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A topological version of Bertini's theorem

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Abstract. We give a topological version of a Bertini type theorem due to Abhyankar. A new definition of a branched covering is given. If the restriction $\pi_V : V \to Y$ of the natural projection $\pi : Y \times Z \to Y$ to a closed set $V \subset Y \times Z$ is a branched covering then, under certain assumptions, we can obtain generators of the fundamental group $\pi_1((Y \times Z) \setminus V)$.

Introduction. In his book [1, pp. 349–356], Abhyankar proves an interesting theorem called by him a "Bertini theorem" or a "Lefschetz theorem". The theorem expresses a topological fact in complex analytic geometry. The purpose of this paper is to restate this theorem and its proof in purely topological language. Our formulation reads as follows:

Theorem 1. *Let Z be a connected topological manifold* (*without boundary*) *modeled on a real normed space E of dimension at least* 2 *and let Y be a simply connected and locally simply connected topological space. Suppose that V is a closed subset of $Y \times Z$ and $\pi : Y \times Z \to Y$ denotes the natural projection. Assume that $\pi_V = \pi|V : V \to Y$ is a branched covering whose regular fibers are finite and whose singular set $\Delta = \Delta(\pi_V)$ does not disconnect Y at any of its points. Set $X = (Y \times Z) \setminus V$ and $L = \{p\} \times Z$, where $p \in Y \setminus \Delta$. If there exists a continuous mapping $h : Y \to Z$ whose graph is contained in X, then the inclusion $i : L \setminus V \hookrightarrow X$ induces an epimorphism $i_* : \pi_1(L \setminus V) \to \pi_1(X)$.*

We have adopted the following definition. For any topological spaces Y and Y^*, a continuous, surjective mapping $\psi : Y^* \to Y$ is a (*topological*) *branched covering* if there exists a nowhere dense subset Δ of Y such that $\psi|Y^* \setminus \psi^{-1}(\Delta) : Y^* \setminus \psi^{-1}(\Delta) \to Y \setminus \Delta$ is a covering mapping. Notice that the *singular set* Δ of the branched covering ψ is not unique, but there is a smallest $\Delta(\psi)$ among such sets. Clearly, $\Delta(\psi)$ is a closed subset of Y. Thus

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the set $Y \setminus \Delta(\psi)$ of *regular points* is open. Topological branched coverings are studied in [2] .

The assumption in Theorem 1 that the set Δ *does not disconnect* Y *at any point of* Δ means that for each $y \in \Delta$ and every connected neighborhood U of y in Y there exists a smaller neighborhood W of y for which $W \setminus \Delta$ is connected.

2. An equivalent version and a straightening property

THEOREM 2 (cf. [1, (39.7)]). *Suppose the assumptions of Theorem* 1 *are satisfied. Then, for every connected covering* $\varphi : X^* \to X$ (*i.e.* X^* *is connected*), *the set* $\varphi^{-1}(L \setminus V)$ *is connected.*

Theorems 1 and 2 are equivalent due to the following simple but useful observation.

LEMMA 1 (cf. [1, (39.3)]). *Let a topological space* A *as well as its subspace* B *be connected and locally simply connected. Then the following are equivalent*:

(1.1) *the induced homomorphism* $\pi_1(B) \to \pi_1(A)$ *is an epimorphism*,
(1.2) *if* $\eta : A^* \to A$ *is any connected covering*, *then* $\eta^{-1}(B)$ *is connected*,
(1.3) *if* $\eta : A^* \to A$ *is the universal covering*, *then* $\eta^{-1}(B)$ *is connected.*

While Abhyankar deals with the second version (Theorem 2), we prefer to prove Theorem 1 directly. We will use the following lemmas from Abhyankar's proof.

LEMMA 2 (cf. [1, (39.2′)]). *Let* $\overline{B}$ *be the closed unit ball centered at* 0 *in any real normed space and let* B *and* S *be the corresponding open ball and sphere. Assume that* $l : [0,1] \to B$ *is a continuous mapping such that* $l(0) = 0$. *Then there exists a homeomorphism* $\tau : [0,1] \times \overline{B} \to [0,1] \times \overline{B}$ *such that* $\beta \circ \tau = \beta$, $\tau|([0,1] \times S) \cup (\{0\} \times \overline{B}) = \mathrm{id}$ *and* $\tau(\operatorname{graph} l) = [0,1] \times \{0\}$, *where* $\beta : [0,1] \times \overline{B} \to [0,1]$ *is the natural projection.*

Proof. Take $(t,b) \in [0,1] \times \overline{B}$. If $b \neq l(t)$ then we can find a unique positive number $e(t,b)$ such that $\|e(t,b)b+(1-e(t,b))l(t)\| = 1$. The mapping $E : ([0,1] \times \overline{B}) \setminus \operatorname{graph} l \ni (t,b) \mapsto e(t,b) \in [1,\infty)$ is locally bounded and its graph is closed, so it is continuous. We define

$$\tau(t,b) = \begin{cases} (t,0) & \text{if } b = l(t), \\ \left(t, b + \frac{1-e(t,b)}{e(t,b)} l(t)\right) & \text{if } b \neq l(t). \end{cases}$$

The inverse mapping τ^{-1} is

$$\tau^{-1}(t,b) = (t, b + (1 - \|b\|)l(t)).$$

Clearly, τ is the desired homeomorphism.

