Between the Paley–Wiener theorem
and the Bochner tube theorem

by Zofia Szmydt and Bogdan Ziemian (Warszawa)

Abstract. We present the classical Paley–Wiener–Schwartz theorem [1] on the Laplace transform of a compactly supported distribution in a new framework which arises naturally in the study of the Mellin transformation. In particular, sufficient conditions for a function to be the Mellin (Laplace) transform of a compactly supported distribution are given in the form resembling the Bochner tube theorem [2].

1. Notation. We employ the usual notation of set theory. \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}_+ \) the set of positive real numbers, and \( \mathbb{R}_+^n = (\mathbb{R}_+)^n \). For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we set \( \langle x \rangle = 1 + |x_1| + \ldots + |x_n| \). We write \( x < y \) \( (x \leq y) \) for \( x, y \in \mathbb{R}^n \) to denote \( x_j < y_j \) \( (x_j \leq y_j \) resp.) for \( j = 1, \ldots, n \), and we set \( I = (0, t] = \{ x \in \mathbb{R}^n : 0 < x \leq t \} \), where \( t \in \mathbb{R}_+^n \). By 1 we denote \( (1, \ldots, 1) \). \( \mathbb{N} \) is the set of positive integers and \( \mathbb{N}_0 \) the set of non-negative integers. We write \( |\alpha| = \alpha_1 + \ldots + \alpha_n \) for \( \alpha \in \mathbb{N}_0^n \). For \( x \in \mathbb{R}^n \) and \( \alpha \in \mathbb{N}_0^n \) we write \( x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \).

We employ the usual notation of distribution theory. \( D'(\Omega) \) denotes the space of distributions on an open set \( \Omega \subset \mathbb{R}^n \), and \( D'_A(\Omega) \) the space of distributions on \( \Omega \) with support in \( A \subset \Omega \). The value of a distribution \( u \) on a test function \( \varphi \) is denoted by \( u[\varphi] \).

2. Auxiliary theorems

Theorem 1. Let \( u \in D'_K(\mathbb{R}^n) \) where \( K \) is a connected compact set in \( \mathbb{R}^n \) such that any two points \( x, y \in K \) can be joined by a rectifiable curve in \( K \) of length \( \leq \tilde{C}|x - y| \), \( \tilde{C} < \infty \). Then there exists a constant \( C < \infty \) and \( k \in \mathbb{N}_0 \) such that

1991 Mathematics Subject Classification: Primary 46F12.
Key words and phrases: Mellin distributions, Bochner tube theorem.
\[ |u[\psi]| \leq C \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}} \left| \left( \frac{\partial}{\partial x} \right)^\alpha \psi(x) \right| \quad \text{for } \psi \in C^k(\mathbb{R}^n). \]

The proof of this theorem, based on the Whitney extension theorem, is given in [1].

Now following [3] we recall the spaces of Mellin distributions. Denote by \( \mu : \mathbb{R}^n \rightarrow \mathbb{R}^n \) the diffeomorphism
\[ \mu(y) = e^{-y} := (e^{-y_1}, \ldots, e^{-y_n}). \]

We define the space of Mellin distributions on \( \mathbb{R}_+^n \) for every \( \alpha \in \mathbb{R}^n \) as the dual of the space
\[ \mathcal{M}_\alpha = \mathcal{M}_\alpha(\mathbb{R}_+^n) = \{ \sigma \in C^\infty(\mathbb{R}_+^n) : (x^{\alpha+1} \sigma) \circ \mu \in S(\mathbb{R}^n) \}, \]
with the natural topology in \( \mathcal{M}_\alpha \) induced from \( S(\mathbb{R}^n) \).

Now the set \( C^\infty_{(0)}(I) \) (of restrictions to \( I \) of functions in \( C^\infty(\mathbb{R}^n) \)) is not dense in \( M_\alpha(I) \).

