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# Resultant and the Łojasiewicz exponent

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**Abstract.** An effective formula for the Łojasiewicz exponent of a polynomial mapping of $\mathbb{C}^2$ into $\mathbb{C}^2$ at an isolated zero in terms of the resultant of its components is given.

**1. Introduction.** Let $H = (f, g) : U \to \mathbb{C}^2$, $0 \in U \subset \mathbb{C}^2$, be a holomorphic mapping having an isolated zero at the origin. The *Łojasiewicz exponent* of $H$ at 0 is the number

$$\mathcal{L}_0(H) = \inf \{\nu \in \mathbb{R} : \exists A > 0,\ \exists B > 0,\ \forall |z| < B,\ A|z|^\nu \le |H(z)|\},$$

where $|z| = \max(|x|, |y|)$ for $z = (x, y) \in \mathbb{C}^2$. This exponent plays an important role in the theory of singularities and has been studied by several authors. Information on this subject can be found in [$\mathrm{CK}_1$].

The aim of the present paper is to give an effective formula for $\mathcal{L}_0(H)$. The previous results have not given such possibilities. The formula obtained in [$\mathrm{CK}_1$] and, in another way, in [CP] needs parametrizations of the branches of the curve $\{fg = 0\}$, whereas the formula in [P] uses the characteristic polynomials of both $x$ and $y$ relative to $H$. The formulae in [LT] are not effective.

The main result of our paper is Theorem 3.1 which enables us to find $\mathcal{L}_0(H)$ effectively for a polynomial mapping $H$ in terms of the resultant. The restriction to polynomial mappings is inspired only by the wish of preparing a computer programme for calculating the Łojasiewicz exponent. A possible extension of Theorem 3.1 to the whole class of holomorphic mappings is given in Remark 3.4.

We follow our paper [$\mathrm{CK}_2$] in which an effective formula for the Łojasiewicz exponent at infinity was given.

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**2. Notations and definitions.** We use the same notations and definitions as in [CK$_1$], except for the symbol $\mathcal{L}_0(H)$.

**3. The main result.** Let $H = (f,g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping satisfying the following conditions:

(i) $H^{-1}(0)$ is a finite fibre,
(ii) $H(0,y) = 0$ if and only if $y = 0$,
(iii) $\deg_y f = \deg f(0,y)$ or $\deg_y g = \deg g(0,y)$.

Let $w = (x,y) \in \mathbb{C}^2$ be an arbitrary point and let $Q(w,x) = \mathrm{Res}_y(f(x,y) - u, g(x,y) - v)$ be the resultant of $f(x,y) - u$ and $g(x,y) - v$ with respect to $y$. Put

$$Q(w,x) = Q_N(w)x^N + \ldots + Q_0(w). \tag{1}$$

Since $H^{-1}(0)$ is finite, not all $Q_i$ vanish for $w = 0$. Since $Q_0(0) = 0$, there exists $\mu$, $1 \le \mu \le N$, such that $Q_0(0) = \ldots = Q_{\mu-1}(0) = 0$ and $Q_\mu(0) \ne 0$.

Let $H = (f,g)$ satisfy (i)–(iii).

(3.1) Theorem. *If*

(iv) $\mathrm{ord}\, f = \mathrm{ord}\, f(0,y)$ *and* $\mathrm{ord}\, g = \mathrm{ord}\, g(0,y)$,

*then*

$$\mathcal{L}_0(H) = \left[\min_{i=0}^{\mu-1} \frac{\mathrm{ord}\, Q_i}{\mu - i}\right]^{-1}$$

*and* $\mu$ *is the multiplicity of* $H$ *at* 0.

(3.2) Remark. Assumptions (ii)–(iv) have simple geometric interpretations. Condition (ii) means that $H$ has only one zero on the $y$-axis, at the origin. Condition (iii) means that the point at infinity lying on the $y$-axis does not belong to at least one of the curves $\{f = 0\}$ or $\{g = 0\}$. Finally, (iv) means that the $y$-axis is tangent at the origin neither to $\{f = 0\}$ nor to $\{g = 0\}$. The assumptions do not restrict our considerations because, under the general assumptions that $H$ has a finite number of zeros and $H(0) = 0$, one can get them by using a linear automorphism of the domain of $H$. $\mathcal{L}_0(H)$ is invariant with respect to such mappings.

(3.3) Remark. The following example shows that assumption (iv) cannot be weakened. Let $H(z) = (f(x,y), g(x,y)) = ((y^3 - x)^2, y^2x)$. One can easily find, by using the main theorem of [CK$_1$], that $\mathcal{L}_0(H) = 6$, whereas $Q(w,x) = x^{10} - 2ux^8 + u^2x^6 - 2v^3x^5 - 2uv^3x^3 + v^6$ and $\left[\min_{i=0}^{\mu-1} \frac{\mathrm{ord}\, Q_i}{\mu-i}\right]^{-1} = 2$.

