omega}\,dt'\Big)Y_2(\widetilde\Omega) + \|\gamma\|^2_{2,\tilde\Omega}(\|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t})\Big].
\end{aligned}
$$

Consider now the Stokes problem

$$
\begin{aligned}
&\mu\nabla_u^2\widetilde u - \nu\nabla_u\nabla_u\cdot\widetilde u + p_{\sigma\eta}\nabla_u\widetilde{\overline\eta}_{\Omega_t}\\
&\qquad = \eta\widetilde g - \eta\widetilde u_t - \widetilde p_{\sigma\Gamma}\nabla_u\gamma_0 - p_{\sigma\eta}\nabla_u\zeta\overline\eta_{\Omega_t} + k_1,\\
&\nabla_u\cdot\widetilde u = \nabla_u\cdot\widetilde u,\\
&\widetilde u|_{\partial\tilde\Omega} = 0.
\end{aligned}
\tag{3.57}
$$

For $\widetilde u$ and $\widetilde{\overline\eta}_{\Omega_t}$ satisfying (3.57) we have

$$
\begin{aligned}
&\|\widetilde u\|^2_{2,\tilde\Omega} + \|\widetilde{\overline\eta}_{\Omega_t}\|^2_{1,\tilde\Omega}\\
&\quad\le C_1(|u|^2_{1,0,\tilde\Omega} + \|\gamma\|^2_{0,\tilde\Omega} + \|\gamma_{0\xi}\|^2_{0,\tilde\Omega} + \|\overline\eta_{\Omega_t}\|^2_{0,\tilde\Omega} + \|\widetilde g\|^2_{0,\tilde\Omega})\\
&\qquad + C_2\Big[(\|u\|^2_{2,\tilde\Omega} + \|\gamma_0\|^2_{2,\tilde\Omega} + \|\eta_\sigma\|^2_{1,\tilde\Omega})\Big\|\int_0^t u\,dt'\Big\|^2_{3,\tilde\Omega}\Big] + c\|\nabla_u\cdot\widetilde u\|^2_{1,\tilde\Omega}.
\end{aligned}
\tag{3.58}
$$

Summing up inequalities (3.49)–(3.51), (3.53)–(3.55), (3.56), (3.58) and using Lemma 5.1 of [21] in the case $G = \overline\Omega$, $v = \widetilde u_\xi$ we obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\tilde\Omega}\Big(\eta\widetilde u_\xi^2 + \frac{p_{\sigma\eta}}{\eta}\widetilde{\overline\eta}^2_{\Omega_t\xi} + \frac{\eta c_v}{\Gamma}\widetilde\gamma_\xi^2\Big)A\,d\xi\\
&\quad + \frac12\mu\|\widetilde u_\xi\|^2_{1,\tilde\Omega} + \frac{\kappa}{\theta^*}\|\widetilde\gamma_{\xi\xi}\|^2_{0,\tilde\Omega} + \|\widetilde{\overline\eta}_{\Omega_t}\|^2_{1,\tilde\Omega}\\
&\le \varepsilon(\|\widetilde u_{\xi\xi}\|^2_{0,\tilde\Omega} + \|\eta_{\sigma\xi}\|^2_{0,\tilde\Omega} + \|\widetilde\gamma_{\xi\xi}\|^2_{0,\tilde\Omega})\\
&\quad + C_1(|u|^2_{1,0,\tilde\Omega} + \|v\|^2_{1,\Omega_t} + \|\gamma_{0\xi}\|^2_{0,\tilde\Omega} + \|\gamma\|^2_{0,\tilde\Omega}\\
&\quad + \|\vartheta_{0t}\|^2_{0,\Omega_t} + \|\overline\eta_{\Omega_t}\|^2_{0,\tilde\Omega} + \|\widetilde g\|^2_{0,\tilde\Omega} + \|\widetilde k\|^2_{0,\tilde\Omega})\\
&\quad + C_2\Big[\Big(X_2(\widetilde\Omega) + \int_0^t \|u\|^2_{3,\tilde\Omega}\,dt'\Big)Y_2(\widetilde\Omega) + \|\gamma\|^2_{2,\tilde\Omega}(\|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t})\Big].
\end{aligned}
\tag{3.59}
$$

Now we consider subdomains near the boundary. Differentiate $(3.47)_1$ with respect to τ, multiply the result by $\widetilde u_\tau J$ and integrate over $\widehat\Omega$ (J is the Jacobian of $x=x(z)$). Next, divide $(3.47)_3$ by $\widehat\Gamma$, differentiate the result with respect to τ, multiply by $\widetilde\gamma_\tau J$ and integrate over $\widehat\Omega$. Hence using Lemma 5.1 of [21] and equation $(3.47)_2$ we get

$$
\begin{aligned}
(3.60)\quad & \frac{1}{2}\frac{d}{dt}\int\limits_{\hat\Omega}\Big(\widehat\eta\widetilde u_\tau^2+\frac{p_{\sigma\hat\eta}}{\widehat\eta}\widetilde{\overline\eta}_{\Omega_t\tau}^2+\frac{\widehat\eta c_v}{\widehat\Gamma}\widetilde\gamma_\tau^2\Big)J\,dz+c_0(\|\widetilde u_\tau\|_{1,\hat\Omega}^2+\|\widetilde\gamma_{\tau z}\|_{0,\hat\Omega}^2)\\
&-\int\limits_{\hat S}(\widehat{\mathbb T}(\widetilde u,\widetilde p_\sigma)\widehat n)_{,\tau}\widetilde u_\tau J\,dz'-\kappa\int\limits_{\hat S}(\widehat n\cdot\widehat\Gamma^{-1}\widehat\nabla\widetilde\gamma)_{,\tau}\widetilde\gamma_\tau J\,dz'\\
&\le\varepsilon(\|\widetilde u_{zz}\|_{0,\hat\Omega}^2+\|\widehat\eta_{\sigma z}\|_{0,\hat\Omega}^2+\|\widehat\gamma_{0zz}\|_{0,\hat\Omega}^2)\\
&\quad+C_1(|\widehat u|_{1,0,\hat\Omega}^2+\|v\|_{1,\Omega_t}^2+\|\widehat\gamma_{0\tau}\|_{0,\hat\Omega}^2+\|\widehat\gamma\|_{0,\hat\Omega}^2\\
&\quad+\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|\widehat{\overline\eta}_{\Omega_t}\|_{0,\hat\Omega}^2+\|\widetilde g\|_{1,\hat\Omega}^2+\|\widetilde k\|_{1,\hat\Omega}^2)\\
&\quad+C_2\Big[\Big(X_2(\widehat\Omega)+\int\limits_0^t\|\widehat u\|_{3,\hat\Omega}^2\,dt'\Big)Y_2(\widehat\Omega)+\|\widehat\gamma\|_{2,\hat\Omega}^2(\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^2)\Big],
\end{aligned}
$$

where $X_2(\widehat\Omega)$ and $Y_2(\widehat\Omega)$ are defined analogously to $X_2(\widetilde\Omega)$ and $Y_2(\widetilde\Omega)$.

Using the boundary conditions $(3.47)_4$ and $(3.47)_5$ we have

$$
\begin{aligned}
(3.61)\quad & -\int\limits_{\hat S}(\widehat{\mathbb T}(\widetilde u,\widetilde p_\sigma)\widehat n)_{,\tau}\widetilde u_\tau J\,dz'\\
&\qquad\le\varepsilon\|\widehat u_{zz}\|_{0,\hat\Omega}^2+C_1\|\widehat u\|_{1,\hat\Omega}^2+C_2\|\widehat u\|_{2,\hat\Omega}^2\Big\|\int\limits_0^t\widehat u\,dt'\Big\|_{3,\hat\Omega}^2
\end{aligned}
$$

and

$$
\begin{aligned}
(3.62)\quad & -\kappa\int\limits_{\hat S}(\widehat n\cdot\widehat\Gamma^{-1}\widehat\nabla\widehat\gamma)_{,\tau}\widetilde\gamma_\tau J\,dz'\\
&\le\varepsilon\|\widehat\gamma_{0zz}\|_{0,\hat\Omega}^2+C_1(\|\widehat\gamma\|_{0,\hat\Omega}^2+\|\widehat\gamma_{0z}\|_{0,\hat\Omega}^2+\|\widetilde\Gamma_1\|_{2,\hat\Omega}^2)\\
&\quad+C_2\|\widehat\gamma\|_{2,\hat\Omega}^2\Big(\|\widehat\gamma_0\|_{2,\hat\Omega}^2+\|\widehat\gamma\|_{2,\hat\Omega}^2+\|\widehat\eta_\sigma\|_{2,\hat\Omega}^2+\Big\|\int\limits_0^t\widehat u\,dt'\Big\|_{3,\hat\Omega}^2\Big).
\end{aligned}
$$

To obtain (3.61) and (3.62) we have applied the interpolation inequality (see Lemma 2.1).

Writing equation (3.52) in the coordinates z we obtain

$$
(3.63)\qquad \partial_t\widetilde{\overline\eta}_{\Omega_t}+\widehat\eta\widehat\nabla\cdot\widetilde u=\widehat\eta\widehat u\cdot\widehat\nabla\widehat\zeta-\widehat\zeta\partial_t\varrho_{\Omega_t}.
$$

Applying now the operator $(\mu+\nu)\nabla_{z_i}$ to (3.63), dividing the result by $\widehat\eta$, adding to $(3.47)_1$ and multiplying both sides of the result by $p_{\sigma\hat\eta}$ gives

$$
\begin{aligned}
(3.64)\quad & \frac{(\mu+\nu)p_{\sigma\hat\eta}}{\widehat\eta}\nabla_{z_i}\partial_t\widetilde{\overline\eta}_{\Omega_t}+p_{\sigma\hat\eta}^2\widehat\nabla_i\widetilde{\overline\eta}_{\Omega_t}\\
&=p_3p_{\sigma\hat\eta}\widehat{\overline\eta}_{\Omega_t}\widehat\nabla_i\widehat\zeta-p_{\sigma\hat\eta}p_{\sigma\hat\Gamma}\widehat\zeta\widehat\nabla_i\widehat\gamma_0-p_{\sigma\hat\eta}^2\widehat\nabla_i\widehat\zeta\,\widehat{\overline\eta}_{\Omega_t}-p_3p_{\sigma\hat\eta}\widehat\nabla_i\widehat\zeta\widehat\gamma-p_{\sigma\hat\eta}\widehat\eta\widetilde u_{it}
\end{aligned}
$$

$$+ p_{\sigma\hat\eta}\widehat{\eta}\widetilde{g}_i - \frac{(\mu+\nu)}{\widehat{\eta}}p_{\sigma\hat\eta}\nabla_{z_i}\widehat{\eta}\widehat{\nabla}\cdot\widetilde{u} + \frac{(\mu+\nu)}{\widehat{\eta}}p_{\sigma\hat\eta}\nabla_{z_i}(\widehat{\eta}\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta})$$
$$+ \mu p_{\sigma\hat\eta}(\widehat{\nabla}^2\widetilde{u}_i - \widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u}) + (\mu+\nu)p_{\sigma\hat\eta}(\widehat{\nabla}_i - \nabla_{z_i})\widehat{\nabla}\cdot\widetilde{u}$$
$$- \frac{(\mu+\nu)}{\widehat{\eta}}p_{\sigma\hat\eta}\nabla_{z_i}\widehat{\zeta}\partial_t\varrho_{\Omega_t} + p_{\sigma\hat\eta}k_4^i.$$

Multiplying the normal component of (3.64) by $\widetilde{\eta}_{\sigma n}J$ and integrating over $\widehat{\Omega}$ implies

$$\frac{1}{2}\frac{d}{dt}\int_{\hat\Omega}\frac{p_{\sigma\hat\eta}}{\widehat{\eta}}\widetilde{\widetilde{\eta}}^2_{\Omega_t n}J\,dz + c_0\|\widetilde{\widetilde{\eta}}_{\Omega_t n}\|^2_{0,\hat\Omega} \tag{3.65}$$
$$\le (\varepsilon + cd)(\|\widetilde{\widetilde{\eta}}_{\Omega_t n}\|^2_{0,\hat\Omega} + \|\widetilde{u}_{nn}\|^2_{0,\hat\Omega})$$
$$+ C_1(\|\widehat{\widetilde{\eta}}_{\Omega_t}\|^2_{0,\hat\Omega} + \|\widehat{\gamma}_{0z}\|^2_{0,\hat\Omega} + \|\widehat{\gamma}\|^2_{0,\hat\Omega} + \|\vartheta_{0t}\|^2_{0,\Omega_t}$$
$$+ |\widehat{u}|^2_{1,0,\hat\Omega} + \|v\|^2_{1,\Omega_t} + \|\widetilde{u}_{z\tau}\|^2_{0,\hat\Omega} + \|\widetilde{g}\|^2_{0,\hat\Omega})$$
$$+ C_2\Big[\|\widehat{\eta}_\sigma\|^2_{2,\hat\Omega}\|\widehat{u}\|^2_{2,\hat\Omega} + \|\widehat{\widetilde{\eta}}_{\Omega_t}\|^2_{2,\Omega_t}$$
$$\times\Big(\|\widehat{\eta}_{\sigma t}\|^2_{1,\hat\Omega} + \|\widehat{u}\|^2_{2,\hat\Omega} + \|\widehat{\gamma}_{0t}\|^2_{1,\hat\Omega} + \Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{3,\hat\Omega}\Big)$$
$$+ \|\widehat{u}\|^2_{2,\hat\Omega}\Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{3,\hat\Omega} + \|\widehat{u}\|^2_{3,\hat\Omega}\Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{2,\hat\Omega}\Big].$$

Now, write $(3.47)_1$ in the form

$$\widehat{\eta}\widetilde{u}_{it} - \mu\Delta\widetilde{u}_i - \nabla_{z_i}\nabla\cdot\widetilde{u} = \widehat{\nabla}_i\widetilde{p}_\sigma + \widehat{\eta}\widetilde{g}_i + k_4^i - k_9^i, \tag{3.66}$$

where

$$k_9^i = (\mu\Delta\widetilde{u}_i + \nu\nabla_{z_i}\nabla\cdot\widetilde{u}) - (\mu\widehat{\nabla}^2\widetilde{u}_i + \nu\widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u}).$$

Multiplying the third component of (3.66) by $\widetilde{u}_{3nn}J$ and integrating over $\widehat{\Omega}$ yields

$$\frac{1}{2}\frac{d}{dt}\int_{\hat\Omega}\widehat{\eta}\widetilde{u}^2_{3n}J\,dz + c_0\|\widetilde{u}_{3nn}\|^2_{0,\hat\Omega} \tag{3.67}$$
$$\le (\varepsilon + cd)\|\widetilde{u}_{nn}\|^2_{0,\hat\Omega} + \varepsilon\|\widetilde{u}_{3nt}\|^2_{0,\hat\Omega}$$
$$+ C_1(\|\widetilde{u}_{z\tau}\|^2_{0,\hat\Omega} + \|\widehat{u}\|^2_{1,\hat\Omega} + \|\widetilde{u}_t\|^2_{0,\hat\Omega} + \|\widetilde{\widetilde{\eta}}_{\Omega_t n}\|^2_{0,\hat\Omega}$$
$$+ \|\widehat{\widetilde{\eta}}_{\Omega_t}\|^2_{0,\hat\Omega} + \|\widehat{\gamma}_{0z}\|^2_{0,\hat\Omega} + \|\widehat{\gamma}\|^2_{0,\hat\Omega} + \|\widetilde{g}\|^2_{0,\hat\Omega})$$
$$+ C_2\Big(\|\widehat{\eta}_{\sigma t}\|^2_{1,\hat\Omega}\|\widetilde{u}\|^2_{2,\hat\Omega} + \|\widetilde{u}\|^4_{2,\hat\Omega} + \|\widetilde{u}\|^2_{2,\hat\Omega}\Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{3,\hat\Omega}\Big).$$

To estimate $\widetilde{u}_{inn}$ $(i = 1,2)$ and $\widetilde{\widetilde{\eta}}_{\Omega_t\tau}$ we rewrite (3.66) as

$$
\begin{aligned}
-\mu\Delta\widetilde{u}_i + \nabla_{z_i}(p_{\sigma\hat{\eta}}\widetilde{\widetilde{\eta}}_{\Omega_t}) &= \widehat{\eta}\widetilde{g}_i - \widehat{\eta}\widetilde{u}_{it} + k_4^i - k_9^i + \nabla_{z_i}(p_{\sigma\hat{\eta}}\widetilde{\widetilde{\eta}}_{\Omega_t}) \\
&\quad - \widehat{\nabla}_i\widetilde{p}_\sigma + \nu\nabla_{z_i}\operatorname{div}\widetilde{u} \\
&\equiv \widetilde{f}_i + \nu\nabla_{z_i}\operatorname{div}\widetilde{u}
\end{aligned}
\tag{3.68}
$$

and the boundary condition $(3.47)_4$ as

$$
\frac{\partial\widetilde{u}_i}{\partial z_3} = -\frac{\partial\widetilde{u}_3}{\partial z_i} + \left(\frac{\partial\widetilde{u}_i}{\partial z_3} + \frac{\partial\widetilde{u}_3}{\partial z_i} - \mu^{-1}\widehat{\tau}_i\widehat{T}\widehat{n}\right) + \mu^{-1}k_7\cdot\widehat{\tau}_i \equiv \widetilde{h}_i, \quad i = 1,2, \quad z_3 = 0,
\tag{3.69}
$$

where we have also used the fact that $\widehat{\tau}_i\cdot\widehat{n} = 0$, $i = 1,2$. When considering problem (3.68)–(3.69) in $\widehat{\Omega}$ we have to add the boundary conditions

$$
\begin{aligned}
&\widetilde{u}_i|_{|z'|=d} = 0, \quad \widetilde{u}_i|_{z_3=d} = 0, \quad i = 1,2,\\
&\widetilde{\widetilde{\eta}}_{\Omega_t}|_{|z'|=d} = 0, \quad \widetilde{\widetilde{\eta}}_{\Omega_t}|_{z_3=d} = 0.
\end{aligned}
\tag{3.70}
$$

Multiplying (3.68) by $\widetilde{u}_i$, summing over $i = 1,2$, integrating over $\widehat{\Omega}$ and using the boundary conditions (3.69) and (3.70) yields

$$
\|\widetilde{u}'_z\|^2_{0,\hat{\Omega}} \le \varepsilon\|\widetilde{\widetilde{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + c(\|\widetilde{f}'\|^2_{0,\hat{\Omega}} + \|\widetilde{h}'\|^2_{0,\hat{\Omega}}) + C_1\|\operatorname{div}\widetilde{u}\|^2_{0,\hat{\Omega}},
\tag{3.71}
$$

where the prime indicates that only two components $(i = 1,2)$ are taken into account.