COROLLARY. *Let* $B, \overline{B}$ *and* S *be as in Lemma* 2. *Assume that* $l : [a,b] \to B$, *where* $a \le b$, *is a continuous mapping and take* $c \in [a,b]$. *Then there exists a homeomorphism* $\tau : [a,b] \times \overline{B} \to [a,b] \times \overline{B}$ *such that* $\beta \circ \tau = \beta$, $\tau|(([a,b] \times S) \cup (\{c\} \times \overline{B}) = \mathrm{id}$ *and* $\tau(\operatorname{graph} l) = [a,b] \times \{l(c)\}$, *where* $\beta : [0,1] \times \overline{B} \to [0,1]$ *is the natural projection.*

LEMMA 3 (cf. [1, (39.2)]). *Every manifold* M (*without boundary*) *modeled on any real normed space* E *has the following straightening property*: *For each set* $J \subset [0,1] \times M$ *such that the natural projection* $\beta : [0,1] \times M \to [0,1]$ *restricted to* J *is a covering of finite degree*, *there exists a homeomorphism* $\tau : [0,1] \times M \to [0,1] \times M$ *which satisfies the following three conditions*:

(2.1) $\beta \circ \tau = \beta$,

(2.2) $\tau|\{0\} \times M = \mathrm{id}$,

(2.3) $\tau(J) = [0,1] \times \alpha(J \cap (\{0\} \times M))$, *where* $\alpha : [0,1] \times M \to M$ *is the natural projection.*

Remark. Such a homeomorphism τ will be called a *straightening homeomorphism.* The segment $[0,1]$ can be replaced by any other segment $[a,b]$, where $a \le b$.

Proof of Lemma 3. Let d denote the degree of the covering $\beta|J$. Notice that $J = \bigcup_{j=1}^{d} \operatorname{graph} l_j$, where $l_j : [0,1] \to M$ are continuous mappings with pairwise disjoint graphs. For each $t \in [0,1]$, choose a family $U_{1,t}, \ldots, U_{d,t}$ of neighborhoods of $l_1(t), \ldots, l_d(t)$, respectively, and a family of homeomorphisms $f_{j,t}$ from $\overline{U}_{j,t}$ onto the closed unit ball $\overline{B}$ in E such that the sets $\overline{U}_{j,t}$ $(j = 1, \ldots, d)$ are pairwise disjoint and $f_{j,t}(U_{j,t}) = \operatorname{int} \overline{B} = B$. For every $t \in [0,1]$ there exists $\delta(t) > 0$ such that $l_j(t') \in U_{j,t}$ for every $j = 1, \ldots, d$ and $t' \in [0,1] \cap (t - \delta(t), t + \delta(t))$. Set $V_t = [0,1] \cap (t - \delta(t), t + \delta(t))$. Take a finite set $\{\overline{t}_1, \ldots, \overline{t}_n\}$ such that $\{V_{\overline{t}_i}\}_{i=1}^{n}$ covers $[0,1]$ and a finite sequence $0 = t_0 < \ldots < t_n = 1$ such that $I_k = [t_{k-1}, t_k] \subset V_{\overline{t}_k}$. Thus, $l_j(I_k) \subset U_{j,\overline{t}_k}$ for $k = 1, \ldots, n$ and $j = 1, \ldots, d$.

For every $k = 1, \ldots, n$, we define a straightening homeomorphism $\tau_k : I_k \times M \to I_k \times M$ using the Corollary on each $\overline{U}_{j,\overline{t}_k}$ $(j = 1, \ldots, d)$ and setting $\tau_k(t,m) = (t,m)$ for $m \in M \setminus \bigcup_{j=1}^{d} \overline{U}_{j,\overline{t}_k}$. Let $H_k = [0, t_k]$. For every $k = 1, \ldots, n$, we can define a straightening homeomorphism $\zeta_k : H_k \times M \to H_k \times M$ as follows:

1) $\zeta_1 = \tau_1$,

2) if ζ_{k-1} is defined then $\zeta_k = \zeta_{k-1} \cup ((\mathrm{id} \times \xi_k) \circ \tau_k)$, where $\xi_k : M \ni m \mapsto (\alpha \circ \zeta_{k-1})(t_{k-1}, m) \in M$ and α is the natural projection on M.

It is easy to check that $\tau = \zeta_n$ is the desired straightening homeomorphism.

3. Proof of Theorem 1. Clearly, X is a connected and locally simply connected space. Let $j : L\setminus V \hookrightarrow X\setminus(\Delta\times Z)$ and $k : X\setminus(\Delta\times Z)\hookrightarrow X$ be the inclusions. Then the proof falls naturally into two parts.