(3.4) Remark. The formula in Theorem 3.1 can easily be generalized (the proof runs actually without any changes) to mappings whose components are pseudopolynomials. Namely, let $H = (f,g) : V \times \mathbb{C} \to \mathbb{C}^2$ where

$f(x,y)$, $g(x,y)$ are polynomials in $y$ with coefficients (functions of $x$) holomorphic in a neighbourhood $V$ of the origin in $\mathbb{C}$. Instead of (i), (ii) we assume that $H^{-1}(0)=\{0\}$, and instead of (iii) that at least one of $f$ and $g$ is monic with respect to $y$. For $w\in\mathbb{C}^2$, we define $Q(w,x)$ as in Theorem 3.1. Now, let

$$Q(w,x)=\sum_{j=0}^{\infty}Q_j(w)x^j,\qquad (w,x)\in\mathbb{C}^2\times V.$$

As before (see [C], Lemma 1), there exists $\mu\ge 1$ such that $Q_0(0)=\ldots=Q_{\mu-1}(0)=0$, $Q_\mu(0)\ne 0$, thus the theorem analogous to Theorem 3.1 holds.

This theorem can also be used for an arbitrary holomorphic mapping with an isolated zero. For if $\widetilde{H}=(\widetilde{f},\widetilde{g}):\widetilde{U}\to\mathbb{C}^2$, $0\in\widetilde{U}\subset\mathbb{C}^2$, is a holomorphic mapping having an isolated zero at the origin, then, using a linear automorphism of $\mathbb{C}^2$, we may assume that $\widetilde{f}$ and $\widetilde{g}$ are regular in $y$. Let $f,g$ be distinguished pseudopolynomials associated with $\widetilde{f},\widetilde{g}$, respectively, by the Weierstrass preparation theorem. Then $\mathcal{L}_0(H)=\mathcal{L}_0(\widetilde{H})$, where $H=(f,g)$ and satisfies the assumptions at the beginning of the remark.

**4. Auxiliary lemma.** Let $Q(w,x)=Q_N(w)x^N+\ldots+Q_0(w)$ be a polynomial with coefficients holomorphic in a neighbourhood of the origin in $\mathbb{C}^n$ and let $Q_0(0)=\ldots=Q_{\mu-1}(0)=0$, $Q_\mu(0)\ne 0$, $0<\mu\le N$. Put

$$\delta(Q)=\left[\min_{i=0}^{\mu-1}\frac{\operatorname{ord}Q_i}{\mu-i}\right]^{-1}. \tag{2}$$

(4.1) LEMMA. *$\delta(Q)$ is the least real number $\nu$ for which there exist positive numbers $A,B$ such that*

$$\{(w,x):|w|<B,\ Q(w,x)=0\}\subset\{(w,x):|w|<B,\ A|x|^\nu\le|w|\}.$$

Proof. By the Weierstrass preparation theorem, there exist $\varrho>0$ and a distinguished pseudopolynomial $P(w,x)$ of the form

$$P(w,x)=x^\mu+a_{\mu-1}(w)x^{\mu-1}+\ldots+a_0(w),\qquad a_i(0)=0,$$

such that, for $|w|<\varrho$, $|x|<\varrho$, we have

$$Q(w,x)=P(w,x)R(w,x),\qquad R(w,x)\ne 0. \tag{3}$$

From Lemma 8.1 of [CK$_2$] we have

$$\delta(Q)=\left[\min_{i=0}^{\mu-1}\frac{\operatorname{ord}a_i}{\mu-i}\right]^{-1}. \tag{4}$$

Now, we show that there exist $A,B>0$ such that

$$\{(w,x):|w|<B,\ Q(w,x)=0\}\subset\{(w,x):|w|<B,\ A|x|^{\delta(Q)}\le|w|\}. \tag{5}$$

Indeed, by Proposition 2.2 of [P] and by (4), there exist $A_1, B_1 > 0$ such that

$$\{(w,x) : |w| < B_1,\ P(w,x) = 0\} \subset \{(w,x) : |w| < B_1,\ A_1|x|^{\delta(Q)} \le |w|\}.$$

Hence and from (3) we get, for $\varrho < B_1$,

$$(6)\quad \{(w,x) : |w| < \varrho,\ |x| < \varrho,\ Q(w,x) = 0\} \subset \{(w,x) : |w| < \varrho,\ |x| < \varrho,\ A_1|x|^{\delta(Q)} \le |w|\}.$$

This gives (5) for $A = \min(A_1, \varrho^{-\delta(Q)+1})$ and $B = \varrho$.