In order to estimate $\|\widetilde{\widetilde{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}$ consider the problem

$$
\begin{aligned}
&\operatorname{div} w = p_{\sigma\hat{\eta}}\widetilde{\widetilde{\eta}}_{\Omega_t}, \quad w_3|_{z_3=0} = \chi(z')\int_{\hat{\Omega}} p_{\sigma\hat{\eta}}\widetilde{\widetilde{\eta}}_{\Omega_t}\,dz,\\
&w|_{\partial\hat{\Omega}\setminus\hat{S}} = 0, \quad w_i|_{z_3=0} = 0, \quad i = 1,2,
\end{aligned}
\tag{3.72}
$$

where $\chi(z')$ is a smooth function such that $\int_{\hat{S}}\chi(z')\,dz' = 1$, $\chi(z') \ge 0$, $\chi|_{|z'|=d} = 0$, $1 \le 4d^2|\chi|_{\infty,\hat{S}}$. Moreover, we assume that χ vanishes only in a neighbourhood of $\widehat{S}$, $\min_{|z'|\le d/2}\chi(z') > 0$ and $\chi(z') \le c/d^2$. By [21] (Lemma 4.4) there exists a solution of (3.72) such that $w \in W^1_2(\widehat{\Omega})$ and

$$
\|w\|_{1,\hat{\Omega}} \le C_1\|\widetilde{\widetilde{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}.
\tag{3.73}
$$

Now, multiply (3.68) by w and integrate over $\widehat{\Omega}$ to get

$$
-\mu\int_{\hat{\Omega}}\Delta\widetilde{u}\cdot w\,dz + \int_{\hat{\Omega}}\nabla(p_{\sigma\hat{\eta}}\widetilde{\widetilde{\eta}}_{\Omega_t})\cdot w\,dz = \int_{\hat{\Omega}}\widetilde{f}\cdot w\,dz + \nu\int_{\hat{\Omega}}\nabla\operatorname{div}\widetilde{u}\cdot w\,dz.
\tag{3.74}
$$

Applying the same argument as in [21] (see the proof of Lemma 4.4) and using (3.73) we get

$$\|\widetilde{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} \le \varepsilon\|\widetilde{\overline{\eta}}_{\Omega_t z}\|^2_{0,\hat{\Omega}} + C_1(\|\widetilde{f}\|^2_{0,\hat{\Omega}} + \|\widetilde{u}_z\|^2_{0,\hat{\Omega}} + \|\operatorname{div}\widetilde{u}\|^2_{1,\hat{\Omega}}). \tag{3.75}$$

Now, instead of problem (3.68)–(3.70) we consider the problem

$$\begin{aligned} &-\mu\Delta\widetilde{u}_{i\tau} + \nabla_{z_i}(p_{\sigma\hat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t})_{,\tau} = \widetilde{f}_{i\tau} + \nu\nabla_{z_i}\operatorname{div}\widetilde{u}_\tau, \quad i=1,2,3,\\ &\partial_{z_3}\widetilde{u}_{iz} = \widetilde{h}_{i\tau}, \quad i=1,2. \end{aligned} \tag{3.76}$$

Multiplying $(3.76)_1$ by $\widetilde{u}_{i\tau}$, summing over $i=1,2$ and integrating over $\widehat{\Omega}$ yields

$$\begin{aligned} \|\widetilde{u}'_{z\tau}\|^2_{0,\hat{\Omega}} &\le \varepsilon\|\widetilde{\overline{\eta}}_{\Omega_t\tau}\|^2_{0,\hat{\Omega}} + c(\|\widetilde{f}'\|^2_{0,\hat{\Omega}} + \|\widetilde{h}'_{z'}\|^2_{0,\hat{\Omega}}) + C_1\|\operatorname{div}\widetilde{u}_\tau\|^2_{0,\hat{\Omega}}\\ &\quad + C_2\|\widetilde{\overline{\eta}}_{\Omega_t}\|^2_{1,\hat{\Omega}}(\|\widetilde{\overline{\eta}}_{\Omega_t}\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}}). \end{aligned} \tag{3.77}$$

Next, consider the problem

$$\operatorname{div} w_1 = (p_{\sigma\hat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t})_{,\tau}, \quad w_1|_{\partial\hat{\Omega}} = 0. \tag{3.78}$$

Since $\int_{\hat{\Omega}}(p_{\sigma\hat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t})_{,\tau}\,dz = 0$ there exists a solution $w_1 \in W^1_2(\widehat{\Omega})$ of problem (3.78) such that

$$\|w_1\|_{1,\hat{\Omega}} \le C_1[\|\widetilde{\overline{\eta}}_{\Omega_t\tau}\|_{0,\hat{\Omega}} + (\|\widehat{\eta}_{\sigma\tau}\|_{1,\hat{\Omega}} + \|\widehat{\gamma}_{0\tau}\|_{1,\hat{\Omega}})\|\widetilde{\overline{\eta}}_{\Omega_t}\|_{1,\hat{\Omega}}]. \tag{3.79}$$

Multiplying $(3.76)_1$ by w_1 and integrating over $\widehat{\Omega}$ gives

$$\begin{aligned} \|\widetilde{\overline{\eta}}_{\Omega_t\tau}\|^2_{0,\hat{\Omega}} &\le C_1(\|\widetilde{f}\|^2_{0,\hat{\Omega}} + \|\widetilde{u}_{z\tau}\|^2_{0,\hat{\Omega}} + \|\operatorname{div}\widetilde{u}\|^2_{1,\hat{\Omega}})\\ &\quad + C_2(\|\widehat{\eta}_\sigma\|^2_{2,\hat{\Omega}}\|\widetilde{\overline{\eta}}_{\Omega_t}\|^2_{1,\hat{\Omega}} + \|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}}\|\widetilde{\overline{\eta}}_{\Omega_t}\|^2_{1,\hat{\Omega}}). \end{aligned} \tag{3.80}$$

Now we estimate $\|\widetilde{u}'_{nn}\|^2_{0,\hat{\Omega}}$. From (3.68) we obtain

$$\begin{aligned} \|\widetilde{u}'_{nn}\|^2_{0,\hat{\Omega}} &\le C_1(\|\widetilde{\overline{\eta}}_{\Omega_t\tau}\|^2_{0,\hat{\Omega}} + \|\overline{\eta}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0\tau}\|^2_{0,\hat{\Omega}}\\ &\quad + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\widetilde{u}_{z\tau}\|^2_{0,\hat{\Omega}} + \|\widetilde{f}\|^2_{0,\hat{\Omega}} + \|\operatorname{div}\widetilde{u}\|^2_{1,\hat{\Omega}}). \end{aligned} \tag{3.81}$$

From the form of $\widetilde{f}$ and $\widetilde{h}'$ we have

$$\begin{aligned} \|\widetilde{f}\|^2_{0,\hat{\Omega}} &\le (\varepsilon + cd)(\|\widetilde{u}_{zz}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t z}\|^2_{0,\hat{\Omega}})\\ &\quad + C_1(|\widehat{u}|^2_{1,0,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\widehat{g}\|^2_{0,\hat{\Omega}})\\ &\quad + C_2\Big[\|\widehat{u}\|^2_{2,\hat{\Omega}}\Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{3,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t}\|^2_{1,\hat{\Omega}}(\|\widehat{\overline{\eta}}_\sigma\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}})\Big] \end{aligned} \tag{3.82}$$

and

$$\|\widetilde{h}'\|^2_{1,\hat{\Omega}} \le C_1\Big(\|\widetilde{u}_{3z\tau}\|^2_{0,\hat{\Omega}} + \|\widetilde{u}\|^2_{2,\hat{\Omega}}\Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{3,\hat{\Omega}} + \|\widehat{u}\|^2_{1,\hat{\Omega}}\Big). \tag{3.83}$$

In order to estimate $\widetilde{\gamma}_{nn}$ we rewrite $(3.47)_3$ as

$$\widehat{\eta}c_v\widetilde{\gamma}_t-\kappa\varDelta\widetilde{\gamma}=\kappa\widehat{\nabla}^2\widetilde{\gamma}-\kappa\varDelta\widetilde{\gamma}-\widehat{\varGamma}p_{\widehat{\varGamma}}\widehat{\nabla}\cdot\widetilde{u}+\widehat{\eta}\widetilde{k}+k_6. \tag{3.84}$$

Dividing (3.84) by $\widehat{\varGamma}$, multiplying the result by $\widetilde{\gamma}_{nn}J$ and integrating over $\widehat{\varOmega}$ we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\varOmega}}\frac{\widehat{\eta}c_v}{\widehat{\varGamma}}\widetilde{\gamma}_n^2J\,dz+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{nn}\|_{0,\hat{\varOmega}}^2 \qquad (3.85)\\
&\le(\varepsilon+cd)\|\widetilde{\gamma}_{nn}\|_{0,\hat{\varOmega}}^2+\varepsilon(\|\widehat{\widetilde{\eta}}_{\varOmega_tn}\|_{0,\hat{\varOmega}}^2+\|\widetilde{\gamma}_{nt}\|_{0,\hat{\varOmega}}^2)\\
&\quad+C_1(\|\widetilde{u}\|_{1,\hat{\varOmega}}^2+\|\widehat{\gamma}_{0t}\|_{0,\hat{\varOmega}}^2+\|\widehat{\gamma}_{0z}\|_{0,\hat{\varOmega}}^2+\|\widehat{\gamma}\|_{0,\hat{\varOmega}}^2+\|\widetilde{\gamma}_{z\tau}\|_{0,\hat{\varOmega}}^2+\|\vartheta_{0t}\|_{0,\varOmega_t}^2+\|v\|_{1,\varOmega_t}^2)\\
&\quad+C_2\Big[\|\widehat{\gamma}_{0t}\|_{1,\hat{\varOmega}}^2\|\widehat{\gamma}\|_{2,\hat{\varOmega}}^2+\|\widehat{\gamma}\|_{2,\hat{\varOmega}}^2\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|_{3,\hat{\varOmega}}^2+\|\widetilde{u}\|_{2,\hat{\varOmega}}^4\\
&\quad+\|\widehat{u}\|_{2,\hat{\varOmega}}^2\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|_{3,\hat{\varOmega}}^2+\|\widehat{\gamma}\|_{3,\hat{\varOmega}}^2\|\widehat{u}\|_{2,\hat{\varOmega}}^2+\|\widehat{\eta}_{\sigma t}\|_{1,\hat{\varOmega}}^2\|\widehat{\gamma}\|_{2,\hat{\varOmega}}^2\\
&\quad+\|\widehat{\gamma}\|_{3,\hat{\varOmega}}^2\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|_{2,\hat{\varOmega}}^2+\|\widehat{\gamma}\|_{2,\hat{\varOmega}}^2(\|v\|_{1,\varOmega_t}^2+\|\vartheta_{0t}\|_{0,\varOmega_t}^2)\Big].
\end{aligned}$$

Finally, we have

$$\frac{d}{dt}\int\limits_{\hat{\varOmega}}\widehat{\eta}\widetilde{u}_n^2J\,dz\le\varepsilon\|\widetilde{u}_{nt}\|_{0,\hat{\varOmega}}^2+c\|u\|_{1,\hat{\varOmega}}^2. \tag{3.86}$$

Now, taking into account inequalities (3.60)–(3.62), (3.65), (3.67), (3.71), (3.72), (3.77), (3.80)–(3.83), (3.85) and (3.86) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\varOmega}}\Big(\widehat{\eta}\widetilde{u}_z^2+\frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\widetilde{\eta}}_{\varOmega_tz}^2+\frac{\widehat{\eta}c_v}{\widehat{\varGamma}}\widetilde{\gamma}_z^2\Big)J\,dz \qquad (3.87)\\
&\qquad+c_0(\|\widetilde{u}_z\|_{1,\hat{\varOmega}}^2+\|\widetilde{\widetilde{\eta}}_{\varOmega_t}\|_{0,\hat{\varOmega}}^2+\|\widetilde{\eta}_{\sigma z}\|_{0,\hat{\varOmega}}^2+\|\widetilde{\gamma}_{zz}\|_{0,\hat{\varOmega}}^2)\\
&\le\varepsilon(\|\widetilde{u}_{zt}\|_{0,\hat{\varOmega}}^2+\|\widetilde{\gamma}_{0zt}\|_{0,\hat{\varOmega}}^2)\\
&\qquad+C_1(|\widehat{u}|_{1,0,\hat{\varOmega}}^2+\|\widehat{\widetilde{\eta}}_{\varOmega_t}\|_{0,\hat{\varOmega}}^2+\|\widehat{\gamma}\|_{0,\hat{\varOmega}}^2+\|\widehat{\gamma}_{0z}\|_{0,\hat{\varOmega}}^2\\
&\qquad+\|\widehat{\gamma}_{0t}\|_{0,\hat{\varOmega}}^2+\|\widetilde{g}\|_{0,\hat{\varOmega}}^2+\|\widetilde{k}\|_{0,\hat{\varOmega}}^2+\|\widetilde{\varGamma}_1\|_{2,\hat{\varOmega}}^2)\\
&\qquad+C_2\Big[\Big(X_2(\widehat{\varOmega})+\int\limits_0^t\|\widehat{u}\|_{3,\hat{\varOmega}}^2\,dt'\Big)\cdot Y_2(\widehat{\varOmega})\\
&\qquad+\|\widehat{\gamma}\|_{2,\hat{\varOmega}}^2(\|\vartheta_{0t}\|_{0,\varOmega_t}^2+\|v\|_{1,\varOmega_t}^2+\|\widehat{\gamma}\|_{2,\hat{\varOmega}}^2)\Big].
\end{aligned}$$

Going back to the variables ξ in (3.87), summing over all neighbourhoods of the partition of unity (where we use (3.59) for interior subdomains), assuming that ε and d are sufficiently small and using (3.34), we obtain (3.48). ∎

LEMMA 3.5. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of problem* (3.3). *Then*

$$\begin{aligned}(3.88)\quad & \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\Big(\varrho v_{xx}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma xx}^2+\frac{\varrho c_v}{\theta}\vartheta_{0xx}^2\Big)\,dx \\ &\quad + c_0(\|v_{xx}\|_{1,\Omega_t}^2+\|\varrho_{\sigma x}\|_{1,\Omega_t}^2+\|\vartheta_{0xxx}\|_{0,\Omega_t}^2) \\ &\le \varepsilon(\|v_{xxt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxt}\|_{0,\Omega_t}^2) \\ &\quad + C_1(|v|_{2,1,\Omega_t}^2+\|\varrho_{\sigma x}\|_{0,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{1,\Omega_t}^2 \\ &\quad + \|\vartheta_{0t}\|_{1,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2+\|f\|_{1,\Omega_t}^2+\|r\|_{1,\Omega_t}^2+\|\theta_1\|_{3,\Omega_t}^2) \\ &\quad + C_2\Big(X_3+\int_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1+X_3)Y_3,\end{aligned}$$

where

$$v_{xx}^2=\sum_{i,j,k=1}^3 v_{ix_jx_k}^2,\quad \varrho_{\sigma xx}^2=\sum_{j,k=1}^3\varrho_{\sigma x_jx_k}^2,\quad \vartheta_{0xx}^2=\sum_{j,k=1}^3\vartheta_{0x_jx_k}^2,$$

$$X_3=\|v\|_{3,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+|\varrho_\sigma|_{2,1,\Omega_t}^2+|\vartheta_0|_{2,1,\Omega_t}^2+\|\vartheta_0\|_{3,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2,$$

$$\begin{aligned}Y_3&=\|v\|_{4,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+\|\varrho_{\sigma x}\|_{2,\Omega_t}^2+|\varrho_\sigma|_{2,1,\Omega_t}^2+\|\vartheta_{0x}\|_{3,\Omega_t}^2+\|\vartheta_{0t}\|_{1,\Omega_t}^2 \\ &\quad +\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2.\end{aligned}$$