Part 1. *The mapping* $j_* : \pi_1(L\setminus V)\to\pi_1(X\setminus(\Delta\times Z))$ *is an epimorphism.*

Let $u=(f,g):[0,1]\to X\setminus(\Delta\times Z)$ be any loop at $(p,h(p))$. We define a new loop $w=(\widetilde{f},\widetilde{g}):[0,1]\to X\setminus(\Delta\times Z)$ by

$$w(t)=\begin{cases} u(2t) & \text{for } 0\le t\le 1/2,\\ (f(2-2t),h(f(2-2t))) & \text{for } 1/2<t\le 1.\end{cases}$$

Since Y is simply connected, we have $[w]=[u]$. Define

$$A:[0,1]\ni t\mapsto (t,\widetilde{g}(t))\in[0,1]\times Z,$$
$$\Omega:[0,1]\times Z\ni(t,z)\mapsto(\widetilde{f}(t),z)\in Y\times Z.$$

The restriction $\beta|\Omega^{-1}(V)$ of the natural projection $\beta:[0,1]\times Z\to[0,1]$ is a covering of finite degree. By Lemma 3, it has a straightening homeomorphism $\tau:[0,1]\times Z\to[0,1]\times Z$. Set $\widehat{t}=1/2-|t-1/2|$ and $\tau_t=(\alpha\circ\tau)(t,\cdot)$, where $\alpha:[0,1]\times Z\to Z$ is the natural projection. We can assume that $\tau_t=\tau_{\widehat{t}}$ because $\widetilde{f}(t)=\widetilde{f}(\widehat{t})$. The homotopy $H(t,s)=(\widetilde{f}(\widehat{t}(1-s)),(\tau^{-1}_{\widehat{t}(1-s)}\circ\tau_t\circ\widetilde{g})(t))$ joins the loop $w=H(\cdot,0)$ to the loop $H(\cdot,1)$ whose image is in $L\setminus V$. Notice that $H(0,s)=H(1,s)=(p,h(p))$ for every $s\in[0,1]$. This implies that $[u]=[H(\cdot,1)]\in j_*(\pi_1(L\setminus V))$ and completes the proof of Part 1.

Part 2. *The mapping* $k_*:\pi_1(X\setminus(\Delta\times Z))\to\pi_1(X)$ *is an epimorphism.*

Let $u=(f,g):[0,1]\to X$ be any loop at $(p,h(p))$. For every $t\in[0,1]$, there exists a neighborhood $U_t\times W_t$ of $u(t)$, where U_t and W_t are simply connected and $U_t\times W_t\subset X$. The family $V_t=u^{-1}(U_t\times W_t)$ $(t\in[0,1])$ is an open covering of $[0,1]$. Choose a finite subcover V_k $(k=1,\ldots,n)$ and a sequence $0=t_0<\ldots<t_n=1$ such that $[t_{k-1},t_k]\subset V_k$ $(k=1,\ldots,n)$. Let $V_k=u^{-1}(U_k\times W_k)$ for every k.

The arc component C_k of $U_k\cap U_{k+1}$ which contains $f(t_k)$ is open in Y. Since $\Delta\cap C_k$ is nowhere dense in C_k, there is a point p_k in $C_k\setminus\Delta$ for $k=1,\ldots,n-1$. Let $p_0=p_n=p$. For every k, there exists an arc $v_k:[t_{k-1},t_k]\to U_k\setminus\Delta$ which joins p_{k-1} to p_k, because U_k is a connected and locally arcwise connected space with a closed, nowhere dense and nowhere disconnecting subspace $U_k\cap\Delta$ (see [1, (14.5)]). Similarly, there exist arcs $c_k:[0,1]\to C_k\setminus\Delta$ joining $c_k(0)=f(t_k)$ to $c_k(1)=p_k$ for $k=1,\ldots,n-1$. Let $c_0=c_n:[0,1]\ni t\mapsto p\in(U_1\cap U_n)\setminus\Delta$. For every k, there exists a homotopy $H_k:[0,1]\times[t_{k-1},t_k]\to U_k$ joining $H_k(0,t)=f(t)$ to $H_k(1,t)=v_k(t)$ which satisfies $H_k(s,t_{k-1})=c_{k-1}(s)$ and $H_k(s,t_k)=c_k(s)$. Set $v=\bigcup_{k=1}^n v_k:[0,1]\to Y\setminus\Delta$ and $H=\bigcup_{k=1}^n H_k:[0,1]\times[0,1]\to Y$. Then the homotopy $\widetilde{H}(s,t) = (H(s,t), g(t))$ joins the loop $\widetilde{H}(0,t) = u(t)$ to the loop $\widetilde{H}(1,t) = (v(t), g(t))$ whose image is in $X \setminus (\Delta \times Z)$. Since the image of $\widetilde{H}$ is in X and $\widetilde{H}(s,0) = \widetilde{H}(s,1) = (p, h(p))$, $[u] \in k_*(\pi_1(X \setminus (\Delta \times Z)))$.

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