Starlikeness of functions satisfying a differential inequality

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Abstract. In a recent paper Fournier and Ruscheweyh established a theorem related to a certain functional. We extend their result differently, and then use it to obtain a precise upper bound on \( \alpha \) so that for \( f \) analytic in \( |z| < 1 \), \( f(0) = f'(0) - 1 = 0 \) and satisfying \( \text{Re}\{zf''(z)\} > -\lambda \), the function \( f \) is starlike.

1. Introduction and statement of results. Let \( U \) be the unit disk \( |z| < 1 \), and let \( \mathcal{H} \) be the space of analytic functions in \( U \) with the topology of local uniform convergence. The subclasses \( A \) and \( A_0 \) of \( \mathcal{H} \) consist of functions \( f \in \mathcal{H} \) such that \( f(0) = f'(0) - 1 = 0 \) and \( f(0) = 1 \) respectively. By \( S \), \( C \), \( St \) and \( K \) we denote, respectively, the well known subsets of \( A \) of univalent, close-to-convex, starlike (with respect to origin) and convex functions. Further, for \( \beta < 1 \), we introduce

\[ P_\beta = \{ f \in A_0 : \text{Re} f(z) > \beta, \ z \in U \} \]

and

\[ P_\beta = \{ f \in A : \exists \alpha \in \mathbb{R} \text{ such that } \text{Re}\{e^{i\alpha}(f'(z) - \beta)\} > 0, \ z \in U \}. \]

If \( f \) and \( g \) are in \( \mathcal{H} \) and have the power series

\[ f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} \beta_k z^k, \]

the convolution or Hadamard product of \( f \) and \( g \) is defined by

\[ h(z) = (f * g)(z) = \sum_{k=0}^{\infty} a_k \beta_k z^k. \]

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For \( V \subset A_0 \) the dual \( V^* \) of \( V \) is the set of functions \( g \in A_0 \) such that 
\((f \ast g)(z) \neq 0 \) for every \( f \in V \), and \( V^{**} = (V^*)^* \).

We define functions \( h_T \) in \( A \) by
\[
h_T(z) = \frac{1}{1+iT} \left[ iT \frac{z}{1-z} + \frac{z}{(1-z)^2} \right], \quad T \in \mathbb{R},
\]
and the subclass \( V_\beta \) of \( A_0 \) by
\[
V_\beta = \left\{ (1-\beta) \frac{1-xz}{1-yz} + \beta : |x| \leq 1, \ |y| \leq 1, \ \beta < 1 \right\}.
\]

We refer to [2, 3] for results in duality theory.

For a suitable \( \Lambda : [0, 1] \to \mathbb{R} \) define
\[
L_\Lambda(f) = \inf_{z \in U} \int_0^1 \Lambda(t) \left[ \text{Re} \frac{f(tz)}{tz} - \frac{1}{(1+t)^2} \right] dt, \quad f \in C,
\]
and
\[
L_\Lambda(C) = \inf_{f \in C} L_\Lambda(f).
\]

In a recent paper [1] Fournier and Ruscheweyh have established the following

**Theorem A.** Let \( \Lambda \) be integrable on \([0, 1]\) and positive on \((0, 1)\). If 
\( \Lambda(t)/(1-t^2) \) is decreasing on \((0, 1)\) then \( L_\Lambda(C) = 0 \).

The functions
\[
A_c(t) = \begin{cases} 
(1-t^c)/c, & -1 < c \leq 2, \ c \neq 0, \\
\log(1/t), & c = 0,
\end{cases}
\]
satisfy the conditions of Theorem A.

It is clear that Theorem A can be extended to the case of \( t\Lambda(t) \) integrable on \([0, 1]\), positive on \((0, 1)\), and \( t\Lambda(t)/(1-t^2) \) decreasing on \((0, 1)\). Indeed,
\[
\int_0^1 A(t) \text{Re} \left\{ \frac{h_T(tz)}{tz} - \frac{1}{(1+t)^2} \right\} dt
\]
\[
= \int_0^1 t\Lambda(t) \text{Re} \left\{ \frac{1}{1+iT} \left[ iTz \frac{1}{1-tz} + \frac{2-tz}{(1-tz)^2} \right] + \frac{2+t}{(1+t)^2} \right\} dt,
\]
which shows that integrability of \( t\Lambda(t) \) is enough for the existence of the integral. Further, if \( t\Lambda(t)/(1-t^2) \) is decreasing, so is \( \Lambda(t)/(1-t^2) \) and hence the treatment in [1] gives the result. Thus the functions
\[
A_c(t) = (1-t^c)/c, \quad -2 < c \leq -1,
\]
satisfy the above conditions.