P r o o f. The proof is similar to that of Lemma 3.4. We use the introduced partition of unity. Differentiating $(3.46)_1$ and $(3.46)_3$ (divided by Γ) twice with respect to ξ, multiplying the results by $u_{\xi\xi}A$ and $\widetilde{\gamma}_{\xi\xi}A$, respectively, next integrating over $\widetilde{\Omega}$ and summing up we get

$$\begin{aligned}(3.89)\quad & \frac{1}{2}\frac{d}{dt}\int_{\tilde\Omega}\Big(\eta\widetilde{u}_{\xi\xi}^2+\frac{p_{\sigma\eta}}{\eta}\widetilde{\widetilde{\eta}}_{\Omega_t\xi\xi}+\frac{\eta c_v}{\Gamma}\widetilde{\gamma}_{\xi\xi}^2\Big)A\,d\xi \\ &\qquad + \frac{1}{2}\mu\|\widetilde{u}_{\xi\xi}\|_{1,\tilde\Omega}^2+\|\widetilde{\widetilde{\eta}}_{\Omega_t}\|_{2,\tilde\Omega}^2+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{\xi\xi\xi}\|_{0,\tilde\Omega}^2 \\ &\quad \le \varepsilon(\|\widetilde{u}_{\xi\xi\xi}\|_{0,\tilde\Omega}^2+\|\widetilde{\gamma}_{\xi\xi\xi}\|_{0,\tilde\Omega}^2+\|\widetilde{\eta}_{\sigma\xi\xi}\|_{0,\tilde\Omega}^2) \\ &\qquad + C_1(|u|_{2,1,\tilde\Omega}^2+\|\gamma_{0\xi}\|_{1,\tilde\Omega}^2+\|\gamma\|_{0,\tilde\Omega}^2+\|\eta_{\sigma\xi}\|_{0,\tilde\Omega}^2\end{aligned}$$

$$
\begin{aligned}
&+\|\overline{\eta}_{\Omega_t}\|^2_{0,\tilde\Omega}+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widetilde g\|^2_{1,\tilde\Omega}+\|\widetilde k\|^2_{1,\tilde\Omega})\\
&+C_2\Big[\Big(X_3(\widetilde\Omega)+\int_0^t\|u\|^2_{3,\tilde\Omega}\,dt'\Big)(1+X_3(\widetilde\Omega))Y_3(\widetilde\Omega)\\
&+\|\gamma\|^2_{3,\tilde\Omega}(\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t})\Big],
\end{aligned}
$$

where we have used equation (3.52), Lemma 5.1 of [21] and the estimate for the solution u, $\overline{\eta}_{\Omega_t}$ of the Stokes problem (3.57), i.e. the estimate of $\|\widetilde u\|_{3,\tilde\Omega}$ and $\|\widetilde{\overline{\eta}}_{\Omega_t}\|_{2,\tilde\Omega}$, respectively and

$$
\begin{aligned}
X_3(\widetilde\Omega)&=\|u\|^2_{3,\tilde\Omega}+\|u_t\|^2_{1,\tilde\Omega}\\
&\quad+|\eta_\sigma|^2_{2,1,\tilde\Omega}+\|\gamma_0\|^2_{3,\tilde\Omega}+|\gamma_0|^2_{2,1,\tilde\Omega}+\|\overline{\eta}_{\Omega_t}\|^2_{0,\tilde\Omega},\\
Y_3(\widetilde\Omega)&=\|u\|^2_{4,\tilde\Omega}+\|u_t\|^2_{1,\tilde\Omega}+\|\eta_{\sigma x}\|^2_{2,\tilde\Omega}+|\eta_\sigma|^2_{2,1,\tilde\Omega}\\
&\quad+\|\gamma_{0x}\|^2_{3,\tilde\Omega}+\|\gamma_{0t}\|^2_{1,\tilde\Omega}+\|\overline{\eta}_{\Omega_t}\|^2_{0,\Omega_t}+\|\gamma\|^2_{0,\Omega_t}.
\end{aligned}
\tag{3.90}
$$

In the same way as (3.60) we obtain the following inequality:

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\hat\Omega}\Big(\widehat\eta\widetilde u^2_{\tau\tau}+\frac{p_{\sigma\hat\eta}}{\widehat\eta}\widetilde{\overline{\eta}}^2_{\Omega_t\tau\tau}+\frac{\widehat\eta c_v}{\widehat\Gamma}\widetilde\gamma^2_{\tau\tau}\Big)J\,dz+\frac12\mu\|\widetilde u_{\tau\tau}\|^2_{1,\hat\Omega}+\frac{\kappa}{\theta^*}\|\widetilde\gamma_{\tau\tau z}\|^2_{0,\hat\Omega}\\
&\le\varepsilon(\|\widehat u_{zzz}\|^2_{0,\hat\Omega}+\|\widehat\gamma_{0zzz}\|^2_{0,\hat\Omega}+\|\widehat\eta_{\sigma zz}\|^2_{0,\hat\Omega})\\
&\quad+C_1(|\widehat u|^2_{2,1,\hat\Omega}+\|\widehat\gamma_{0z}\|^2_{1,\hat\Omega}+\|\widehat\gamma\|^2_{0,\hat\Omega}+\|\eta_{\sigma z}\|^2_{0,\hat\Omega}+\|\overline{\eta}_{\Omega_t}\|^2_{0,\hat\Omega}\\
&\quad+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widetilde g\|^2_{1,\hat\Omega}+\|\widetilde k\|^2_{1,\hat\Omega}+\|\widetilde\Gamma_1\|^2_{3,\hat\Omega})\\
&\quad+C_2\Big[\Big(X_3(\widehat\Omega)+\int_0^t\|\widehat u\|^2_{3,\hat\Omega}\,dt'\Big)(1+X_3(\widehat\Omega))Y_3(\widehat\Omega)\\
&\quad+\|\widehat\gamma\|^2_{3,\hat\Omega}(\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widehat\gamma\|^2_{3,\hat\Omega})\Big],
\end{aligned}
\tag{3.91}
$$

where we have used the boundary conditions $(3.47)_4$ and $(3.47)_5$, and where $X_3(\widehat\Omega)$ and $Y_3(\widehat\Omega)$ are defined by (3.90) with $\widetilde\Omega$, u, $\overline{\eta}_{\Omega_t}$, γ replaced by $\widehat\Omega$, $\widehat u$, $\widehat{\overline{\eta}}_{\Omega_t}$, $\widehat\gamma$, respectively.

Differentiating the third component of (3.64) with respect to τ, multiplying the result by $\widetilde{\overline{\eta}}_{\Omega_tn\tau}J$ and integrating over $\widehat\Omega$ yields

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\hat\Omega}\frac{p_{\sigma\hat\eta}}{\widehat\eta}\widetilde{\overline{\eta}}^2_{\Omega_tn\tau}J\,dz+c_0\|\widetilde{\overline{\eta}}_{\Omega_tn\tau}\|^2_{0,\hat\Omega}\\
&\le(\varepsilon+cd)(\|\widetilde{\overline{\eta}}_{\Omega_tzz}\|^2_{0,\hat\Omega}+\|\widehat u_{zzz}\|^2_{0,\hat\Omega})+C_1(|\widehat u|^2_{2,1,\hat\Omega}+\|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat\Omega}+\|\widehat\eta_{\sigma z}\|^2_{0,\hat\Omega}
\end{aligned}
\tag{3.92}
$$

$$+ \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{1,\hat{\Omega}}^2 + \|\widetilde{u}_{z\tau}\|_{0,\hat{\Omega}}^2 + \|\vartheta_{0t}\|_{0,\Omega_t}^2 + \|v\|_{1,\Omega_t}^2 + \|\widetilde{g}\|_{1,\hat{\Omega}}^2)$$

$$+ C_2\Big(X_3(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|_{3,\hat{\Omega}}^2 dt'\Big)(1 + X_3(\widehat{\Omega}))Y_3(\widehat{\Omega}).$$

Next, differentiating the third component of (3.66) with respect to τ, multiplying the result by $\widetilde{u}_{3nn\tau}J$ and integrating over $\widehat{\Omega}$ gives

$$(3.93)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \widehat{\eta}|\widetilde{u}_{3n\tau}|^2 J\,dz + c_0\|\widetilde{u}_{3nn\tau}\|_{0,\hat{\Omega}}^2$$
$$\le (\varepsilon + cd)(\|\widetilde{u}_{zzz}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t zz}\|_{0,\hat{\Omega}}^2)$$
$$+ C_1(\|\widetilde{u}_{z\tau\tau}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t n\tau}\|_{0,\hat{\Omega}}^2 + \|\widehat{u}\|_{2,\hat{\Omega}}^2 + \|\widetilde{u}_t\|_{1,\hat{\Omega}}^2$$
$$+ \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{1,\hat{\Omega}}^2 + \|\widetilde{g}\|_{1,\hat{\Omega}}^2)$$
$$+ C_2\Big(X_3(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|_{3,\hat{\Omega}}^2\,dt'\Big)(1 + X_3(\widehat{\Omega}))Y_3(\widehat{\Omega}).$$

Similarly, we obtain the estimate

$$(3.94)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{n\tau}^2 J\,dz + \frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{nn\tau}\|_{0,\hat{\Omega}}^2$$
$$\le (\varepsilon + cd)(\|\widetilde{\gamma}_{zzz}\|_{0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma zz}\|_{0,\hat{\Omega}}^2) + C_1(\|\widetilde{u}\|_{2,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{1,\hat{\Omega}}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2$$
$$+ \|\widehat{\gamma}_{0t}\|_{0,\hat{\Omega}}^2 + \|\vartheta_{0t}\|_{0,\Omega_t}^2 + \|v\|_{1,\Omega_t}^2 + \|\widetilde{k}\|_{0,\hat{\Omega}})$$
$$+ C_2\Big[\Big(X_3(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|_{3,\hat{\Omega}}^2\,dt'\Big)(1 + X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})$$
$$+ (\|\widehat{\eta}_\sigma\|_{1,\hat{\Omega}}^2 + \|\widehat{\gamma}_0\|_{1,\hat{\Omega}}^2)(\|\vartheta_{0t}\|_{0,\Omega_t}^2 + \|v\|_{1,\Omega_t}^2)\Big].$$

Next, using problem (3.68)–(3.69) we have

$$(3.95)\quad \|\widetilde{u}'_{z\tau\tau}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t\tau\tau}\|_{0,\hat{\Omega}}^2$$
$$\le (\varepsilon + cd)(\|\widetilde{u}_{zzz}\|_{0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma zz}\|_{0,\hat{\Omega}}^2)$$
$$+ C_1(\|\operatorname{div}\widetilde{u}_\tau\|_{1,\hat{\Omega}}^2 + \|\widetilde{u}_t\|_{0,\hat{\Omega}}^2 + \|\widehat{u}\|_{2,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{1,\hat{\Omega}}^2$$
$$+ \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{0,\hat{\Omega}}^2 + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + \|\widetilde{g}\|_{1,\hat{\Omega}}^2)$$
$$+ C_2\Big(X_3(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|_{3,\hat{\Omega}}^2\,dt'\Big)(1 + X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})$$

and

$$\begin{aligned}(3.96)\qquad \|\widetilde{u}'_{nn\tau}\|^2_{0,\hat\Omega} \le{}& (\varepsilon+cd)(\|\widehat{u}_{zzz}\|^2_{0,\hat\Omega}+\|\widehat{\overline{\eta}}_{\Omega_t zz}\|^2_{0,\hat\Omega})\\ &+C_1(\|\widehat{u}\|^2_{2,\hat\Omega}+\|\widetilde{u}_t\|^2_{1,\hat\Omega}+\|\widehat{\eta}_{\sigma z}\|^2_{0,\hat\Omega}\\ &+\|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat\Omega}+\|\widehat{\gamma}_{0z}\|^2_{1,\hat\Omega}+\|\widehat{\gamma}\|^2_{0,\hat\Omega}+\|\widehat{g}\|^2_{1,\hat\Omega})\\ &+C_2\Big(X_3(\widehat{\Omega})+\int_0^t\|\widehat{u}\|^2_{3,\hat\Omega}\,dt'\Big)(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega}).\end{aligned}$$

Hence, taking into account (3.91)–(3.96) we get

$$\begin{aligned}(3.97)\qquad &\frac12\frac{d}{dt}\int_{\hat\Omega}\bigg[\widehat{\eta}(\widetilde{u}^2_{\tau\tau}+\widetilde{u}^2_{3n\tau})+\frac{p_{\sigma\hat\eta}}{\widehat{\eta}}(\widetilde{\overline{\eta}}^2_{\Omega_t\tau\tau}+\widetilde{\overline{\eta}}^2_{\Omega_t n\tau})+\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}(\widetilde{\gamma}^2_{\tau\tau}+\widetilde{\gamma}^2_{n\tau})\bigg]J\,dz\\ &\quad+c_0(\|\widetilde{u}_\tau\|^2_{2,\hat\Omega}+\|\widetilde{\overline{\eta}}_{\Omega_t\tau}\|^2_{1,\hat\Omega}+\|\widetilde{\gamma}_{zz\tau}\|^2_{0,\hat\Omega})\\ &\le(\varepsilon+cd)(\|\widehat{u}_{zzz}\|^2_{0,\hat\Omega}+\|\widehat{\gamma}_{0zzz}\|^2_{0,\hat\Omega}+\|\widehat{\eta}_{\sigma zz}\|^2_{0,\hat\Omega})\\ &\quad+C_1(|\widehat{u}|^2_{2,1,\hat\Omega}+\|\widehat{\gamma}\|^2_{0,\hat\Omega}+\|\widehat{\gamma}_{0z}\|^2_{1,\hat\Omega}+\|\widehat{\gamma}_{0t}\|^2_{1,\hat\Omega}+\|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat\Omega}\\ &\quad+\|\widehat{\eta}_{\sigma z}\|^2_{0,\hat\Omega}+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widehat{g}\|^2_{1,\hat\Omega}+\|\widetilde{k}\|^2_{1,\tilde\Omega})\\ &\quad+C_2\Big[\Big(X_3(\widehat{\Omega})+\int_0^t\|\widehat{u}\|^2_{3,\hat\Omega}\,dt'\Big)(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\\ &\quad+(\|\widehat{\eta}_\sigma\|^2_{1,\hat\Omega}+\|\widehat{\gamma}_0\|^2_{1,\hat\Omega}+\|\widehat{\gamma}\|^2_{3,\hat\Omega})(\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t})+\|\widehat{\gamma}\|^4_{3,\hat\Omega}\Big].\end{aligned}$$

Differentiating the third component of (3.64) with respect to n, multiplying the result by $\widetilde{\overline{\eta}}_{\Omega_t nn}J$ and next integrating over $\widehat{\Omega}$ implies

$$\begin{aligned}(3.98)\qquad &\frac12\frac{d}{dt}\int_{\hat\Omega}\frac{(\mu+\nu)p_{\sigma\hat\eta}}{\widehat{\eta}}\widetilde{\overline{\eta}}^2_{\Omega_t nn}J\,dz+c_0\|\widetilde{\overline{\eta}}_{\Omega_t nn}\|^2_{0,\hat\Omega}\\ &\quad\le(\varepsilon+cd)(\|\widetilde{u}_{zzz}\|^2_{0,\hat\Omega}+\|\widetilde{\overline{\eta}}_{\Omega_t zz}\|^2_{0,\hat\Omega})\\ &\quad\quad+C_1(|\widehat{u}|^2_{2,1,\hat\Omega}+\|\widetilde{u}_\tau\|^2_{2,\hat\Omega}+\|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat\Omega}+\|\widehat{\eta}_{\sigma z}\|^2_{0,\hat\Omega}\\ &\quad\quad+\|\widehat{\gamma}\|^2_{0,\hat\Omega}+\|\widehat{\gamma}_{0z}\|^2_{1,\hat\Omega}+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widetilde{g}\|^2_{1,\hat\Omega})\\ &\quad\quad+C_2\Big(X_3(\widehat{\Omega})+\int_0^t\|\widehat{u}\|^2_{3,\hat\Omega}\,dt'\Big)(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega}).\end{aligned}$$

Now, we rewrite $(3.47)_1$ in the form

$$(3.99)\quad (\mu+\nu)\nabla_{z_i}\operatorname{div}\widetilde{u} = -\mu(\Delta\widetilde{u}_i - \nabla_{z_i}\operatorname{div}\widetilde{u}) + \widehat{\eta}\widetilde{u}_{it} - \widehat{\eta}\widetilde{g}_i - k_4^i - (\mu\nabla^2\widetilde{u}_i + \nu\nabla_{z_i}\operatorname{div}\widetilde{u} - \mu\widehat{\nabla}^2\widetilde{u}_i - \nu\widehat{\nabla}\widehat{\operatorname{div}}\widetilde{u}) - (p_{\sigma\hat{\eta}}\widehat{\nabla}_i\widetilde{\overline{\eta}}_{\Omega_t} - p_{\sigma\hat{\eta}}\widehat{\overline{\eta}}_{\Omega_t}\widehat{\nabla}_i\widehat{\zeta} + \widetilde{p}_{\sigma\hat{\Gamma}}\widehat{\nabla}_i\widehat{\gamma}_0).$$

Differentiating the third component of (3.99) with respect to n gives

$$(3.100)\quad \|(\operatorname{div}\widetilde{u})_{,nn}\|^2_{0,\hat{\Omega}} \le (\varepsilon+cd)\|\widetilde{u}\|^2_{3,\hat{\Omega}} + C_1(|\widehat{u}|^2_{2,1,\hat{\Omega}} + \|\widetilde{u}_\tau\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0nn}\|^2_{0,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t nn}\|^2_{0,\hat{\Omega}} + \|\widetilde{g}\|^2_{0,\hat{\Omega}}) + C_2\Big(X_3(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega}).$$

Next, differentiating (3.68) with respect to n yields

$$(3.101)\quad \|\widetilde{u}_{nnn}\|^2_{0,\hat{\Omega}} \le (\varepsilon+cd)\|\widetilde{u}\|^2_{3,\hat{\Omega}} + C_1(|\widehat{u}|^2_{2,1,\hat{\Omega}} + \|\widetilde{u}_{\tau\tau}\|^2_{1,\hat{\Omega}} + \|(\operatorname{div}\widetilde{u})_{,n}\|^2_{1,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t n}\|^2_{1,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{1,\hat{\Omega}} + \|\widetilde{g}\|^2_{1,\hat{\Omega}}) + C_2\Big(X_3(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega}).$$

In order to estimate $\|\widetilde{\gamma}_{nnn}\|^2_{0,\hat{\Omega}}$ we use (3.84). We get

$$(3.102)\quad \|\widetilde{\gamma}_{nnn}\|^2_{0,\hat{\Omega}} \le (\varepsilon+cd)\|\widetilde{\gamma}\|^2_{3,\hat{\Omega}} + C_1(\|\widehat{u}\|^2_{2,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{1,\hat{\Omega}} + \|\widehat{\gamma}_{0t}\|^2_{1,\hat{\Omega}} + \|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t}) + C_2\Big(X_3(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega}).$$

