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Exposed points in the set of representing measures for the disc algebra

by Alexander J. Izzo (Bowling Green, Ohio)

Abstract. It is shown that for each nonzero point x in the open unit disc D, there is a measure whose support is exactly $\partial D \cup \{x\}$ and that is also a weak*-exposed point in the set of representing measures for the origin on the disc algebra. This yields a negative answer to a question raised by John Ryff.

The *disc algebra* (on the disc) is the algebra of continuous functions on the closed unit disc $\overline{D}$ that are holomorphic on the open unit disc D. Let M_0 denote the set of representing measures for the origin on the disc algebra, that is, those probability measures μ on $\overline{D}$ such that

$$\int f\,d\mu = f(0)$$

for every function f in the disc algebra. It is well known that the only representing measure for the origin supported on the unit *circle* is ordinary Lebesgue measure divided by 2π, but there are many representing measures for the origin supported on the unit disc.

For each point x in $\overline{D}$, let δ_x denote the unit point mass at x. John Ryff [R] proved the following result.

1. Theorem. *Suppose $\mu \in M_0$, and $\mu \neq \delta_0$. Then there is a simply connected domain Ω that contains the origin such that*

$$\partial\overline{\Omega} \subset \operatorname{supp}(\mu) \subset \overline{\Omega}$$

where $\partial\overline{\Omega}$ is the boundary of $\overline{\Omega}$.

Ryff observed that with Ω as above, there is a unique measure ν in M_0 supported by $\partial\overline{\Omega}$, and that this measure is an extreme point of M_0. In fact, he observed that this measure is not only an extreme point of M_0, but also a weak*-exposed point of M_0. (A point p is an *exposed point* of a convex set C

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in a topological vector space V if there is a continuous real-linear functional Λ on V such that $\Lambda(q) < \Lambda(p)$ for every q in $C \setminus \{p\}$. By a *weak*-exposed point* we mean an exposed point relative to the weak*-topology. Every exposed point of a convex set is an extreme point, but not conversely.)

Ryff's observations led him to ask whether the measures ν arising as above are the only extreme points of M_0, or at least the only weak*-exposed points of M_0. Raymond Brummelhuis and Jan Wiegerinck [B–W] showed that for extreme points the answer is negative by producing for each point x in D with $x \neq 0$, an extreme point of M_0 with support exactly $\partial D \cup \{x\}$. (Note that for such a measure, the domain Ω of Theorem 1 is the open unit disc D.) The purpose of the present paper is to show that these extreme points are also weak*-exposed points, and thus that the answer to Ryff's question is negative not only for extreme points but also for weak*-exposed points. That is, we will prove the following:

2. Theorem. *Suppose $x \in D$ and $x \neq 0$. Then there exists a weak*-exposed point of M_0 with support exactly $\partial D \cup \{x\}$.*

Before proving the theorem, we recall some material from [B–W]. As in that paper, we define for each finite positive measure μ on D a function P_μ on ∂D by

$$P_\mu(\zeta) = \frac{1}{2\pi} \int\limits_D \frac{1 - |z|^2}{|\zeta - z|^2}\, d\mu(z).$$

Using Fubini's theorem and the Poisson integral formula, one can easily show that P_μ is in $L^1(\partial D)$. In addition, it is clear that P_μ is nonnegative.

Let m denote Lebesgue measure on the circle ∂D. We will need the following lemma which is proved in [B–W].

3. Lemma. *Suppose μ is in M_0, and write $\mu = \mu_1 + \mu_2$, where μ_1 is concentrated on D and μ_2 is concentrated on ∂D. Then*

$$\mu_2 = \left(\frac{1}{2\pi} - P_{\mu_1}\right) dm. \tag{$*$}$$

In particular, $0 \leq P_{\mu_1} \leq 1/(2\pi)$. On the other hand, if μ_1 is an arbitrary finite positive measure on D and the measure μ_2 defined by $(*)$ is also positive, then $\mu_1 + \mu_2$ is in M_0.

Proof of Theorem 2. Without loss of generality x is a real number and satisfies $0 < x < 1$. Then the maximum of P_{δ_x} over the circle ∂D is taken on at the point 1. Let

$$\varepsilon_0 = \sup\left\{\varepsilon > 0 : \frac{1}{2\pi} - \varepsilon P_{\delta_x} \geq 0\right\},$$

or equivalently let $\varepsilon_0 = 1/(2\pi P_{\delta_x}(1))$. Now if we let

$$\sigma = \varepsilon_0\delta_x + \left(\frac{1}{2\pi} - \varepsilon_0 P_{\delta_x}\right) dm$$

then obviously the support of σ is exactly $\partial D \cup \{x\}$, and by Lemma 3, σ is in M_0. To show that σ is a weak*-exposed point of M_0, let w be a nonnegative continuous function on $\overline{D}$ such that

$$\begin{aligned} &w(x) = \frac{1-|x|^2}{|1-x|^2}, \\ &w(z) < \frac{1-|z|^2}{|1-z|^2} \quad \text{for all } z \text{ in } D\setminus\{x\}, \quad \text{and} \\ &w(z) = 0 \qquad\qquad \text{for all } z \text{ in } \partial D. \end{aligned}$$

Note that

$$\int w\,d\sigma = \varepsilon_0 w(x) = \varepsilon_0 \int \frac{1-|z|^2}{|1-z|^2}\,d\delta_x = 2\pi\varepsilon_0 P_{\delta_x}(1) = 1.$$

Note also that if μ is in M_0, and we write $\mu = \mu_1 + \mu_2$, where μ_1 is concentrated on D and μ_2 is concentrated on ∂D then

$$\begin{aligned} \int w\,d\mu &= \int w\,d\mu_1 \\ &\le \int \frac{1-|z|^2}{|1-z|^2}\,d\mu_1(z) &(**)\\ &= 2\pi P_{\mu_1}(1) \\ &\le 1 \quad \text{(by Lemma 3).} &(***) \end{aligned}$$

The inequality $(**)$ is strict unless all the mass of μ_1 lies at the point x. In that case Lemma 3 shows that

$$\mu = \varepsilon\delta_x + \left(\frac{1}{2\pi} - \varepsilon P_{\delta_x}\right) dm$$

for some ε satisfying $0 \le \varepsilon \le 1/(2\pi P_{\delta_x}(1))$, and then the inequality $(***)$ is strict unless $\varepsilon = \varepsilon_0$. Of course, if $\varepsilon = \varepsilon_0$, then $\mu = \sigma$. Thus

$$\int w\,d\mu \le 1 \quad \text{for all } \mu \text{ in } M_0,$$

with equality if and only if $\mu = \sigma$, so σ is a weak*-exposed point of M_0.

References

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DEPARTMENT OF MATHEMATICS AND STATISTICS
BOWLING GREEN STATE UNIVERSITY
BOWLING GREEN, OHIO 43403
U.S.A.
E-mail: AIZZO@MATH.BGSU.EDU

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