In the present paper we extend Theorem A in the following form.
Theorem 1. For \( \Lambda \) not integrable on \([0, 1] \), let \( t\Lambda(t) \) be integrable on \([0, 1] \), positive on \((0, 1) \), and suppose
\[
\Lambda(t)/(1 - t^2) \text{ is decreasing on } (0, 1).
\]
Then \( L_\Lambda(C) = 0 \).

We use the theorem to establish the following:

Theorem 2. Suppose \( \alpha : [0, 1] \to \mathbb{R} \) is non-negative with \( \int_0^1 \alpha(t) \, dt = 1 \),
\[
\Lambda(t) = \frac{\int_0^1 \alpha(t)}{1 - t^2} \, dt
\]
satisfies the conditions of Theorem 1 and for \( \lambda > 0 \), define
\[
V_\alpha(f) = z \int_0^1 \left( 1 + \frac{\lambda z}{1 - t^2} \right) \alpha(t) \, dt * f(z), \quad f \in A.
\]
Then for \( \lambda \) given by
\[
2\lambda \int_0^1 \frac{\alpha(t)}{1 + t} \, dt = 1
\]
we have \( V_\alpha(P_0) \subset S \), and
\[
V_\alpha(P_0) \subset St \iff L_\Lambda(C) = 0.
\]
For any larger value of \( \lambda \) there exists an \( f \in P_0 \) with \( V_\alpha(f) \) not even locally univalent.

As a special case of the above theorem we obtain a result which is interesting enough to be stated as a theorem.

Theorem 3. If \( \lambda > 0 \) and \( f \in A \) satisfies the differential inequality
\[
\text{Re } zf''(z) > -\lambda,
\]
then \( f \in St \) if
\[
0 < \lambda \leq 1/\log 4.
\]
For any larger value of \( \lambda \), a function \( f \in A \) satisfying (3) need not even be locally univalent.

Theorem 4. Let \( \alpha : [0, 1] \to \mathbb{R} \) be non-negative with \( \int_0^1 \alpha(t) \, dt = 1 \) and suppose
\[
\Lambda(t) = \alpha(t)/t
\]
satisfies the conditions of Theorem 1. If \( V_\alpha(f) \) is defined by (1), then
\[
V_\alpha(P_0) \subset K \iff L_\Lambda(C) = 0
\]
and \( \lambda \) is given by
\[
2\lambda \int_0^1 \frac{2 + t}{(1 + t)^2} \alpha(t) \, dt = 1.
\]
2. Proof of Theorem 1. For a fixed $f \in C$ and $z \in U$ let
\[ tg(t) = \text{Re} \frac{f(tz)}{tz} = \frac{1}{(1+t)^2}. \]
Then $g$ is analytic in $t$. Let
\[ A_n(t) = \begin{cases} A(t), & 1/n \leq t \leq 1, \\ \frac{(1-t^2)A(1/n)}{1-1/n^2}, & 0 \leq t \leq 1/n. \end{cases} \]
From Theorem A we get
\[ 0 \leq \frac{n^2}{n^2 - 1} A\left(\frac{1}{n}\right) \int_0^{\frac{1}{n}} (1-t^2)tg(t) dt + \int_{\frac{1}{n}}^{1} tA(t)g(t) dt = H_n + G_n. \]
Now
\[ |H_n| \leq \frac{A(1/n)}{2(n^2 - 1)} M_1 \to 0 \text{ as } n \to \infty. \]
Let $\chi_n(t)$ be the characteristic function of $[1/n, 1]$. For each $n$,
\[ |tA(t)g(t)\chi_n(t)| \leq M_2 tA(t). \]
Since $tA(t)$ is integrable, it follows that
\[ \lim_{n \to \infty} G_n = \lim_{n \to \infty} \int_0^{1/n} tA(t)g(t)\chi_n(t) dt = \int_0^{1/n} tA(t)g(t) dt. \]
Hence $L_A(f) \geq 0$ for $z \in U$. This completes the proof.

We are thankful to Prof. S. Ruscheweyh for his help with the proof of Theorem 1.