Finally, we have

$$(3.103)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \widehat{\eta}\widetilde{u}_{zz}^2 J\,dz \le \varepsilon\|\widetilde{u}_{zzt}\|^2_{0,\hat{\Omega}} + C_1\|\widetilde{u}_{zz}\|^2_{0,\hat{\Omega}}$$

and

$$(3.104)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{zz}^2 J\,dz \le \varepsilon\|\widetilde{\gamma}_{zzt}\|^2_{0,\hat{\Omega}} + C_1(\|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{1,\hat{\Omega}}) + C_2(\|\widehat{u}\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}_{0t}\|^2_{1,\hat{\Omega}} + \|\widehat{\eta}_{\sigma t}\|^2_{1,\hat{\Omega}})\|\widehat{\gamma}\|^2_{3,\hat{\Omega}},$$

where we have used the relations

$$\widehat{\eta}_{\sigma t} + \widehat{\eta}\widehat{\nabla}\cdot\widehat{u} = 0 \quad\text{and}\quad J_t = J\widehat{\nabla}\cdot\widehat{u}.$$

From (3.97)–(3.104) we obtain

$$(3.105)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\hat\Omega}\Big(\widehat\eta\widetilde u_{zz}^2+\frac{p_{\sigma\hat\eta}}{\widehat\eta}\widetilde\eta_{\sigma zz}^2+\frac{\widehat\eta c_v}{\widehat\Gamma}\widetilde\gamma_{zz}^2\Big)J\,dz$$
$$+c_0(\|\widetilde u\|_{3,\hat\Omega}^2+\|\widetilde{\widetilde\eta}_{\Omega_t z}\|_{1,\hat\Omega}^2+\|\widetilde\gamma_{zzz}\|_{0,\hat\Omega}^2)$$
$$\le(\varepsilon+cd)(\|\widetilde u\|_{3,\hat\Omega}^2+\|\widehat\gamma_{0zzz}\|_{0,\hat\Omega}^2+\|\widehat\eta_{\sigma zz}\|_{0,\hat\Omega}^2+\|\widehat u_{zzt}\|_{0,\hat\Omega}^2+\|\widehat\gamma_{0zzt}\|_{0,\hat\Omega}^2)$$
$$+C_1(|\widehat u|_{2,1,\hat\Omega}^2+\|\widehat\gamma\|_{0,\hat\Omega}^2+\|\widehat\gamma_{0z}\|_{1,\hat\Omega}^2+\|\widehat\gamma_{0t}\|_{1,\hat\Omega}^2+\|\widehat{\overline\eta}_{\Omega_t}\|_{0,\hat\Omega}^2$$
$$+\|\widehat\eta_{\sigma z}\|_{0,\hat\Omega}^2+\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^2+\|\widetilde g\|_{1,\hat\Omega}^2+\|\widetilde k\|_{1,\hat\Omega}^2)$$
$$+C_2\Big[\Big(X_3(\widehat\Omega)+\int\limits_0^t\|\widehat u\|_{3,\hat\Omega}^2\,dt'\Big)(1+X_3(\widehat\Omega))Y_3(\widehat\Omega)$$
$$+(\|\eta_\sigma\|_{1,\hat\Omega}^2+\|\widehat\gamma_0\|_{1,\hat\Omega}^2+\|\widehat\gamma\|_{3,\hat\Omega}^2)(\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^2)+\|\widehat\gamma\|_{3,\hat\Omega}^4\Big].$$

Hence, applying the same argument as in Lemma 3.4 we get (3.88). ∎

LEMMA 3.6. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of problem* (3.3). *Then*

$$(3.106)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\Big(\varrho v_{xt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{xt}^2+\frac{\varrho c_v}{\theta}\vartheta_{0xt}^2\Big)dx$$
$$+c_0(\|v_t\|_{2,\Omega_t}^2+\|\varrho_{\sigma t}\|_{1,\Omega_t}^2+\|\vartheta_{0xxt}\|_{0,\Omega_t}^2)$$
$$\le\varepsilon(\|v_{xtt}\|_{0,\Omega_t}^2+\|\vartheta_{0xttt}\|_{0,\Omega_t}^2)+C_1(|v|_{2,0,\Omega_t}^2+\|\varrho_{\sigma x}\|_{0,\Omega_t}^2$$
$$+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|\overline\varrho_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{1,\Omega_t}^2+\|\vartheta_{0t}\|_{1,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2$$
$$+|f|_{1,0,\Omega_t}^2+|r|_{1,0,\Omega_t}^2+\|\theta_{1t}\|_{2,\Omega_t}^2+\|\theta_1\|_{1,\Omega_t}^2)$$
$$+C_2\Big(X_4+\int\limits_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1+X_4)Y_4,$$

where

$$X_4=|v|_{3,1,\Omega_t}^2+|\varrho_\sigma|_{2,0,\Omega_t}^2+|\vartheta_0|_{3,1,\Omega_t}^2+\|\overline\varrho_{\Omega_t}\|_{0,\Omega_t}^2,$$
$$Y_4=|v|_{4,2,\Omega_t}^2+|\varrho_\sigma|_{3,1,\Omega_t}^2+|\vartheta_{0t}|_{3,2,\Omega_t}^2+\|\vartheta_{0x}\|_{3,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2+\|\overline\varrho_{\Omega_t}\|_{0,\Omega_t}^2.$$

Proof. We use the partition of unity introduced in Lemma 3.4. First we consider interior subdomains. Differentiating $(3.46)_1$ with respect to t and

ξ, multiplying the result by $\widetilde{u}_{t\xi}A$ and integrating over $\widetilde{\Omega}$ yields

$$
\begin{aligned}
(3.107)\quad & \frac{1}{2}\frac{d}{dt}\int_{\tilde\Omega}\eta\widetilde{u}_{t\xi}^2A\,d\xi+\frac{1}{2}\mu\int_{\tilde\Omega}(\nabla_{u_i}\widetilde{u}_{jt\xi}+\nabla_{u_j}\widetilde{u}_{it\xi})^2A\,d\xi\\
&\quad+(\nu-\mu)\|\nabla_u\cdot\widetilde{u}_{t\xi}\|_{0,\tilde\Omega}^2-\int_{\tilde\Omega}\widetilde{p}_{\sigma t\xi}\nabla_u\cdot\widetilde{u}_{t\xi}A\,d\xi\\
&\le\varepsilon(\|u_{t\xi}\|_{1,\tilde\Omega}^2+\|\eta_{\sigma t\xi}\|_{0,\tilde\Omega}^2+\|\gamma_{0t\xi}\|_{0,\tilde\Omega}^2)\\
&\quad+C_1(\|u_t\|_{1,\tilde\Omega}^2+\|\eta_{\sigma\xi}\|_{0,\tilde\Omega}^2+\|\eta_{\sigma t}\|_{0,\tilde\Omega}^2+\|\overline{\eta}_{\Omega_t}\|_{0,\tilde\Omega}^2\\
&\quad+\|\gamma\|_{0,\tilde\Omega}^2+\|\gamma_{0\xi}\|_{0,\tilde\Omega}^2+\|\gamma_{0t}\|_{0,\tilde\Omega}^2+|\widetilde{g}|_{1,0,\tilde\Omega}^2)\\
&\quad+C_2\Big(X_4(\widetilde{\Omega})+\int_0^t\|u\|_{4,\tilde\Omega}^2\,dt'\Big)(1+X_4(\widetilde{\Omega}))Y_4(\widetilde{\Omega}),
\end{aligned}
$$

where

$$
\begin{aligned}
X_4(\widetilde{\Omega})&=|u|_{3,1,\tilde\Omega}^2+|\overline{\eta}_{\Omega_t}|_{2,0,\tilde\Omega}^2+|\gamma|_{3,1,\tilde\Omega}^2+|\eta_\sigma|_{2,0,\tilde\Omega}^2+|\gamma_0|_{3,1,\tilde\Omega}^2,\\
Y_4(\widetilde{\Omega})&=|u|_{4,2,\tilde\Omega}^2+|\overline{\eta}_{\Omega_t}|_{3,1,\tilde\Omega}^2+|\gamma|_{4,2,\tilde\Omega}^2+|\eta_{\sigma t}|_{2,1,\tilde\Omega}^2+|\gamma_{0t}|_{3,2,\tilde\Omega}^2.
\end{aligned}
$$

Next, dividing $(3.46)_3$ by Γ, differentiating with respect to t and ξ, multiplying the result by $\widetilde{\gamma}_{t\xi}A$ and integrating over $\widetilde{\Omega}$ we obtain

$$
\begin{aligned}
(3.108)\quad & \frac{1}{2}\frac{d}{dt}\int_{\tilde\Omega}\frac{\eta c_v}{\Gamma}\widetilde{\gamma}_{t\xi}^2A\,d\xi+\int_{\tilde\Omega}p_{\sigma\Gamma}\nabla_u\cdot\widetilde{u}_{t\xi}\widetilde{\gamma}_{t\xi}A\,d\xi+\frac{\kappa}{\theta^*}\int_{\tilde\Omega}|\nabla_u\widetilde{\gamma}_{t\xi}|^2A\,d\xi\\
&\le\varepsilon(\|\widetilde{u}_{t\xi}\|_{1,\tilde\Omega}^2+\|\widetilde{\gamma}_{t\xi}\|_{1,\tilde\Omega}^2+\|\vartheta_{0xtt}\|_{0,\Omega_t}^2)\\
&\quad+C_1(\|u_t\|_{1,\tilde\Omega}^2+\|\gamma_{0t}\|_{1,\tilde\Omega}^2+\|\eta_{\sigma\xi}\|_{0,\tilde\Omega}^2+\|\eta_{\sigma t}\|_{0,\tilde\Omega}^2\\
&\quad+\|\overline{\eta}_{\Omega_t}\|_{0,\tilde\Omega}^2+\|\gamma_{0\xi}\|_{0,\tilde\Omega}^2+\|\gamma\|_{0,\tilde\Omega}^2+\|\vartheta_{0t}\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2\\
&\quad+\|v\|_{1,\Omega_t}^2+\|r_t\|_{0,\Omega_t}^2+\|r\|_{0,\Omega_t}^2+\|\theta_{1t}\|_{1,\Omega_t}^2+|\widetilde{k}|_{1,0,\tilde\Omega}^2)\\
&\quad+C_2\Big(X_4+\int_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1+X_4)Y_4,
\end{aligned}
$$

where to estimate $\int_{\tilde\Omega}\big(\frac{\eta c_v}{\Gamma}\zeta\partial_t\theta_{\Omega_t}\big)_{,t\xi}\widetilde{\gamma}_{t\xi}A\,d\xi$ we have used

$$
\begin{aligned}
(3.109)\quad \|\partial_t^2\theta_{\Omega_t}\|_{0,\tilde\Omega}^2&\le\varepsilon\|\vartheta_{0xtt}\|_{0,\Omega_t}^2+C_1(\|v_t\|_{1,\Omega_t}^2+\|v\|_{1,\Omega_t}^2\\
&\quad+\|\vartheta_{0t}\|_{1,\Omega_t}^2+\|r_t\|_{0,\Omega_t}^2+\|r\|_{0,\Omega_t}^2+\|\theta_{1t}\|_{1,\Omega_t}^2)\\
&\quad+C_2\Big(X_4+\int_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1+X_4)Y_4.
\end{aligned}
$$

Since

$$\begin{aligned}\widetilde{p}_{\sigma t\xi} &= \widetilde{p}_{\sigma\eta\eta}\eta_{\sigma\xi}\eta_{\sigma t} + \widetilde{p}_{\sigma\eta\Gamma}(\eta_{\sigma\xi}\gamma_{0\xi} + \eta_{\sigma t}\gamma_{0t}) + \widetilde{p}_{\sigma\Gamma\Gamma}\gamma_{0\xi}\gamma_{0t}\\ &\quad + p_{\sigma\eta}\widetilde{\overline{\eta}}_{\Omega_t\xi t} + p_{\sigma\Gamma}\widetilde{\gamma}_{\xi t} - (p_{\sigma\eta}\eta_\sigma + p_{\sigma\Gamma}\gamma)\zeta_{\xi t} - p_{\sigma\eta}(\zeta_t\eta_{\sigma\xi} + \zeta_\xi\eta_{\sigma t})\\ &\quad - p_{\sigma\Gamma}(\zeta_t\gamma_{0\xi} + \zeta_\xi\gamma_{0t}),\end{aligned}$$

using (3.107), (3.108), equation (3.52), (3.109) and Lemma 5.1 of [21] with $G = \widetilde{\Omega}$, $v = \widetilde{u}_{t\xi}$ we get

$$\begin{aligned}(3.110)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\tilde\Omega}\Big(\eta\widetilde{u}_{t\xi}^2 + \frac{p_{\sigma\eta}}{\eta}\widetilde{\overline{\eta}}_{\Omega_t t\xi}^2 + \frac{\eta c_v}{\Gamma}\widetilde{\gamma}_{t\xi}^2\Big)A\,d\xi\\ &\qquad + c_0(\|\widetilde{u}_t\|_{2,\tilde\Omega}^2 + \|\widetilde{\gamma}_{t\xi\xi}\|_{0,\tilde\Omega}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t t\xi}\|_{0,\tilde\Omega}^2)\\ &\quad \le \varepsilon(\|u_{t\xi}\|_{1,\tilde\Omega}^2 + \|\eta_{\sigma t\xi}\|_{0,\tilde\Omega}^2 + \|\gamma_{0t\xi\xi}\|_{0,\tilde\Omega}^2 + \|\vartheta_{0xtt}\|_{0,\Omega_t}^2)\\ &\qquad + C_1(|u|_{2,0,\tilde\Omega}^2 + \|\eta_{\sigma\xi}\|_{0,\tilde\Omega}^2 + \|\eta_{\sigma t}\|_{0,\tilde\Omega}^2 + \|\overline{\eta}_{\Omega_t}\|_{0,\tilde\Omega}^2 + \|\gamma_{0\xi}\|_{1,\tilde\Omega}^2\\ &\qquad + \|\gamma_{0t}\|_{1,\tilde\Omega}^2 + \|\gamma\|_{0,\tilde\Omega}^2 + \|v_t\|_{1,\Omega_t}^2 + \|v\|_{1,\Omega_t}^2 + \|\vartheta_{0t}\|_{1,\Omega_t}^2\\ &\qquad + |\widetilde{g}|_{1,0,\tilde\Omega}^2 + |\widetilde{k}|_{1,0,\tilde\Omega}^2 + \|r_t\|_{0,\Omega_t}^2 + \|r\|_{0,\Omega_t}^2 + \|\theta_{1t}\|_{1,\Omega_t}^2)\\ &\qquad + C_2\Big(X_4 + \int\limits_0^t \|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1 + X_4)Y_4,\end{aligned}$$

where we have also used the following estimate for a solution $\widetilde{u}$, η_σ of the Stokes problem (3.57):

$$\begin{aligned}\|\widetilde{u}_t\|_{2,\tilde\Omega}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t t\xi}\|_{0,\tilde\Omega}^2 &\le C_1(\|\widetilde{u}_{tt}\|_{0,\tilde\Omega}^2 + |u|_{2,1,\tilde\Omega}^2 + |\eta_\sigma|_{1,0,\tilde\Omega}^2 + \|\gamma_{0\xi}\|_{1\tilde\Omega}^2\\ &\quad + \|\gamma_{0t}\|_{1,\tilde\Omega}^2 + \|\gamma\|_{0,\tilde\Omega}^2 + |\widetilde{g}|_{1,0,\tilde\Omega}^2)\\ &\quad + C_2\Big(X_4 + \int\limits_0^t \|v\|_{4,\Omega_t,}^2\,dt'\Big)Y_4 + c\|(\nabla_u\cdot\widetilde{u})_{,t}\|_{1,\tilde\Omega}^2.\end{aligned}$$

For subdomains near the boundary we obtain the inequality

$$\begin{aligned}(3.111)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\hat\Omega}\Big(\widehat{\eta}\widetilde{u}_{t\tau}^2 + \frac{p_{\sigma\hat\eta}}{\widehat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t t\tau}^2 + \frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{t\tau}^2\Big)J\,dz\\ &\quad + c_0(\|\widetilde{u}_{\tau\tau}\|_{1,\hat\Omega}^2 + \|\widetilde{\gamma}_{t\tau z}\|_{0,\hat\Omega}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t t\tau}\|_{0,\tilde\Omega}^2)\\ &\le \varepsilon(\|\widetilde{u}_{tz}\|_{1,\hat\Omega}^2 + \|\widehat{\eta}_{\sigma tz}\|_{0,\hat\Omega}^2 + \|\widetilde{\gamma}_{tz}\|_{1,\hat\Omega}^2 + \|\vartheta_{0xtt}\|_{0,\Omega_t}^2)\\ &\quad + C_1(|\widehat{u}|_{2,0,\hat\Omega}^2 + \|\widehat{\eta}_{\sigma z}\|_{0,\hat\Omega}^2 + \|\widehat{\eta}_{\sigma t}\|_{0,\hat\Omega}^2 + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat\Omega}^2 + \|\widehat{\gamma}_{0z}\|_{1,\hat\Omega}^2\\ &\quad + \|\widehat{\gamma}_{0t}\|_{1,\hat\Omega}^2 + \|\widehat{\gamma}\|_{0,\hat\Omega}^2 + \|v_t\|_{1,\Omega_t}^2 + \|v\|_{1,\Omega_t}^2 + \|\vartheta_{0t}\|_{1,\Omega_t}^2 + |\widetilde{g}|_{1,0,\hat\Omega}^2\end{aligned}$$

$$+ |\widetilde{k}|^2_{1,0,\hat{\Omega}} + \|r_t\|^2_{0,\Omega_t} + \|r\|^2_{0,\Omega_t} + \|\vartheta_{1t}\|^2_{1,\Omega_t} + \|\widetilde{\Gamma}_{1t}\|^2_{2,\hat{\Omega}} + \|\widetilde{\Gamma}_1\|^2_{1,\hat{\Omega}})$$

$$+ C_2\Big(X_4 + \int\limits_0^t \|v\|^2_{4,\Omega_{t'}}\,dt'\Big)(1 + X_4)Y_4,$$

where we have used the following estimates:

$$\int\limits_{\hat{S}} |(\widehat{\mathbb{T}}(\widetilde{u}, \widetilde{p}_\sigma)\widehat{n})_{,t\tau}\widetilde{u}_{t\tau}J|\,dz'$$

$$\le \varepsilon(\|\widehat{u}_{t\tau}\|^2_{1,\hat{\Omega}} + \|\widehat{u}_{tzz}\|^2_{0,\hat{\Omega}}) + C_1|\widehat{u}|^2_{2,1,\hat{\Omega}}$$

$$+ C_2\Big[\|\widehat{u}\|^4_{2,\hat{\Omega}} + |\widehat{u}|^2_{3,2,\hat{\Omega}}\Big\|\int\limits_0^t \widehat{u}\,dt'\Big\|^2_{3,\hat{\Omega}}(1 + \|\widehat{u}\|^2_{3,\hat{\Omega}})\Big]$$

and

$$\int\limits_{\hat{S}} |(\widehat{n}\cdot\widehat{\Gamma}^{-1}\widehat{\nabla}\widetilde{\gamma})_{,t\tau}\widetilde{\gamma}_{t\tau}J|\,dz'$$

$$\le \varepsilon(\|\widetilde{\gamma}_{t\tau}\|^2_{1,\hat{\Omega}} + \|\widetilde{\gamma}_{tzz}\|^2_{0,\hat{\Omega}})$$

$$+ C_1(\|\widehat{\gamma}_{0t}\|^2_{1,\hat{\Omega}} + \|v\|^2_{2,\Omega_t} + \|\vartheta_{0t}\|^2_{1,\Omega_t} + \|\widetilde{\Gamma}_{1t}\|^2_{2,\hat{\Omega}} + \|\widetilde{\Gamma}_1\|^2_{1,\hat{\Omega}})$$

$$+ C_2\Big[\Big(X_4(\widehat{\Omega}) + \int\limits_0^t \|\widehat{u}\|^2_{4,\Omega_{t'}}\,dt'\Big)(1 + X_4(\widehat{\Omega}))Y_4(\widehat{\Omega})$$

$$+ \int\limits_0^t \|\widehat{u}\|^2_{4,\hat{\Omega}}\,dt'(\|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t})\Big].$$