It remains to show that if there exist $A, B > 0$ and $\nu \in \mathbb{R}$ such that

$$(7)\quad \{(w,x) : |w| < B,\ Q(w,x) = 0\} \subset \{(w,x) : |w| < B,\ A|x|^{\nu} \le |w|\},$$

then $\nu \ge \delta(Q)$. In fact, from (7) we get, for $\varrho < B$,

$$(8)\quad \{(w,x) : |w| < \varrho,\ |x| < \varrho,\ Q(w,x) = 0\} \subset \{(w,x) : |w| < \varrho,\ |x| < \varrho,\ A|x|^{\nu} \le |w|\}.$$

Take a sufficiently small $\varepsilon > 0$ such that all the roots of the equations $P(w,x) = 0$ for $|w| < \varepsilon$ lie in the disc $\{x : |x| < \varrho\}$. Then, from (8) we get

$$\{(w,x) : |w| < \varepsilon,\ P(w,x) = 0\} \subset \{(w,x) : |w| < \varepsilon,\ A|x|^{\nu} \le |w|\}.$$

Now, Lemma 2.4 of [P] and (4) yield $\nu \ge \delta(P)$.

**5. The set $N(H,x)$.** In the sequel, let $H$ be a polynomial mapping satisfying conditions (i)–(iii). We define

$$N(H,x) = \{\nu \in \mathbb{R} : \exists A > 0,\ \exists B > 0,\ \forall |x| < B,\ A|x|^{\nu} \le |H(z)|\},$$

where $z = (x,y)$.

Let $Q$ be defined as in (1) and $\delta(Q)$ as in (2).

(5.1) PROPOSITION. *$\delta(Q)$ is the least real number belonging to $N(H,x)$.*

Proof. From the property of the resultant we have $Q(H(z),x) \equiv 0$. Then, by Lemma 4.1, we have $\delta(Q) \in N(H,x)$.

Take now $\nu \in N(H,x)$. Then there exist $A, B > 0$ such that $A|x|^{\nu} \le |H(z)|$ for $|x| < B$. Take $w, x$ such that $|x| < B$ and $Q(w,x) = 0$. By the property of the resultant, there exists $z = (x,y)$ such that $w = H(z)$. Hence $A|x|^{\nu} \le |w|$. Then from Lemma 4.1 we get $\delta(Q) \le \nu$.

**6. Proof of Theorem 3.1.** We begin with a proposition following directly from the main theorem in [$CK_1$].

(6.1) PROPOSITION. *If $H = (f,g) : U \to \mathbb{C}^2$, $0 \in U \subset \mathbb{C}^2$, is a holomorphic mapping having an isolated zero at the origin, then*

(a) *there exist positive numbers $A, B$ such that*

$$A|z|^{\mathcal{L}_0(H)} \le |H(z)| \quad \textit{for } |z| < B,$$

(b) *if* $\operatorname{ord} f = \operatorname{ord} f(0,y)$ *and* $\operatorname{ord} g = \operatorname{ord} g(0,y)$, *then there exists a branch* $\Gamma$ *of the curve* $\{fg = 0\}$ *in a neighbourhood of the origin such that*

$$|x| \sim |z|, \quad |z|^{\mathcal{L}_0(H)} \sim |H(z)| \quad \textit{for } |z| \to 0 \textit{ and } z \in \Gamma.$$

Let now $H$ be a polynomial mapping satisfying (i)–(iii). First, we show

(6.2) LEMMA. *Under the above assumptions*, $\mathcal{L}_0(H) \in N(H,x)$.

Proof. Since $H$ has an isolated zero at the origin, by Proposition 6.1(a) there exist $A_1, B_1 > 0$ such that

$$A_1|z|^{\mathcal{L}_0(H)} \le |H(z)| \quad \text{for } |z| < B_1. \tag{8}$$

Now, we claim that there exist $A_2, \eta > 0$ such that

$$|H(z)| \ge A_2 \quad \text{for } |x| < \eta \text{ and } |y| \ge B_1. \tag{9}$$

Indeed, otherwise there would exist a sequence $\{(x_n, y_n)\}$ such that $x_n \to 0$, $|y_n| > B_1$ and $H(x_n, y_n) \to 0$. Then, taking a subsequence if necessary, we may assume that $y_n \to \infty$ or $y_n \to y_0$, $|y_0| \ge B_1$. In the first case, we obtain a contradiction with (iii), whereas in the second case, we have $H(0, y_0) = 0$, which contradicts (ii).

From (8) and (9), taking $B = \min(B_1, \eta, 1)$ and $A = \min(A_1, A_2)$ and noting that $|x| \le |z|$, we obtain

$$A|x|^{\mathcal{L}_0(H)} \le |H(z)| \quad \text{for } |x| < B,$$

which concludes the proof.

Proof of Theorem 3.1. From Theorem 5.3 in Ch. IV of [W] it follows that $\mu = \mu(f,g)$.

Let $\delta(Q)$ be defined as in (2). From Lemma 6.2 and Proposition 5.1 we get $\delta(Q) \le \mathcal{L}_0(H)$.

Proposition 5.1 implies that there exist $A, B > 0$ such that

$$A|x|^{\delta(Q)} \le |H(z)| \quad \text{for } |x| < B. \tag{10}$$

Considering (10) on the branch from Proposition 6.1(b), we easily conclude that $\mathcal{L}_0(H) \le \delta(Q)$. This ends the proof.

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