3. Proof of Theorems 2 and 3. For $f \in P_0$ let $F(z) = V_\alpha(f)$. We then have
\[ F'(z) = \int_0^1 \left( 1 + \frac{\lambda z}{1-zt} \right) \alpha(t) dt * f'(z), \quad f \in P_0. \]
Since $V_0^* = P_{1/2}$ and $V_{0}^{**} = \{ f' : f \in P_0 \}$, $F'(z) \neq 0$ if and only if
\[ \frac{1}{2} < \text{Re} \int_0^1 \left( 1 + \frac{\lambda z}{1-zt} \right) \alpha(t) dt. \]
This gives
\[ \lambda \int_0^1 \frac{\alpha(t)}{1+t} dt \leq \frac{1}{2}. \]
Further, because $\text{Re} e^{i\alpha} f'(z) > 0$, (5) also ensures that $\text{Re} e^{i\alpha} F'(z) > 0$ and hence $F$ is univalent.
For starlikeness we use the easily verifiable property that $F \in A$ is in $St$ if and only if

\[(6) \quad \frac{1}{z}(F * h_T)(z) \neq 0, \quad T \in \mathbb{R}, \ z \in U.\]

This gives

\[
0 \neq \int_0^1 \left(1 + \frac{\lambda z}{1 - tz}\right) \alpha(t) dt * \frac{h_T(z)}{z} * \frac{f(z)}{z}
\]

\[
= \int_0^1 \left[1 + \frac{\lambda}{t} \left(\frac{1}{z} \int_0^z \left(\frac{h(tw)}{tw} - 1\right) dw\right)\right] \alpha(t) dt * f'(z), \quad f \in P_0.
\]

This implies that $F \in St$ if and only if

\[
\frac{1}{2} < \text{Re} \int_0^1 \left[1 + \frac{\lambda}{t} \left(\frac{1}{z} \int_0^z \left(\frac{h(tw)}{tw} - 1\right) dw\right)\right] \alpha(t) dt.
\]

On substituting the value of $\lambda$ from (2) in the above inequality, we obtain

\[
0 < \text{Re} \int_0^1 \frac{\alpha(t)}{t^2} \left(\frac{1}{z} \int_0^z \left(\frac{h(tw)}{tw} - \frac{t}{1+t}\right) dw\right) dt.
\]

This is similar to the last equation in [1]. Hence we need

\[
A(t) = \int_0^1 \frac{\alpha(t)}{t^2} dt
\]

in order to use Theorem A. This completes the proof.

For the proof of Theorem 3 we take $\alpha(t) \equiv 1$. Then $A(t) = 1/t - 1$ satisfies the conditions of Theorem 1 and $F$ satisfies (3). For $\alpha(t) \equiv 1$ the value of $\lambda$ obtained from (2) gives (4).

Notice that in (3), $\lambda = 0$ only if $f(z) \equiv z$. Thus functions of the form

\[
\varphi z + (1 - \varphi)f(z), \quad \varphi < 1,
\]

where $f$ satisfies (3), are in $St$ for $(1 - \varphi)\lambda \leq 1/\log 4$.

Further, if $f \in A$ satisfies (3), then for a non-negative $\alpha$ satisfying $\int_0^1 \alpha(t) dt = 1$, the functions

\[
\phi(z) = \int_0^1 \frac{\alpha(t)}{t} f(tz) dt
\]

also satisfy (3) and hence are starlike for the same value of $\lambda$. 
4. Proof of Theorem 4. We need to prove that \( zF'(z) \in St, F(z) = V_\alpha(f) \). Hence (6) gives

\[
0 \neq F'(z) \ast \frac{h_T(z)}{z} = \int_0^1 \left( 1 + \frac{\lambda z}{1 - tz} \right) \alpha(t) \, dt \ast \frac{h_T(z)}{z} \ast f'(z)
\]

\[
= \int_0^1 \left[ 1 + \frac{\lambda}{t} \left( \frac{h(tz)}{tz} - 1 \right) \right] \alpha(t) \, dt \ast f'(z), \quad f \in P_0.
\]

This holds if and only if

\[
\frac{1}{2} < \text{Re} \int_0^1 \left[ 1 + \frac{\lambda}{t} \left( \frac{h(tz)}{tz} - 1 \right) \right] \alpha(t) \, dt.
\]

Substitution of the value of \( \lambda \) in the theorem gives

\[
0 < \text{Re} \int_0^1 \left[ \frac{h(tz)}{tz} - \frac{1}{(1 + t)^2} \right] \alpha(t) \, dt.
\]

Hence with \( A(t) = \alpha(t)/t \), Theorem 1 gives the result.

The choice of \( \alpha(t) = 2(1 - t) \) gives the result of Theorem 3 with \( \lambda \) replaced by \( 2\lambda \).

References