Next, differentiating the third component of (3.63) with respect to t, multiplying the result by $\widetilde{\overline{\eta}}_{\Omega_t n}J$ and integrating over $\widehat{\Omega}$ yields

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}} \frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}(\nu+\mu)\widetilde{\overline{\eta}}^2_{\Omega_t nt}J\,dz + c_0\|\widetilde{\overline{\eta}}_{\Omega_t nt}\|^2_{0,\hat{\Omega}} \tag{3.112}$$

$$\le (\varepsilon + cd)(\|\widetilde{u}_{zzt}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t zt}\|^2_{0,\hat{\Omega}}) + \varepsilon\|\vartheta_{0xtt}\|^2_{0,\Omega_t}$$

$$+ C_1(\|\widetilde{u}_{t\tau}\|^2_{0,\hat{\Omega}} + |\widehat{u}|^2_{2,0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma t}\|^2_{0,\hat{\Omega}}$$

$$+ \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0t}\|^2_{1,\hat{\Omega}} + \|v_t\|^2_{1,\Omega_t} + \|\vartheta_{0t}\|^2_{1,\Omega_t}$$

$$+ \|r_t\|^2_{0,\Omega_t} + \|r\|^2_{0,\Omega_t} + \|\theta_{1t}\|^2_{1,\Omega_t} + |\widetilde{g}|^2_{1,0,\hat{\Omega}})$$

$$+ C_2\Big(X_4 + \int\limits_0^t \|v\|^2_{4,\Omega_{t'}}\,dt'\Big)(1 + X_4)Y_4,$$

where we have used (3.109).

Differentiating the third component of (3.66) with respect to t, multiplying the result by $\widetilde{u}_{3nnt}J$ and integrating over $\widehat{\Omega}$ implies

$$(3.113)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}} \widehat{\eta}|\widetilde{u}_{3nt}|^2 J\,dz + c_0\|\widetilde{u}_{3nnt}\|^2_{0,\hat{\Omega}}$$
$$\le (\varepsilon + cd)(\|\widetilde{u}_{zzt}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t zt}\|^2_{0,\hat{\Omega}}) + \varepsilon\|\widetilde{u}_{ztt}\|^2_{0,\hat{\Omega}}$$
$$+ C_1(\|\widetilde{u}_{z\tau t}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t nt}\|^2_{0,\hat{\Omega}} + |\widehat{u}|^2_{2,0,\hat{\Omega}}$$
$$+ \|\widehat{\eta}_{\sigma t}\|^2_{0,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0t}\|^2_{1,\hat{\Omega}}$$
$$+ \|\widehat{\gamma}_{0z}\|^2_{1,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + |\widetilde{g}|^2_{1,0,\hat{\Omega}} + |\widetilde{k}|^2_{1,0,\hat{\Omega}})$$
$$+ C_2\Big(X_4(\widehat{\Omega}) + \int\limits_0^t \|\widehat{u}\|^2_{4,\hat{\Omega}}\,dt'\Big)(1 + X_4(\widehat{\Omega}))Y_4(\widehat{\Omega}).$$

Differentiating (3.68) with respect to t and τ, multiplying by $\widetilde{u}'_{t\tau}J$, integrating over $\widehat{\Omega}$ and using (3.69) gives

$$(3.114)\quad \|\widetilde{u}'_{zt\tau}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t t\tau}\|^2_{0,\hat{\Omega}}$$
$$\le (\varepsilon + cd)(\|\widetilde{u}_{zzt}\|^2_{0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma zt}\|^2_{0,\hat{\Omega}})$$
$$+ C_1(\|(\operatorname{div}\widetilde{u}')_{,\tau t}\|^2_{0,\hat{\Omega}} + |\widehat{u}|^2_{2,0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma t}\|^2_{0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\hat{\Omega}}$$
$$+ \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{1,\hat{\Omega}} + \|\widehat{\gamma}_{0t}\|^2_{1,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + |\widetilde{g}|^2_{1,0,\hat{\Omega}})$$
$$+ C_2\Big(X_4(\widehat{\Omega}) + \int\limits_0^t \|\widehat{u}\|^2_{4,\hat{\Omega}}\,dt'\Big)(1 + X_4(\widehat{\Omega}))Y_4(\widehat{\Omega}).$$

Moreover, from (3.68) we get

$$(3.115)\quad \|\widetilde{u}'_{nnt}\|^2_{0,\hat{\Omega}} \le (\varepsilon + cd)(\|\widetilde{u}_{zzt}\|^2_{0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma zt}\|^2_{0,\hat{\Omega}})$$
$$+ C_1(\|\widetilde{u}_{zt\tau}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t t\tau}\|^2_{0,\hat{\Omega}} + |\widehat{u}|^2_{2,0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\hat{\Omega}} + \|\widehat{\eta}_{\sigma t}\|^2_{0,\hat{\Omega}}$$
$$+ \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{1,\hat{\Omega}} + \|\widehat{\gamma}_{0t}\|^2_{1,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + |\widetilde{g}|^2_{1,0,\hat{\Omega}})$$
$$+ C_2\Big(X_4(\widehat{\Omega}) + \int\limits_0^t \|\widehat{u}\|^2_{0,\hat{\Omega}}\,dt'\Big)(1 + X_4(\widehat{\Omega}))Y_4(\widehat{\Omega}).$$

Next, we have

$$(3.116)\quad \frac{d}{dt}\int\limits_{\hat{\Omega}} \widehat{\eta}\widetilde{u}^2_{zt} J\,dz \le \varepsilon\|\widetilde{u}_{ztt}\|^2_{0,\hat{\Omega}} + C_1\|\widetilde{u}_t\|^2_{0,\hat{\Omega}}.$$

Finally, by using (3.109) we obtain

$$
\begin{aligned}
(3.117)\quad & \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{nt}^2 J\,dz + \frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{nnt}\|_{0,\hat{\Omega}}^2 \\
&\le (\varepsilon + cd)(\|\widetilde{\gamma}_{zzt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t zt}\|_{0,\hat{\Omega}}^2) + \varepsilon(\|\widehat{\gamma}_{0ztt}\|_{0,\hat{\Omega}}^2 + \|\vartheta_{0xtt}\|_{0,\Omega_t}^2) \\
&\quad + C_1(\|\widetilde{\gamma}_{t\tau}\|_{1,\hat{\Omega}}^2 + |\widehat{u}|_{2,1,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma t}\|_{0,\hat{\Omega}}^2 \\
&\quad + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{1,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0t}\|_{1,\hat{\Omega}}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 \\
&\quad + \|v_t\|_{1,\Omega_t}^2 + \|\vartheta_{0t}\|_{1,\Omega_t}^2 + \|r_t\|_{0,\Omega_t}^2 + \|r\|_{0,\Omega_t}^2) \\
&\quad + C_2\Big(X_4 + \int_0^t \|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1 + X_4)Y_4.
\end{aligned}
$$

Taking into account inequalities (3.111)–(3.117) we get

$$
\begin{aligned}
(3.118)\quad & \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \Big(\widehat{\eta}\widetilde{u}_{zt}^2 + \frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t zt}^2 + \frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{zt}^2\Big)J\,dz \\
& + c_0(\|\widetilde{u}_{tz}\|_{1,\hat{\Omega}}^2 + \|\widetilde{\gamma}_{tzz}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t zt}\|_{0,\hat{\Omega}}^2) \\
&\le \varepsilon(\|\widetilde{u}_{tz}\|_{1,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma zt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\gamma}_{tz}\|_{1,\hat{\Omega}}^2 + \|\widetilde{u}_{ztt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\gamma}_{ztt}\|_{0,\hat{\Omega}}^2 + \|\vartheta_{0xtt}\|_{0,\Omega_t}^2) \\
& + (\varepsilon + cd)(\|\widetilde{\gamma}_{zzt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t zt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{u}_{zzt}\|_{0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma zt}\|_{0,\hat{\Omega}}^2) \\
& + C_1(|\widehat{u}|_{2,0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma t}\|_{0,\hat{\Omega}}^2 + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{1,\hat{\Omega}}^2 \\
& + \|\widehat{\gamma}_{0t}\|_{1,\hat{\Omega}}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + \|\widetilde{u}_{t\tau}\|_{1,\hat{\Omega}}^2 + \|v_t\|_{1,\Omega_t}^2 + \|\vartheta_{0t}\|_{1,\Omega_t}^2 \\
& + \|v\|_{1,\Omega_t}^2 + |\widetilde{g}|_{1,0,\hat{\Omega}}^2 + |\widetilde{k}|_{1,0,\hat{\Omega}}^2 + \|\widetilde{\Gamma}_{1t}\|_{2,\hat{\Omega}}^2 \\
& + \|\widetilde{\Gamma}_1\|_{1,\hat{\Omega}}^2 + \|\theta_{1t}\|_{1,\Omega_t}^2 + \|r_t\|_{0,\Omega_t}^2 + \|r\|_{0,\Omega_t}^2) \\
& + C_2\Big(X_4 + \int_0^t \|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1 + X_4)Y_4.
\end{aligned}
$$

Inequalities (3.110) and (3.118) yield the assertion of the lemma. ■

LEMMA 3.7. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of problem* (3.3). *Then*

$$
\begin{aligned}
(3.119)\quad & \frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \Big(\varrho v_{tt}^2 + \frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma tt}^2 + \frac{\varrho c_v}{\theta}\vartheta_{0tt}^2\Big)\,dx \\
&\qquad\qquad + c_0(\|v_{tt}\|_{1,\Omega_t}^2 + \|\varrho_{\sigma tt}\|_{0,\Omega_t}^2 + \|\vartheta_{0tt}\|_{1,\Omega_t}^2)
\end{aligned}
$$

$$\le C_1(\|v_t\|^2_{1,\Omega_t} + \|\vartheta_{0t}\|^2_{1,\Omega_t} + |f|^2_{1,0,\Omega_t} + \|f_{tt}\|^2_{0,\Omega_t} + |r|^2_{1,0,\Omega_t} + \|r_{tt}\|^2_{0,\Omega_t} + |\theta_1|^2_{3,1,\Omega_t}) + C_2X_5(1+X_5)Y_5,$$

where

$$X_5 = |v|^2_{3,1,\Omega_t} + |\varrho_\sigma|^2_{2,0,\Omega_t} + |\vartheta_0|^2_{3,1,\Omega_t}, \quad Y_5 = |v|^2_{4,2,\Omega_t} + |\varrho_\sigma|^2_{3,1,\Omega_t} + |\vartheta_0|^2_{4,2,\Omega_t}.$$

P r o o f. Differentiating $(3.3)_1$ twice with respect to t, multiplying by v_{tt} and integrating over Ω_t yields

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \varrho v_{tt}^2\,dx + \frac{\mu}{2}E_{\Omega_t}(v_{tt}) + (\nu-\mu)\|\operatorname{div} v_{tt}\|^2_{0,\Omega_t} - \int_{\Omega_t} p_{\sigma\varrho}\varrho_{\sigma tt}\operatorname{div} v_{tt}\,dx - \int_{\Omega_t} p_{\sigma\theta}\vartheta_{0tt}\operatorname{div} v_{tt}\,dx - \int_{S_t}(n_iT^{ij}(v,p_\sigma))_{,tt}v_{itt}\,ds$$
$$\le \varepsilon(\|v_{tt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma tt}\|^2_{0,\Omega_t}) + C_1(|f|^2_{1,0,\Omega_t} + \|f_{tt}\|^2_{0,\Omega_t}) + C_2X_5(1+X_5)Y_5. \tag{3.120}$$

Next, dividing $(3.3)_3$ by θ, differentiating twice with respect to t, multiplying by ϑ_{0tt} and integrating over Ω_t yields

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \frac{\varrho c_v}{\theta}\vartheta_{0tt}\,dx + \frac{\kappa}{\theta^*}\int_{\Omega_t}|\nabla\vartheta_{0tt}|^2\,dx + \int_{\Omega_t} p_{\sigma\theta}\vartheta_{0tt}\operatorname{div} v_{tt}\,dx - \int_{S_t}\left(\frac{n\cdot\nabla\vartheta_0}{\theta}\right)_{,tt}\vartheta_{0tt}\,ds$$
$$\le \varepsilon(\|v_{tt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma tt}\|^2_{0,\Omega_t} + \|\vartheta_{0tt}\|^2_{1,\Omega_t}) + C_1(|r|^2_{1,0,\Omega_t} + \|r_{tt}\|^2_{0,\Omega_t} + \|\vartheta_{1tt}\|^2_{1,\Omega_t}) + C_2X_5(1+X_5)Y_5. \tag{3.121}$$

Moreover, we have

$$\|\varrho_{\sigma tt}\|^2_{0,\Omega_t} \le c\|v_t\|^2_{1,\Omega_t} + C_2X_5(1+X_5)Y_5 \tag{3.122}$$

and

$$\|\vartheta_{0tt}\|^2_{0,\Omega_t} \le \varepsilon\|\vartheta_{0xtt}\|^2_{0,\Omega_t} + C_1(\|v_t\|^2_{1,\Omega_t} + \|\vartheta_{0t}\|^2_{1,\Omega_t} + \|r_t\|^2_{0,\Omega_t} + \|r\|^2_{0,\Omega_t} + \|\theta_{1t}\|^2_{1,\Omega_t}) + C_2\Big(X_5 + \int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt'\Big)(1+X_5)Y_5, \tag{3.123}$$

where we have used the continuity equation $(3.3)_1$ and energy equation $(3.3)_2$, respectively.

From (3.120)–(3.123), using the continuity equation $(3.3)_2$ and Lemma 5.4 of [21] we get (3.119). ■

Summarizing, from Lemmas 3.5–3.7 we obtain

LEMMA 3.8. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of problem* (3.3). *Then*

$$
\begin{aligned}
(3.124)\quad & \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\bigg(\varrho|D^2_{x,t}v|^2+\frac{p_{\sigma\varrho}}{\varrho}|D^2_{x,t}\varrho_\sigma|^2+\frac{\varrho c_v}{\theta}|D^2_{x,t}\vartheta_0|^2\bigg)\,dx\\
&\quad+c_0(|v|^2_{3,1,\Omega_t}+|\varrho_{\sigma t}|^2_{1,0,\Omega_t}+\|\varrho_{\sigma x}\|^2_{0,\Omega_t}+|\vartheta_{0t}|^2_{2,1,\Omega_t}+\|\vartheta_{0xxx}\|^2_{0,\Omega_t})\\
&\le C_1(|v|^2_{2,0,\Omega_t}+\|\varrho_{\sigma x}\|^2_{0,\Omega_t}+\|\varrho_{\sigma t}\|^2_{0,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}+\|\vartheta_{0x}\|^2_{1,\Omega_t}\\
&\quad+\|\vartheta_{0t}\|^2_{1,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}+|f|^2_{1,0,\Omega_t}+\|f_{tt}\|^2_{0,\Omega_t}+|r|^2_{1,0,\Omega_t}\\
&\quad+\|r_{tt}\|^2_{1,0,\Omega_t}+\|r\|_{0,\Omega_t}+|\theta_1|^2_{3,1,\Omega_t}+\|\theta_1\|_{1,\Omega_t})\\
&\quad+C_2\Big(X_6+\int\limits_0^t\|v\|^2_{4,\Omega_{t'}}\,dt'\Big)(1+X_6)Y_6,
\end{aligned}
$$

where

$$X_6=|v|^2_{3,1,\Omega_t}+|\varrho_\sigma|^2_{2,0,\Omega_t}+|\vartheta_0|^2_{3,1,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t},$$
$$Y_6=|v|^2_{4,2,\Omega_t}+|\varrho_\sigma|^2_{3,1,\Omega_t}+|\vartheta_{0t}|^2_{3,2,\Omega_t}+\|\vartheta_{0x}\|^2_{3,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}.$$

Finally, we obtain inequalities for the fourth derivatives.

