Exposed points in the set of representing measures for the disc algebra

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Abstract. It is shown that for each nonzero point \( x \) in the open unit disc \( D \), there is a measure whose support is exactly \( \partial D \cup \{x\} \) and that is also a weak*-exposed point in the set of representing measures for the origin on the disc algebra. This yields a negative answer to a question raised by John Ryff.

The disc algebra (on the disc) is the algebra of continuous functions on the closed unit disc \( D \) that are holomorphic on the open unit disc \( D \). Let \( M_0 \) denote the set of representing measures for the origin on the disc algebra, that is, those probability measures \( \mu \) on \( D \) such that

\[
\int f \, d\mu = f(0)
\]

for every function \( f \) in the disc algebra. It is well known that the only representing measure for the origin supported on the unit circle is ordinary Lebesgue measure divided by \( 2\pi \), but there are many representing measures for the origin supported on the unit disc.

For each point \( x \) in \( D \), let \( \delta_x \) denote the unit point mass at \( x \). John Ryff [R] proved the following result.

1. Theorem. Suppose \( \mu \in M_0 \), and \( \mu \neq \delta_0 \). Then there is a simply connected domain \( \Omega \) that contains the origin such that

\[
\partial \Omega \subset \text{supp}(\mu) \subset \overline{\Omega}
\]

where \( \partial \Omega \) is the boundary of \( \Omega \).

Ryff observed that with \( \Omega \) as above, there is a unique measure \( \nu \) in \( M_0 \) supported by \( \partial \Omega \), and that this measure is an extreme point of \( M_0 \). In fact, he observed that this measure is not only an extreme point of \( M_0 \), but also a weak*-exposed point of \( M_0 \). (A point \( p \) is an exposed point of a convex set \( C \)

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in a topological vector space \( V \) if there is a continuous real-linear functional \( A \) on \( V \) such that \( A(q) < A(p) \) for every \( q \in C \setminus \{p\} \). By a weak\(^*\)-exposed point we mean an exposed point relative to the weak\(^*\)-topology. Every exposed point of a convex set is an extreme point, but not conversely.)

Ryff’s observations led him to ask whether the measures \( \nu \) arising as above are the only extreme points of \( M_0 \), or at least the only weak\(^*\)-exposed points of \( M_0 \). Raymond Brummelhuis and Jan Wiegerinck [B–W] showed that for extreme points the answer is negative by producing for each point \( x \in D \) with \( x \neq 0 \), an extreme point of \( M_0 \) with support exactly \( \partial D \cup \{x\} \).

(Note that for such a measure, the domain \( \Omega \) of Theorem 1 is the open unit disc \( D \).) The purpose of the present paper is to show that these extreme points are also weak\(^*\)-exposed points, and thus that the answer to Ryff’s question is negative not only for extreme points but also for weak\(^*\)-exposed points. That is, we will prove the following:

2. Theorem. Suppose \( x \in D \) and \( x \neq 0 \). Then there exists a weak\(^*\)-exposed point of \( M_0 \) with support exactly \( \partial D \cup \{x\} \).

Before proving the theorem, we recall some material from [B–W]. As in that paper, we define for each finite positive measure \( \mu \) on \( D \) a function \( P_{\mu} \) on \( \partial D \) by

\[
P_{\mu}(\zeta) = \frac{1}{2\pi} \int_D \frac{1 - |z|^2}{|z - \zeta|^2} d\mu(z).
\]

Using Fubini’s theorem and the Poisson integral formula, one can easily show that \( P_{\mu} \) is in \( L^1(\partial D) \). In addition, it is clear that \( P_{\mu} \) is nonnegative.

Let \( m \) denote Lebesgue measure on the circle \( \partial D \). We will need the following lemma which is proved in [B–W].

3. Lemma. Suppose \( \mu \) is in \( M_0 \), and write \( \mu = \mu_1 + \mu_2 \), where \( \mu_1 \) is concentrated on \( D \) and \( \mu_2 \) is concentrated on \( \partial D \). Then

\[
(\ast) \quad \mu_2 = \left( \frac{1}{2\pi} - P_{\mu_1} \right) dm.
\]

In particular, \( 0 \leq P_{\mu_1} \leq 1/(2\pi) \). On the other hand, if \( \mu_1 \) is an arbitrary finite positive measure on \( D \) and the measure \( \mu_2 \) defined by \( (\ast) \) is also positive, then \( \mu_1 + \mu_2 \) is in \( M_0 \).

Proof of Theorem 2. Without loss of generality \( x \) is a real number and satisfies \( 0 < x < 1 \). Then the maximum of \( P_{\delta_x} \) over the circle \( \partial D \) is taken on at the point 1. Let

\[
\varepsilon_0 = \sup \left\{ \varepsilon > 0 : \frac{1}{2\pi} - \varepsilon P_{\delta_x} \geq 0 \right\},
\]
or equivalently let \( \varepsilon_0 = 1/(2\pi P_{\delta_x}(1)) \). Now if we let

\[
\sigma = \varepsilon_0 \delta_x + \left( \frac{1}{2\pi} - \varepsilon_0 P_{\delta_x} \right) dm
\]

then obviously the support of \( \sigma \) is exactly \( \partial D \cup \{ x \} \), and by Lemma 3, \( \sigma \) is in \( M_0 \). To show that \( \sigma \) is a weak\(^*\)-exposed point of \( M_0 \), let \( w \) be a nonnegative continuous function on \( D \) such that

\[
w(x) = \frac{1 - |x|^2}{1 - |x|^2},
\]

\[
w(z) < \frac{1 - |z|^2}{1 - |z|^2} \quad \text{for all } z \in D \setminus \{ x \}, \quad \text{and}
\]

\[
w(z) = 0 \quad \text{for all } z \in \partial D.
\]

Note that

\[
\int w \, d\sigma = \varepsilon_0 w(x) = \varepsilon_0 \int \frac{1 - |z|^2}{1 - |z|^2} d\delta_x = 2\pi \varepsilon_0 P_{\delta_x}(1) = 1.
\]

Note also that if \( \mu \) is in \( M_0 \), and we write \( \mu = \mu_1 + \mu_2 \), where \( \mu_1 \) is concentrated on \( D \) and \( \mu_2 \) is concentrated on \( \partial D \) then

\[
\int w \, d\mu = \int w \, d\mu_1
\]

\[
\leq \int \frac{1 - |z|^2}{1 - |z|^2} \, d\mu_1(z)
\]

\[
= 2\pi \mu_1(1)
\]

\[
\leq 1 \quad \text{(by Lemma 3)}.
\]

The inequality (**) is strict unless all the mass of \( \mu_1 \) lies at the point \( x \). In that case Lemma 3 shows that

\[
\mu = \varepsilon \delta_x + \left( \frac{1}{2\pi} - \varepsilon P_{\delta_x} \right) dm
\]

for some \( \varepsilon \) satisfying \( 0 \leq \varepsilon \leq 1/(2\pi P_{\delta_x}(1)) \), and then the inequality (***) is strict unless \( \varepsilon = \varepsilon_0 \). Of course, if \( \varepsilon = \varepsilon_0 \), then \( \mu = \sigma \). Thus

\[
\int w \, d\mu \leq 1 \quad \text{for all } \mu \text{ in } M_0,
\]

with equality if and only if \( \mu = \sigma \), so \( \sigma \) is a weak\(^*\)-exposed point of \( M_0 \).

References


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