LEMMA 3.9. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of* (3.3). *Then*

$$
\begin{aligned}
(3.125)\quad & \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\bigg(\varrho v^2_{xxx}+\frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xxx}+\frac{\varrho c_v}{\theta}\vartheta^2_{0xxx}\bigg)\,dx\\
&\qquad+c_0(\|v_{xxx}\|^2_{1,\Omega_t}+\|\varrho_{\sigma xxx}\|^2_{0,\Omega_t}+\|\vartheta_{0xxxx}\|^2_{0,\Omega_t})\\
&\quad\le\varepsilon(\|v_{xxxt}\|^2_{0,\Omega_t}+\|\vartheta_{0xxxt}\|^2_{0,\Omega_t})\\
&\qquad+C_1(|v|^2_{3,2,\Omega_t}+\|\varrho_{\sigma x}\|^2_{1,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}+\|\vartheta_{0x}\|^2_{2,\Omega_t}\\
&\qquad+\|\vartheta_{0t}\|^2_{2,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}+\|f\|^2_{2,\Omega_t}+\|r\|^2_{2,\Omega_t}+\|\theta_1\|^2_{4,\Omega_t})\\
&\qquad+C_2\Big(X_7+\int\limits_0^t\|v\|^2_{4,\Omega_{t'}}\,dt'\Big)(1+X_7^2)Y_7,
\end{aligned}
$$

where

$$X_7=|v|^2_{3,2,\Omega_t}+|\varrho_\sigma|^2_{3,2,\Omega_t}+|\vartheta_0|^2_{3,2,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t},$$
$$Y_7=|v|^2_{4,3,\Omega_t}+|\varrho_\sigma|^2_{3,2,\Omega_t}+\|\vartheta_{0x}\|^2_{3,\Omega_t}+\|\vartheta_{0t}\|^2_{3,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}.$$

P r o o f. We use the partition of unity. Differentiating $(3.46)_1$ and $(3.46)_3$ (divided by Γ) three times with respect to ξ, multiplying by $\widetilde{u}_{\xi\xi\xi}A$ and

$\widetilde{\gamma}_{\xi\xi\xi}A$, respectively and next integrating over $\widetilde{\Omega}$ we get the estimate

$$
\begin{aligned}
(3.126)\quad & \frac{1}{2}\frac{d}{dt}\int\limits_{\tilde{\Omega}}\Big(\eta\widetilde{u}^2_{\xi\xi\xi}+\frac{p_{\sigma\eta}}{\eta}\widetilde{\overline{\eta}}^2_{\Omega_t\xi\xi\xi}+\frac{\eta c_v}{\Gamma}\widetilde{\gamma}^2_{\xi\xi\xi}\Big)A\,d\xi \\
&\quad+\frac{1}{2}\mu\|\widetilde{u}_{\xi\xi\xi}\|^2_{1,\tilde{\Omega}}+\|\widetilde{\overline{\eta}}_{\Omega_t\xi\xi\xi}\|^2_{0,\tilde{\Omega}}+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{\xi\xi\xi\xi}\|^2_{0,\tilde{\Omega}} \\
&\le\varepsilon(\|\widetilde{u}_{\xi\xi\xi\xi}\|^2_{0,\tilde{\Omega}}+\|\widetilde{\gamma}_{\xi\xi\xi\xi}\|^2_{0,\tilde{\Omega}}+\|\widetilde{\eta}_{\sigma\xi\xi\xi}\|^2_{0,\tilde{\Omega}}) \\
&\quad+C_1(|u|^2_{3,2,\tilde{\Omega}}+\|\gamma_{0\xi}\|^2_{2,\tilde{\Omega}}+\|\gamma\|^2_{0,\tilde{\Omega}}+\|\eta_{\sigma\xi}\|^2_{1,\tilde{\Omega}} \\
&\quad+\|\overline{\eta}_{\Omega_t}\|^2_{0,\tilde{\Omega}}+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widetilde{g}\|^2_{2,\tilde{\Omega}}+\|\widetilde{k}\|^2_{2,\tilde{\Omega}}) \\
&\quad+C_2\Big[\Big(X_7(\widetilde{\Omega})+\int\limits_0^t\|u\|^2_{3,\tilde{\Omega}}\,dt'\Big)(1+X_7^2(\widetilde{\Omega}))Y_7(\widetilde{\Omega}) \\
&\quad+\|\gamma\|^2_{4,\tilde{\Omega}}(\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t})\Big],
\end{aligned}
$$

where we have used equation (3.52), Lemma 5.1 of [21] in the case $G=\widetilde{\Omega}$, $v=\widetilde{u}_{\xi\xi\xi}$ and the estimate for the solution u, $\overline{\eta}_{\Omega_t}$ of the Stokes problem (3.57), i.e. the estimate of $\|\widetilde{u}\|_{4,\tilde{\Omega}}$ and $\|\widetilde{\overline{\eta}}_{\Omega_t}\|_{3,\tilde{\Omega}}$ respectively and

$$
(3.127)\quad
\begin{aligned}
X_7(\widetilde{\Omega})&=|u|^2_{3,2,\tilde{\Omega}}+|\overline{\eta}_{\Omega_t}|^2_{3,2,\tilde{\Omega}}+|\gamma|^2_{3,2,\tilde{\Omega}}+|\eta_\sigma|^2_{3,2,\tilde{\Omega}}+|\gamma_0|^2_{3,2,\tilde{\Omega}},\\
Y_7(\widetilde{\Omega})&=|u|^2_{4,3,\tilde{\Omega}}+|\overline{\eta}_{\Omega_t}|^2_{3,2,\tilde{\Omega}}+|\gamma|^2_{4,3,\tilde{\Omega}}+|\eta_{\sigma t}|^2_{2,1,\tilde{\Omega}}+\|\gamma_{0t}\|^2_{3,\tilde{\Omega}}.
\end{aligned}
$$

For boundary subdomains we obtain

$$
\begin{aligned}
(3.128)\quad & \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\Big(\widehat{\eta}\widetilde{u}^2_{\tau\tau\tau}+\frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}^2_{\Omega_t\tau\tau\tau}+\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}^2_{\tau\tau\tau}\Big)J\,dz \\
&\quad+\frac{1}{2}\mu\|\widetilde{u}_{\tau\tau\tau}\|^2_{1,\hat{\Omega}}+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{\tau\tau\tau z}\|^2_{0,\hat{\Omega}} \\
&\le\varepsilon(\|\widehat{u}_{zzzz}\|^2_{0,\hat{\Omega}}+\|\widehat{\gamma}_{0zzzz}\|^2_{0,\hat{\Omega}}+\|\widehat{\eta}_{\sigma zzz}\|^2_{0,\hat{\Omega}}) \\
&\quad+C_1(|\widehat{u}|^2_{3,2,\hat{\Omega}}+\|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}}+\|\widehat{\gamma}\|^2_{0,\hat{\Omega}}+\|\widehat{\eta}_{\sigma z}\|^2_{1,\hat{\Omega}}+\|\overline{\eta}_{\Omega_t}\|^2_{0,\hat{\Omega}} \\
&\quad+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widetilde{g}\|^2_{2,\hat{\Omega}}+\|\widetilde{k}\|^2_{2,\hat{\Omega}}+\|\widetilde{\Gamma}_1\|^2_{4,\hat{\Omega}}) \\
&\quad+C_2\Big[\Big(X_7(\widehat{\Omega})+\int\limits_0^t\|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega}) \\
&\quad+\|\widehat{\gamma}\|^2_{4,\hat{\Omega}}(\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t})\Big],
\end{aligned}
$$

where we have used the boundary conditions $(3.47)_4$ and $(3.47)_5$, and where $X_7(\widehat{\Omega})$ and $Y_7(\widehat{\Omega})$ are defined by (3.127) with $\widetilde{\Omega}$, u, $\overline{\eta}_{\Omega_t}$, γ replaced by $\widehat{\Omega}$, $\widehat{u}$, $\widehat{\overline{\eta}}_{\Omega_t}$, $\widehat{\gamma}$, respectively.

In the same way as (3.92) and (3.93) we obtain the following estimates:

$$\tag{3.129} \begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\hat\Omega}\frac{p_{\sigma\hat\eta}}{\widehat\eta}\widetilde{\overline\eta}^2_{\Omega_t n\tau\tau}J\,dz + c_0\|\widetilde{\overline\eta}_{\Omega_t n\tau\tau}\|^2_{0,\hat\Omega} \\ &\le (\varepsilon+cd)(\|\widetilde{\overline\eta}_{\Omega_t n\tau\tau}\|^2_{0,\hat\Omega} + \|\widetilde u_{zz\tau\tau}\|^2_{0,\hat\Omega}) \\ &\quad + C_1(\|\widetilde u_{\tau\tau\tau}\|^2_{1,\hat\Omega} + |\widehat u|^2_{3,2,\hat\Omega} + \|\widehat\eta_{\sigma z}\|^2_{1,\hat\Omega} + \|\widehat{\overline\eta}_{\Omega_t}\|^2_{0,\hat\Omega} \\ &\quad + \|\widehat\gamma_{0z}\|^2_{2,\hat\Omega} + \|\widehat\gamma\|^2_{0,\hat\Omega} + \|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t} + \|\widetilde g\|^2_{2,\hat\Omega}) \\ &\quad + C_2\Big[\Big(X_7(\widehat\Omega) + \int_0^t \|\widehat u\|^2_{3,\hat\Omega}\,dt'\Big)(1+X_7^2(\widehat\Omega))Y_7(\widehat\Omega) \\ &\quad + (\|\widehat\eta_\sigma\|^2_{2,\hat\Omega} + \|\widehat\gamma_0\|^2_{2,\hat\Omega} + \|\widehat\eta_\sigma\|^4_{2,\hat\Omega} \\ &\quad + \|\widehat\eta_\sigma\|^2_{2,\hat\Omega}\|\widehat\gamma_0\|^2_{2,\hat\Omega} + \|\widehat\gamma_0\|^4_{2,\hat\Omega})(\|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t})\Big] \end{aligned}$$

and

$$\tag{3.130} \begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\hat\Omega}\widehat\eta|\widetilde u_{3n\tau\tau}|^2J\,dz + c_0\|\widetilde u_{3nn\tau\tau}\|^2_{0,\hat\Omega} \\ &\le (\varepsilon+cd)(\|\widetilde u_{zz\tau\tau}\|^2_{0,\hat\Omega} + \|\widetilde{\overline\eta}_{\Omega_t z\tau\tau}\|^2_{0,\hat\Omega}) + \varepsilon\|\widetilde u_{n\tau\tau t}\|^2_{0,\hat\Omega} \\ &\quad + C_1(\|\widetilde u_{z\tau\tau\tau}\|^2_{0,\hat\Omega} + \|\widetilde{\overline\eta}_{\Omega_t n\tau\tau}\|^2_{0,\hat\Omega} + |\widehat u|^2_{3,2,\hat\Omega} + \|\widehat\eta_{\sigma z}\|^2_{1,\hat\Omega} \\ &\quad + \|\widehat{\overline\eta}_{\Omega_t}\|^2_{0,\hat\Omega} + \|\widehat\gamma_{0z}\|^2_{2,\hat\Omega} + \|\widehat\gamma\|^2_{0,\hat\Omega} + \|\widehat g\|^2_{2,\hat\Omega}) \\ &\quad + C_2\Big(X_7(\widehat\Omega) + \int_0^t \|\widehat u\|^2_{3,\hat\Omega}\,dt'\Big)(1+X_7^2(\widehat\Omega))Y_7(\widehat\Omega). \end{aligned}$$

Next, differentiating (3.68) three times with respect to τ, multiplying by $\widetilde u'_{\tau\tau\tau}J$, integrating over $\widehat\Omega$ and using the boundary condition (3.69) we get

$$\tag{3.131} \begin{aligned} &\|\widetilde u'_{z\tau\tau\tau}\|^2_{0,\hat\Omega} + \|\widetilde{\overline\eta}_{\Omega_t\tau\tau\tau}\|^2_{0,\hat\Omega} \\ &\le (\varepsilon+cd)(\|\widehat u_{zzzz}\|^2_{0,\hat\Omega} + \|\widehat{\overline\eta}_{\Omega_t zzz}\|^2_{0,\hat\Omega}) + C_1(\|\operatorname{div}\widetilde u_{\tau\tau}\|^2_{1,\hat\Omega} + |\widehat u|^2_{3,2,\hat\Omega} \\ &\quad + \|\widehat\eta_{\sigma z}\|^2_{1,\hat\Omega} + \|\widehat{\overline\eta}_{\Omega_t}\|^2_{0,\hat\Omega} + \|\widehat\gamma_{0z}\|^2_{2,\hat\Omega} + \|\widehat\gamma\|^2_{0,\hat\Omega} + \|\widehat g\|^2_{2,\hat\Omega}) \\ &\quad + C_2\Big(X_7(\widehat\Omega) + \int_0^t \|\widehat u\|^2_{3,\hat\Omega}\,dt'\Big)(1+X_7^2(\widehat\Omega))Y_7(\widehat\Omega). \end{aligned}$$

Moreover, from (3.68) we find

$$(3.132)\quad \|\widetilde{u}'_{nn\tau\tau}\|^2_{0,\hat{\Omega}} \le (\varepsilon + cd)(\|\widetilde{u}_{zzzz}\|^2_{0,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t zzz}\|^2_{0,\hat{\Omega}})$$
$$+ C_1(\|\widetilde{u}'_{\tau\tau\tau\tau}\|^2_{0,\hat{\Omega}} + \|(\operatorname{div}\widetilde{u})_{,\tau\tau\tau}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t\tau\tau\tau}\|^2_{0,\hat{\Omega}} + |\widehat{u}|^2_{3,2,\hat{\Omega}}$$
$$+ \|\widehat{\eta}_{\sigma z}\|^2_{1,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\widehat{g}\|^2_{2,\hat{\Omega}})$$
$$+ C_2\Big(X_7(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1 + X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega}).$$

Next, dividing (3.84) by $\widehat{\Gamma}$, differentiating twice with respect to τ, multiplying the result by $\widetilde{\gamma}_{nn\tau\tau}J$ and integrating over $\widehat{\Omega}$ gives

$$(3.133)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}^2_{n\tau\tau}J\,dz + \frac{\kappa}{\theta^*}\int_{\hat{\Omega}} \widetilde{\gamma}^2_{nn\tau\tau}J\,dz$$
$$\le (\varepsilon + cd)(\|\widetilde{\gamma}_{zzzz}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t zzz}\|^2_{0,\hat{\Omega}}) + \varepsilon\|\widetilde{\gamma}_{n\tau\tau t}\|^2_{0,\hat{\Omega}}$$
$$+ C_1(\|\widetilde{u}\|^2_{3,\hat{\Omega}} + \|\widetilde{\gamma}_{z\tau\tau\tau}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}}$$
$$+ \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t} + \|\widetilde{g}\|^2_{2,\hat{\Omega}})$$
$$+ C_2\Big[\Big(X_7(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1 + X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})$$
$$+ (\|\widehat{\eta}_\sigma\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}} + \|\widehat{\eta}_\sigma\|^4_{2,\hat{\Omega}}$$
$$+ \|\widehat{\eta}_\sigma\|^2_{2,\hat{\Omega}}\|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}_0\|^4_{2,\hat{\Omega}})(\|\vartheta_{0t}\|^2_{1,\Omega_t} + \|v\|^2_{2,\Omega_t})\Big].$$

Differentiating the third component of (3.64) with respect to n and τ, multiplying by $\widetilde{\overline{\eta}}_{\Omega_t nn\tau}J$ and integrating over $\widehat{\Omega}$ yields

$$(3.134)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \frac{\varrho_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t nn\tau}J\,dz + c_0\|\widetilde{\overline{\eta}}_{\Omega_t nn\tau}\|^2_{0,\hat{\Omega}}$$
$$\le (\varepsilon + cd)(\|\widetilde{u}_{zzz\tau}\|^2_{0,\hat{\Omega}} + \|\widetilde{\overline{\eta}}_{\Omega_t zz}\|^2_{0,\hat{\Omega}})$$
$$+ C_1(\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\hat{\Omega}} + |\widehat{u}|^2_{3,2,\hat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{1,\hat{\Omega}} + \|\widehat{\eta}_{\sigma t}\|^2_{1,\hat{\Omega}} + \|\overline{\eta}_{\Omega_t}\|^2_{0,\hat{\Omega}}$$
$$+ \|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t} + \|\widehat{g}\|^2_{2,\hat{\Omega}})$$
$$+ C_2\Big[\Big(X_7(\widehat{\Omega}) + \int_0^t \|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1 + X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})$$
$$+ (\|\widehat{\eta}_\sigma\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}} + \|\widehat{\eta}_\sigma\|^4_{2,\hat{\Omega}}$$
$$+ \|\widehat{\eta}_\sigma\|^2_{2,\hat{\Omega}}\|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}} + \|\widehat{\gamma}_0\|^4_{2,\hat{\Omega}})(\|\vartheta_{0t}\|^2_{1,\Omega_t} + \|v\|^2_{2,\Omega_t})\Big].$$

Differentiating the third component of (3.76) and next (3.68) with respect to n and τ we have respectively

$$\begin{aligned}(3.135)\quad &\|(\operatorname{div}\widetilde{u})_{,nn\tau}\|^2_{0,\hat{\Omega}}\\ &\le(\varepsilon+cd)\|\widetilde{u}_{zzzz}\|^2_{0,\hat{\Omega}}+C_1(\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\hat{\Omega}}+\|\widetilde{\widetilde{\eta}}_{\Omega_t nn\tau}\|^2_{0,\hat{\Omega}}\\ &\quad+|\widehat{u}|^2_{3,2,\hat{\Omega}}+\|\widehat{\eta}_{\sigma z}\|^2_{0,\hat{\Omega}}+\|\widehat{\widehat{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}}+\|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}}+\|\widehat{\gamma}\|^2_{0,\hat{\Omega}})\\ &\quad+C_2\Big(X_7(\widehat{\Omega})+\int_0^t\|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\end{aligned}$$

and

$$\begin{aligned}(3.136)\quad &\|\widetilde{u}_{nnn\tau}\|^2_{0,\hat{\Omega}}\\ &\le(\varepsilon+cd)(\|\widetilde{u}_{zzzz}\|^2_{0,\hat{\Omega}}+\|\widetilde{\widetilde{\eta}}_{\Omega_t zzz}\|^2_{0,\hat{\Omega}})\\ &\quad+C_1(\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\hat{\Omega}}+\|(\operatorname{div}\widetilde{u})_{,zn\tau}\|^2_{0,\hat{\Omega}}+\|\widehat{\widetilde{\eta}}_{\Omega_t zn\tau}\|^2_{0,\hat{\Omega}}+|\widehat{u}|^2_{3,2,\hat{\Omega}}\\ &\quad+\|\widehat{\eta}_{\sigma z}\|^2_{1,\hat{\Omega}}+\|\widehat{\widehat{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}}+\|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}}+\|\widehat{\gamma}\|^2_{0,\hat{\Omega}}+\|\widetilde{g}\|^2_{2,\hat{\Omega}})\\ &\quad+C_2\Big(X_7(\widehat{\Omega})+\int_0^t\|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega}).\end{aligned}$$

Now, we rewrite equation (3.84) as

$$\begin{aligned}(3.137)\quad -\kappa\Delta\widetilde{\gamma}=&-\widehat{\eta}c_v\widetilde{\gamma}_t+\kappa\widehat{\nabla}^2\widetilde{\gamma}-\kappa\Delta\widetilde{\gamma}\\ &-\widehat{\Gamma}p_{\hat{\Gamma}}\widehat{\nabla}\cdot\widetilde{u}+\widehat{\eta}\widetilde{k}+k_6.\end{aligned}$$

Differentiating (3.137) with respect to n and τ and multiplying the result by $\widetilde{\gamma}_{nnn\tau}J$ we have

$$\begin{aligned}(3.138)\quad \|\widetilde{\gamma}_{nnn\tau}\|^2_{0,\hat{\Omega}}&\le(\varepsilon+cd)\|\widetilde{\gamma}_{zzzz}\|^2_{0,\hat{\Omega}}\\ &\quad+C_1(\|\widetilde{\gamma}_{zz\tau\tau}\|^2_{0,\hat{\Omega}}+\|\widehat{u}\|^2_{3,\hat{\Omega}}+\|\widehat{\eta}_{\sigma z}\|^2_{1,\hat{\Omega}}\\ &\quad+\|\widehat{\widehat{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}}+\|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}}+\|\widehat{\gamma}_{0t}\|^2_{2,\hat{\Omega}}+\|\widehat{\gamma}\|^2_{0,\hat{\Omega}}\\ &\quad+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widetilde{k}\|^2_{2,\hat{\Omega}})\\ &\quad+C_2\Big[\Big(X_7(\widehat{\Omega})+\int_0^t\|\widehat{u}\|^2_{3,\hat{\Omega}}\,dt'\Big)(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\\ &\quad+(\|\widehat{\eta}_\sigma\|^2_{2,\hat{\Omega}}+\|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}}+\|\widehat{\eta}_\sigma\|^4_{2,\hat{\Omega}}\\ &\quad+\|\widehat{\eta}_\sigma\|^2_{2,\hat{\Omega}}\|\widehat{\gamma}_0\|^2_{2,\hat{\Omega}}+\|\widehat{\gamma}_0\|^4_{2,\hat{\Omega}})(\|\vartheta_{0t}\|^2_{1,\Omega_t}+\|v\|^2_{2,\Omega_t})\Big].\end{aligned}$$

Next, differentiating the third components of problems (3.64), (3.76), (3.68) and problem (3.137) twice with respect to n we get the estimates for

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\hat\Omega}\frac{p_{\sigma\hat\eta}}{\widehat\eta}\widetilde{\widetilde\eta}_{\Omega_tnnn}J\,dz+\|\widetilde{\widetilde\eta}_{\Omega_tnnn}\|^2_{0,\hat\Omega},\quad \|(\operatorname{div}\widetilde u)_{,nnn}\|^2_{0,\hat\Omega},$$

$$\|\widetilde u_{nnnn}\|^2_{0,\hat\Omega}\quad\text{and}\quad\|\widetilde\gamma_{nnnn}\|^2_{0,\hat\Omega},$$

which are analogous to (3.134)–(3.136) and (3.138), respectively.

Finally, we have

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\hat\Omega}\widehat\eta\widetilde u^2_{zzz}J\,dz\le\varepsilon\|\widetilde u_{zzzt}\|^2_{0,\hat\Omega}+C_1\|\widetilde u\|^2_{3,\hat\Omega}\tag{3.139}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\hat\Omega}\frac{\widehat\eta c_v}{\widehat\Gamma}\widetilde\gamma^2_{zzz}J\,dz&\le\varepsilon\|\widetilde\gamma_{zzzt}\|^2_{0,\hat\Omega}+C_1(\|\widehat\gamma_{0z}\|^2_{2,\hat\Omega}+\|\widehat\gamma\|^2_{0,\hat\Omega})\\&\quad+C_2[(|\widehat\eta_\sigma|^2_{2,1,\hat\Omega}+|\widehat\gamma_0|^2_{2,1,\hat\Omega}+\|\widehat u\|^2_{2,\hat\Omega})\|\widehat\gamma\|^2_{4,\hat\Omega}].\end{aligned}\tag{3.140}$$

The above considerations yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{\hat\Omega}\Big(\widehat\eta\widetilde u^2_{zzz}+\frac{p_{\sigma\hat\eta}}{\widehat\eta}\widetilde\eta^2_{\sigma zzz}+\frac{\widehat\eta c_v}{\widehat\Gamma}\widetilde\gamma^2_{zzz}\Big)J\,dz\\&\quad+c_0(\|\widetilde u_{zzz}\|^2_{1,\hat\Omega}+\|\widetilde{\widetilde\eta}_{\Omega_tzzz}\|^2_{0,\hat\Omega}+\|\widetilde\gamma_{zzzz}\|^2_{0,\hat\Omega})\\&\le(\varepsilon+cd)(\|\widetilde u_{zzzz}\|^2_{0,\hat\Omega}+\|\widetilde{\widetilde\eta}_{\Omega_tzzz}\|^2_{0,\hat\Omega}+\|\widetilde\gamma_{zzzz}\|^2_{0,\hat\Omega})\\&\quad+\varepsilon\|\widetilde u_{zzzt}\|^2_{0,\hat\Omega}+C_1(|\widehat u|^2_{3,2,\hat\Omega}+\|\widehat\gamma_{0z}\|^2_{2,\hat\Omega}+\|\widehat\gamma\|^2_{0,\hat\Omega}+\|\widehat\eta_{\sigma z}\|^2_{1,\hat\Omega}\\&\quad+\|\widehat{\widehat\eta}_{\Omega_t}\|^2_{0,\hat\Omega}+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}+\|\widetilde g\|^2_{2,\hat\Omega}+\|\widetilde k\|^2_{2,\hat\Omega}+\|\widetilde\Gamma_1\|^2_{4,\hat\Omega})\\&\quad+C_2\Big[\Big(X_7(\widehat\Omega)+\int_0^t\|\widehat u\|^2_{3,\hat\Omega}\,dt'\Big)(1+X_7^2(\widehat\Omega))Y_7(\widehat\Omega)\\&\quad+(\|\widehat\eta_\sigma\|^2_{2,\hat\Omega}+\|\widehat\gamma_0\|^2_{2,\hat\Omega}+\|\widehat\eta_\sigma\|^4_{2,\hat\Omega}\\&\quad+\|\widehat\eta_\sigma\|^2_{2,\hat\Omega}\|\widehat\gamma_0\|^2_{2,\hat\Omega}+\|\widehat\gamma_0\|^2_{2,\hat\Omega})(\|\vartheta_{0t}\|^2_{1,\Omega_t}+\|v\|^2_{2,\Omega_t})\Big].\end{aligned}\tag{3.141}$$

By estimates (3.126) and (3.141) we obtain the assertion of the lemma. ■

In order to estimate the first term on the right-hand side of (3.125) we need the following lemma.

Lemma 3.10. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of prob-*

lem (3.3). *Then*

$$(3.142)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\Big(\varrho v_{xxt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma xxt}^2+\frac{\varrho c_v}{\theta}\vartheta_{0xxt}^2\Big)dx$$
$$+c_0(\|v_{xxt}\|_{1,\Omega_t}^2+\|\varrho_{\sigma xxt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxxt}\|_{0,\Omega_t}^2)$$
$$\le\varepsilon(\|v_{xxtt}\|_{0,\Omega_t}^2+\|v_{xxxx}\|_{0,\Omega_t}^2+\|\vartheta_{0xxxx}\|_{0,\Omega_t}^2+\|\vartheta_{0xxtt}\|_{0,\Omega_t}^2)$$
$$+C_1(|v|_{3,1,\Omega_t}^2+\|\varrho_{\sigma x}\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{2,\Omega_t}^2$$
$$+\|\vartheta_{0t}\|_{2,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2+|f|_{2,1,\Omega_t}^2+|r|_{2,1,\Omega_t}^2+\|\theta_{1t}\|_{3,\Omega_t}^2+\|\theta_1\|_{3,\Omega_t}^2)$$
$$+C_2\Big(X_8+\int\limits_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1+X_8^2)Y_8,$$

where

$$X_8=|v|_{3,2,\Omega_t}^2+|\varrho_\sigma|_{3,1,\Omega_t}^2+|\vartheta_0|_{3,1,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2,$$
$$Y_8=|v|_{4,3,\Omega_t}^2+|\varrho_\sigma|_{3,1,\Omega_t}^2+\|\vartheta_{0x}\|_{3,\Omega_t}^2+\|\vartheta_{0t}\|_{3,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2.$$

P r o o f. The proof is analogous to the proofs of Lemmas 3.4–3.6 and 3.9. ■

To estimate the first term on the right-hand side of (3.142) we need the following result.

LEMMA 3.11. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of problem* (3.3). *Then*

$$(3.143)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\Big(\varrho v_{xtt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma xtt}^2+\frac{\varrho c_v}{\theta}\vartheta_{0xtt}^2\Big)dx$$
$$+c_0(\|v_{ttx}\|_{1,\Omega_t}^2+\|\varrho_{\sigma tt}\|_{1,\Omega_t}^2+\|\vartheta_{0ttxx}\|_{0,\Omega_t}^2)$$
$$\le\varepsilon(\|v_{xxxt}\|_{0,\Omega_t}^2+\|v_{xttt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxxt}\|_{0,\Omega_t}^2+\|\vartheta_{0xtt}\|_{0,\Omega_t}^2)$$
$$+C_1(|v|_{3,0,\Omega_t}^2+\|\varrho_{\sigma x}\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{1,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2$$
$$+\|\vartheta_{0x}\|_{2,\Omega_t}^2+|\vartheta_{0t}|_{2,0,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2+|f|_{2,0,\Omega_t}^2$$
$$+|r|_{2,0,\Omega_t}^2+\|\theta_{1tt}\|_{2,\Omega_t}^2+\|\theta_{1t}\|_{2,\Omega_t}^2+\|\theta_1\|_{2,\Omega_t}^2)$$
$$+C_2\Big(X_9+\int\limits_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1+X_9^2)Y_9,$$

where

$$X_9=|v|_{3,0,\Omega_t}^2+|\varrho_\sigma|_{3,0,\Omega_t}^2+|\vartheta_0|_{3,0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2,$$
$$Y_9=|v|_{4,1,\Omega_t}^2+|\varrho_\sigma|_{3,1,\Omega_t}^2+|\vartheta_{0t}|_{3,1,\Omega_t}^2+\|\vartheta_{0x}\|_{3,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2.$$

Proof. We use the partition of unity. Differentiating $(3.46)_1$ and $(3.46)_3$ twice with respect to t and once with respect to ξ, multiplying the results by $\widetilde{u}_{tt\xi}A$ and $\widetilde{\gamma}_{tt\xi}A$, respectively and next integrating the result over $\widetilde{\Omega}$ yields

$$\begin{aligned}(3.144)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\tilde{\Omega}}\left(\eta\widetilde{u}^2_{tt\xi}+\frac{p_{\sigma\eta}}{\eta}\widetilde{\overline{\eta}}^2_{\Omega_t tt\xi}+\frac{\eta c_v}{\Gamma}\widetilde{\gamma}^2_{tt\xi}\right)A\,d\xi\\ &\quad+c_0(\|\widetilde{u}_{tt}\|^2_{2,\tilde{\Omega}}+\|\widetilde{\overline{\eta}}_{\Omega_t tt}\|^2_{1,\tilde{\Omega}}+\|\widetilde{\gamma}_{tt\xi\xi}\|^2_{0,\tilde{\Omega}})\\ &\le\varepsilon\|\vartheta_{0xttt}\|^2_{0,\Omega_t}+C_1(|u|^2_{3,0,\tilde{\Omega}}+\|\gamma\|^2_{0,\tilde{\Omega}}+\|\gamma_{0\xi}\|^2_{2,\tilde{\Omega}}+|\gamma_{0t}|^2_{2,1,\tilde{\Omega}}\\ &\quad+\|\overline{\eta}_{\Omega_t}\|^2_{0,\tilde{\Omega}}+|\eta_\sigma|^2_{1,0,\tilde{\Omega}}+|\widetilde{g}|^2_{2,0,\tilde{\Omega}}+|\widetilde{k}|^2_{2,0,\tilde{\Omega}}\\ &\quad+\|v_{tt}\|^2_{1,\Omega_t}+\|\vartheta_{0tt}\|^2_{1,\Omega_t}+|r|^2_{2,0,\Omega_t}+\|\theta_{1tt}\|^2_{1,\Omega_t})\\ &\quad+C_2X_9(1+X_9^2)Y_9,\end{aligned}$$

where we have used equation (3.52), Lemma 5.4 of [21], the Stokes problem (3.57) and the estimate

$$\begin{aligned}(3.145)\quad \|\vartheta_{0ttt}\|^2_{0,\Omega_t}&\le\varepsilon\|\vartheta_{0xttt}\|^2_{0,\Omega_t}\\ &\quad+C_1(\|v_{tt}\|^2_{1,\Omega_t}+\|\vartheta_{0tt}\|^2_{1,\Omega_t}+|r|^2_{2,0,\Omega_t}+\|\theta_{1tt}\|^2_{1,\Omega_t})\\ &\quad+C_2X_9(1+X_9^2)Y_9.\end{aligned}$$

For boundary subdomains we have

$$\begin{aligned}(3.146)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\left(\widehat{\eta}\widetilde{u}^2_{tt\tau}+\frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}^2_{\Omega_t tt\tau}+\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}^2_{tt\tau}\right)J\,dz\\ &\quad+c_0(\|\widetilde{u}_{tt}\|^2_{2,\hat{\Omega}}+\|\widetilde{\gamma}_{tt\tau z}\|^2_{0,\hat{\Omega}})\\ &\le\varepsilon(\|\widetilde{u}_{ttzz}\|^2_{0,\hat{\Omega}}+\|\widetilde{\overline{\eta}}_{\Omega_t ttz}\|^2_{0,\hat{\Omega}}+\|\widetilde{\gamma}_{ttzz}\|^2_{0,\hat{\Omega}}+\|\vartheta_{0xttt}\|^2_{0,\Omega_t})\\ &\quad+C_1(|\widehat{u}|^2_{3,0,\hat{\Omega}}+\|\widehat{\gamma}\|_{0,\hat{\Omega}}+\|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}}+|\widehat{\gamma}_{0t}|^2_{2,0,\hat{\Omega}}+\|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}}\\ &\quad+|\widehat{\overline{\eta}}_{\Omega_t}|^2_{1,0,\hat{\Omega}}+|\widetilde{g}|^2_{2,0,\hat{\Omega}}+|\widetilde{k}|^2_{2,0,\hat{\Omega}}+\|\widetilde{\Gamma}_{1tt}\|^2_{2,\hat{\Omega}}+\|\widetilde{\Gamma}_{1t}\|^2_{2,\hat{\Omega}}\\ &\quad+\|\widetilde{\Gamma}_1\|^2_{2,\hat{\Omega}}+\|v_{tt}\|^2_{1,\Omega_t}+\|v_t\|^2_{2,\Omega_t}+\|v\|^2_{2,\Omega_t}+|\vartheta_{0t}|^2_{2,0,\Omega_t}+\|\vartheta_{0x}\|^2_{2,\Omega_t}\\ &\quad+\|\vartheta\|^2_{0,\Omega_t}+|r|^2_{2,0,\Omega_t}+\|\theta_{1tt}\|^2_{1,\Omega_t}+\|\theta_{1t}\|^2_{2,\Omega_t}+\|\theta_1\|^2_{2,\Omega_t})\\ &\quad+C_2\Big(X_9+\int\limits_0^t\|v\|^2_{4,\Omega_{t'}}\,dt'\Big)(1+X_9^2)Y_9,\end{aligned}$$

where we have used the boundary conditions $(3.47)_4$ and $(3.47)_5$.

Differentiating the third component of (3.64) twice with respect to t, multiplying the result by $\widetilde{\overline{\eta}}_{\Omega_t ntt} J$ and integrating over $\widehat{\Omega}$ implies

$$
(3.147)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}} \frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}} \widetilde{\overline{\eta}}_{\Omega_t ntt} J\,dz + c_0\|\widetilde{\overline{\eta}}_{\Omega_t ntt}\|_{0,\hat{\Omega}}^2
\\ \le (\varepsilon + cd)(\|\widetilde{u}_{zztt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t ztt}\|_{0,\hat{\Omega}}^2) + \varepsilon\|\vartheta_{0xttt}\|_{0,\Omega_t}^2
\\ + C_1(\|\widetilde{u}_{z\tau tt}\|_{0,\hat{\Omega}}^2 + |\widehat{u}|_{3,0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{1,\hat{\Omega}}^2 + |\widehat{\eta}_{\sigma t}|_{1,0,\hat{\Omega}}^2
\\ + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{2,\hat{\Omega}}^2 + |\widehat{\gamma}_{0t}|_{2,1,\hat{\Omega}}^2 + |\widetilde{g}|_{2,0,\hat{\Omega}}^2 + \|v_{tt}\|_{1,\Omega_t}^2
\\ + \|\vartheta_{0tt}\|_{1,\Omega_t}^2 + |r|_{2,0,\Omega_t}^2 + \|\theta_{1tt}\|_{1,\Omega_t}^2) + C_2X_9(1+X_9^2)Y_9.
$$

Next, differentiating the third component of (3.66) twice with respect to t, multiplying the result by $\widetilde{u}_{3ntt}J$ and integrating over $\widehat{\Omega}$ implies

$$
(3.148)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}} \widehat{\eta}|\widetilde{u}_{3ntt}|^2 J\,dz + c_0\|\widetilde{u}_{3nntt}\|_{0,\hat{\Omega}}^2
\\ \le (\varepsilon + cd)(\|\widetilde{u}_{zztt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t ztt}\|_{0,\hat{\Omega}}^2) + \varepsilon\|\widetilde{u}_{zttt}\|_{0,\hat{\Omega}}^2
\\ + C_1(\|\widetilde{u}_{z\tau tt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t ntt}\|_{0,\hat{\Omega}}^2 + |\widehat{u}|_{3,0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{1,\hat{\Omega}}^2 + |\widehat{\eta}_{\sigma t}|_{1,0,\hat{\Omega}}^2
\\ + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{2,\hat{\Omega}}^2 + |\widehat{\gamma}_{0t}|_{2,1,\Omega_t}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + |\widetilde{g}|_{2,0,\hat{\Omega}}^2)
\\ + C_2X_9(1+X_9^2)Y_9.
$$

From (3.68)–(3.70) we have

$$
(3.149)\quad \|\widetilde{u}'_{z\tau tt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t \tau tt}\|_{0,\hat{\Omega}}^2
\\ \le (\varepsilon + cd)(\|\widetilde{u}_{zztt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t ztt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{u}_{zttt}\|_{0,\hat{\Omega}}^2)
\\ + C_1(\|\mathrm{div}\,\widetilde{u}_{tt}\|_{1,\hat{\Omega}}^2 + |\widehat{u}|_{3,0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{1,\hat{\Omega}}^2 + |\widehat{\eta}_{\sigma t}|_{1,0,\hat{\Omega}}^2
\\ + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{2,\hat{\Omega}}^2 + |\widehat{\gamma}_{0t}|_{2,1,\hat{\Omega}}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + |\widetilde{g}|_{2,0,\hat{\Omega}}^2)
\\ + C_2X_9(1+X_9^2)Y_9.
$$

Next, from (3.68) it follows that

$$
(3.150)\quad \|\widetilde{u}'_{nntt}\|_{0,\hat{\Omega}}^2
\\ \le (\varepsilon + cd)(\|\widetilde{u}_{zztt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t ztt}\|_{0,\hat{\Omega}}^2)
\\ + C_1(\|(\mathrm{div}\,\widetilde{u})_{,\tau tt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t \tau tt}\|_{0,\hat{\Omega}}^2 + |\widehat{u}|_{3,0,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{1,\hat{\Omega}}^2 + |\widehat{\eta}_{\sigma t}|_{1,0,\hat{\Omega}}^2
\\ + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{2,\hat{\Omega}}^2 + |\widehat{\gamma}_{0t}|_{2,1,\hat{\Omega}}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + |\widetilde{g}|_{2,0,\hat{\Omega}}^2)
\\ + C_2X_9(1+X_9^2)Y_9.
$$

Dividing (3.84) by $\widehat{\Gamma}$, differentiating twice with respect to t, multiplying the result by $\widetilde{\gamma}_{nntt}J$ and integrating over $\widehat{\Omega}$ we get

$$
\begin{aligned}
(3.151)\quad & \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{ntt}^2 J\,dz + \frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{nntt}\|_{0,\hat{\Omega}}^2 \\
& \le (\varepsilon + cd)(\|\widetilde{\gamma}_{zztt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t zzt}\|_{0,\hat{\Omega}}^2) + \varepsilon(\|\widetilde{\gamma}_{zttt}\|_{0,\hat{\Omega}}^2 + \|\vartheta_{0xttt}\|_{0,\Omega_t}^2) \\
& \quad + C_1(\|\widetilde{\gamma}_{z\tau tt}\|_{0,\hat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{2,\hat{\Omega}}^2 + |\widehat{\gamma}_{0t}|_{2,1,\hat{\Omega}}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + |\widehat{u}|_{3,1,\hat{\Omega}}^2 \\
& \quad + \|\widehat{\eta}_{\sigma z}\|_{1,\hat{\Omega}}^2 + |\widehat{\eta}_{\sigma t}|_{1,0,\hat{\Omega}}^2 + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 + |\widetilde{k}|_{2,0,\hat{\Omega}}^2 \\
& \quad + \|v_{tt}\|_{1,\Omega_t}^2 + \|\vartheta_{0tt}\|_{1,\Omega_t}^2 + |r|_{2,0,\Omega_t}^2 + \|\theta_{1tt}\|_{1,\Omega_t}^2) \\
& \quad + C_2X_9(1+X_9^2)Y_9.
\end{aligned}
$$

Next, using (3.137) we get

$$
\begin{aligned}
(3.152)\quad \|\widetilde{\gamma}_{z\tau tt}\|_{0,\hat{\Omega}}^2 & \le (\varepsilon + cd)(\|\widetilde{u}_{zztt}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\gamma}_{zztt}\|_{0,\hat{\Omega}}^2) + \varepsilon\|\widetilde{\gamma}_{zttt}\|_{0,\hat{\Omega}}^2 \\
& \quad + C_1(|\widehat{u}|_{3,1,\hat{\Omega}}^2 + \|\widehat{\eta}_{\sigma z}\|_{1,\hat{\Omega}}^2 + |\widehat{\eta}_{\sigma t}|_{1,0,\hat{\Omega}}^2 + \|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\hat{\Omega}}^2 \\
& \quad + \|\widehat{\gamma}_{0z}\|_{2,\hat{\Omega}}^2 + |\widehat{\gamma}_{0t}|_{2,1,\hat{\Omega}}^2 + \|\widehat{\gamma}\|_{0,\hat{\Omega}}^2 + |\widetilde{k}|_{2,0,\hat{\Omega}}^2 \\
& \quad + \|v_{tt}\|_{1,\Omega_t}^2 + \|\vartheta_{0tt}\|_{1,\Omega_t}^2 + |r|_{2,0,\Omega_t}^2 + \|\theta_{1tt}\|_{1,\Omega_t}^2) \\
& \quad + C_2X_9(1+X_9^2)Y_9.
\end{aligned}
$$

Finally, we have

$$
(3.153)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\widehat{\eta}\widetilde{u}_{ztt}^2 J\,dz \le \varepsilon\|\widetilde{u}_{zttt}\|_{0,\hat{\Omega}}^2 + C_1\|\widetilde{u}_{ztt}\|_{0,\hat{\Omega}}^2
$$

and

$$
\begin{aligned}
(3.154)\quad & \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{ztt}^2 J\,dz \le \varepsilon(\|\widetilde{\gamma}_{zttt}\|_{0,\hat{\Omega}}^2 + \|\vartheta_{0xttt}\|_{0,\hat{\Omega}}^2) \\
& \quad + C_1(\|\widehat{\gamma}_{0zz}\|_{0,\hat{\Omega}}^2 + \|v_{tt}\|_{1,\Omega_t}^2 + \|\vartheta_{0tt}\|_{1,\Omega_t}^2 + |r|_{2,0,\Omega_t}^2 + \|\theta_{1tt}\|_{1,\Omega_t}^2) \\
& \quad + C_2X_9(1+X_9^2)Y_9.
\end{aligned}
$$

By estimates (3.146)–(3.154) we get

$$
\begin{aligned}
(3.155)\quad & \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\bigg(\widehat{\eta}\widetilde{u}_{ztt}^2 + \frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t ztt}^2 + \frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{ztt}^2\bigg)J\,dz \\
& + c_0(\|\widetilde{u}_{tt}\|_{2,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t tt}\|_{1,\hat{\Omega}}^2 + \|\widetilde{\gamma}_{zztt}\|_{0,\hat{\Omega}}^2) \\
& \le \varepsilon(\|\widetilde{u}_{ttzz}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\overline{\eta}}_{\Omega_t ttz}\|_{0,\hat{\Omega}}^2 + \|\widetilde{\gamma}_{ttzz}\|_{0,\hat{\Omega}}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\widetilde{u}_{zttt}\|^2_{0,\hat{\Omega}} + \|\widetilde{\gamma}_{zttt}\|^2_{0,\hat{\Omega}} + \|\vartheta_{0xttt}\|^2_{0,\Omega_t}) \\
&+ C_1(|\widehat{u}|^2_{3,0,\hat{\Omega}} + \|\widehat{\gamma}\|^2_{0,\hat{\Omega}} + \|\widehat{\gamma}_{0z}\|^2_{2,\hat{\Omega}} + |\widehat{\gamma}_{0t}|^2_{2,0,\hat{\Omega}} + \|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\hat{\Omega}} \\
&+ |\widehat{\overline{\eta}}_{\Omega_t}|^2_{1,0,\Omega_t} + |\widetilde{g}|^2_{2,0,\hat{\Omega}} + |\widetilde{k}|^2_{2,0,\hat{\Omega}} + \|\widetilde{\Gamma}_1\|^2_{2,\hat{\Omega}} + \|\widetilde{\Gamma}_{1t}\|^2_{2,\hat{\Omega}} \\
&+ \|\widetilde{\Gamma}_{1tt}\|^2_{2,\hat{\Omega}} + \|v_{tt}\|^2_{1,\Omega_t} + \|v\|^2_{2,\Omega_t} + |v_t|^2_{2,1,\Omega_t} + |\vartheta_{0t}|^2_{2,0,\Omega_t} + \|\vartheta_{0x}\|^2_{2,\Omega_t} \\
&+ \|\vartheta\|^2_{0,\Omega_t} + |r|^2_{2,0,\Omega_t} + \|\theta_{1tt}\|^2_{1,\Omega_t} + \|\theta_{1t}\|^2_{2,\Omega_t} + \|\theta_1\|^2_{2,\Omega_t}) \\
&+ C_2\Big(X_9 + \int_0^t \|v\|^2_{4,\Omega_{t'}}\,dt'\Big)(1 + X_9^2)Y_9.
\end{aligned}
$$

Inequalities (3.144) and (3.155) yield the assertion of the lemma. ∎

LEMMA 3.12. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of problem* (3.3). *Then*

$$
\begin{aligned}
(3.156)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\Big(\varrho v_{ttt}^2 + \frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma ttt}^2 + \frac{\varrho c_v}{\theta}\vartheta_{0ttt}^2\Big)\,dx \\
&\quad + c_0(\|v_{ttt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma ttt}\|^2_{0,\Omega_t} + \|\vartheta_{0ttt}\|^2_{1,\Omega_t}) \\
&\le C_1(\|v_{tt}\|^2_{1,\Omega_t} + \|\vartheta_{0tt}\|^2_{1,\Omega_t} + \|f_{ttt}\|^2_{0,\Omega_t} + |f|^2_{2,0,\Omega_t} \\
&\quad + \|r_{ttt}\|^2_{0,\Omega_t} + |r|^2_{2,0,\Omega_t} + |\theta_1|^2_{4,1,\Omega_t}) \\
&\quad + C_2 X_{10}(1 + X_{10}^3)Y_{10},
\end{aligned}
$$

where

$$
X_{10} = |v|^2_{3,0,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t} + |\vartheta_0|^2_{3,0,\Omega_t}, \quad Y_{10} = |v|^2_{4,1,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t} + |\vartheta_0|^2_{4,1,\Omega_t}.
$$

Proof. Differentiating $(3.3)_1$ three times with respect to t, multiplying the result by v_{ttt} and integrating over Ω_t yields

$$
\begin{aligned}
(3.157)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\varrho v_{ttt}^2\,dx + \frac{1}{2}\mu E_{\Omega_t}(v_{ttt}) + (\nu - \mu)\|\mathrm{div}\, v_{ttt}\|^2_{0,\Omega_t} \\
&\quad - \int_{\Omega_t} p_{\sigma\varrho}\varrho_{\sigma ttt}\,\mathrm{div}\, v_{ttt}\,dx - \int_{\Omega_t} p_{\varrho\theta}\vartheta_{0ttt}\,\mathrm{div}\, v_{ttt}\,dx \\
&\quad - \int_{S_t} (n_i T^{ij}(v, \varrho_\sigma))_{,ttt} v_{ittt}\,ds \\
&\le \varepsilon(\|v_{ttt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma ttt}\|^2_{0,\Omega_t}) \\
&\quad + C_1(|f|^2_{2,0,\Omega_t} + \|f_{ttt}\|^2_{0,\Omega_t}) + C_2 X_{10}(1 + X_{10}^3)Y_{10}.
\end{aligned}
$$

Next, dividing $(3.3)_3$ by θ, differentiating three times with respect to t, multiplying by ϑ_{0ttt} and integrating over Ω_t gives

$$\begin{aligned}(3.158)\quad &\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\frac{\varrho c_v}{\theta}\vartheta_{0ttt}^2\,dx+\frac{\kappa}{\theta^*}\int_{\Omega_t}|\nabla\vartheta_{0ttt}|^2\,dx\\ &\quad+\int_{\Omega_t}p_{\varrho\theta}\vartheta_{0ttt}\operatorname{div}v_{ttt}\,dx-\int_{S_t}\bigg(\frac{n\cdot\nabla\vartheta_0}{\theta}\bigg)_{,ttt}\vartheta_{0ttt}\,ds\\ &\le\varepsilon(\|v_{ttt}\|_{1,\Omega_t}^2+\|\varrho_{\sigma ttt}\|_{0,\Omega_t}^2+\|\vartheta_{0ttt}\|_{1,\Omega_t}^2)\\ &\quad+C_1(|r|_{2,0,\Omega_t}^2+\|r_{ttt}\|_{0,\Omega_t}^2+\|\theta_{1ttt}\|_{1,\Omega_t})+C_2X_{10}(1+X_{10}^3)Y_{10}.\end{aligned}$$

Now, using the continuity equation $(3.3)_2$, Lemma 5.5 of [21], inequality (3.145) and the estimate

$$(3.159)\qquad \|\varrho_{\sigma ttt}\|_{0,\Omega_t}^2\le c\|v_{tt}\|_{1,\Omega_t}^2+C_2X_{10}(1+X_{10}^2)Y_{10}$$

we obtain (3.156). ■

Estimating $\|\vartheta\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2$ by $\|\vartheta_{0x}\|_{0,\Omega_t}^2+\|p_\sigma\|_{0,\Omega_t}^2$ (by using (3.19) and (3.27)) from the above lemmas for sufficiently small ε we obtain

Theorem 3.13. *For a sufficiently smooth solution v, ϱ, ϑ_0 of* (3.3) *we have*

$$(3.160)\qquad \frac{d\overline{\phi}}{dt}+c_0\Phi\le c_1P(\phi)\Big(\phi+\int_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1+\phi^3)(\phi+\Phi)+c_2F+c_3\Psi,$$

where

$$\begin{aligned}\overline{\phi}(t)&=\int_{\Omega_t}\varrho\sum_{0\le|\alpha|+i\le3}|D_x^\alpha\partial_t^iv|^2\,dx\\ &\quad+\int_{\Omega_t}\bigg(\frac{p_1}{\varrho}\varrho_\sigma^2+\overline{\varrho}_{\Omega_t}^2+\frac{p_2\varrho c_v}{p_\theta\theta}\vartheta_0^2\bigg)\,dx\\ &\quad+\int_{\Omega_t}\frac{p_{\sigma\varrho}}{\varrho}\sum_{1\le|\alpha|+i\le3}|D_x^2\partial_t^i\varrho_\sigma|^2\,dx\\ (3.161)\qquad&\quad+\int_{\Omega_t}\frac{\varrho c_v}{\theta}\sum_{1\le|\alpha|+i\le3}|D_x^\alpha\partial_t^i\vartheta_0|^2\,dx,\\ \phi(t)&=|v|_{3,0,\Omega_t}^2+|\varrho_\sigma|_{3,0,\Omega_t}^2+|\vartheta_0|_{3,0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2,\\ \Phi(t)&=|v|_{4,1,\Omega_t}^2+|\varrho_\sigma|_{3,0,\Omega_t}^2-\|\varrho_\sigma\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2\\ &\quad+|\vartheta_0|_{4,1,\Omega_t}^2-\|\vartheta_0\|_{0,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2,\\ F(t)&=\|f_{ttt}\|_{0,\Omega_t}^2+|f|_{2,0,\Omega_t}^2+\|r_{ttt}\|_{0,\Omega_t}^2+|r|_{2,0,\Omega_t}^2\\ &\quad+\|r\|_{0,\Omega_t}+|\theta_1|_{4,1,\Omega_t}^2+\|\theta_1\|_{1,\Omega_t},\\ \Psi(t)&=\|v\|_{0,\Omega_t}^2+\|p_\sigma\|_{0,\Omega_t}^2,\end{aligned}$$

P is an increasing continuous function; $c_0 < 1$ *is a positive constant depending on* ϱ_*, ϱ^*, θ_*, θ^*, μ, ν, κ; *and* c_i $(i = 1, 2, 3)$ *are positive constants depending on* ϱ_*, ϱ^*, θ_*, θ^*, $\int_0^t \|v\|_{3,\Omega_{t'}}\, dt'$, $\|S\|_{4-1/2}$, T *and constants from the imbedding Lemma* 2.1 *and the Korn inequalities* (see Section 5 of [21]).

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