Let \( \omega \in (\mathbb{R} \cup \{ \infty \})^n \). We define the function space \( M(\omega)(I) \) as the inductive limit
\[ M(\omega)(I) = \lim_{a \searrow \omega} M_a(I). \]

Now the set \( C^\infty_{(0)}(I) \) is dense in \( M(\omega)(I) \) and the dual space \( M'(\omega) = M(\omega)' \) is a subspace of \( D'_I(\mathbb{R}_+^n) \). Therefore the elements of \( M'(\omega) \) are called Mellin distributions on \( I \). Note that for \( a < b < \omega \) and \( \omega \in (\mathbb{R} \cup \{ \infty \})^n \),
\[ M_{(a)}(I) \subset M_a(I) \subset M_b(I) \subset M(\omega)(I), \]
\[ M(\omega)(I) = \lim_{a \searrow \omega} M_a(I), \]
\[ M'(\omega)(I) = \bigcap_{a < \omega} M'_a(I) = \bigcap_{a < \omega} M'_{(a)}(I). \]
The totality of Mellin distributions is denoted by

\[ M'(I) = \bigcup_{\omega \in (\mathbb{R} \cup \{\infty\})^n} M'_\omega(I) = \bigcup_{\omega \in \mathbb{R}^n} M'_\omega(I). \]

\( M'(I) \) coincides with the space of restrictions to \( \mathbb{R}^n_+ \) of distributions on \( \mathbb{R}^n \) with support in \( I \).

Let \( u \in M'_\omega(I) \) for some \( \omega \in (\mathbb{R} \cup \{\infty\})^n \). We define the Mellin transform of \( u \) by

\[ Mu(z) = u[x^{-z-1}] \quad \text{for Re } z < \omega. \]

This definition differs from the classical one by the change of variable \( z \mapsto -z \).

The following theorem gives a relation between the Mellin transformations \( M \) and \( M_\alpha \) defined by (2) and (1) respectively.

**Theorem 2.** Let \( u \in M'_\omega(I) \). Then \( Mu \) is holomorphic for Re \( z < \omega \) and \( u \in M'_\alpha(\mathbb{R}^n_+) \) for every \( \alpha < \omega \). The tempered distribution \( M_\alpha u \) is a function:

\[ (M_\alpha u)(\beta) = Mu(\alpha + i\beta) = (u \circ \mu)[e^{(\alpha+i\beta)y}] \quad \text{for } \beta \in \mathbb{R}^n. \]

Moreover, \( M_\alpha : M'_\omega \to S' \) is continuous for \( \alpha < \omega \).

**Theorem 3** (Paley–Wiener type theorem). In order that a function \( f(z) = f(z_1, \ldots, z_n) \) be the Mellin transform of a unique Mellin distribution \( u \in M'_\omega(\{0, t]\}) \) it is necessary and sufficient that \( f \) be holomorphic in \( \{z \in \mathbb{C}^n : \text{Re } z < \omega\} \) and that for every \( b < \omega \) and every \( \varrho \in \mathbb{R}_+ \) there exist \( s = s(b) \in \mathbb{N}_0 \) and \( C = C(b, \varrho) < \infty \) such that

\[ |f(\alpha + i\beta)| \leq C(\beta)^s(e^{\varrho t})^{-\alpha} \quad \text{for } \alpha \leq b. \]

**3. The main theorem.** Let \( t^- = (t_1^-, \ldots, t_n^-), t^+ = (t_1^+, \ldots, t_n^+) \), \( 0 < t^- < t^+ \), write \( I = (0, t^+] \) and consider the polyinterval

\[ [t^-, t^+] = \{ x \in \mathbb{R}^n : t^- \leq x \leq t^+ \}. \]

**Theorem 4.** Let \( f \) be a function holomorphic on \( \{z \in \mathbb{C}^n : \text{Re } z < 0\} \cup \{z \in \mathbb{C}^n : \text{Re } z > 0\} \) and such that for every \( b \in \mathbb{R}^n \) with \( b < 0 \) and \( \varrho \in \mathbb{R}_+ \),

\[ |f(\alpha + i\beta)| \leq C(\beta)^s(e^{\varrho t})^{-\alpha} \quad \text{for } \alpha < b, \]

\[ |f(\alpha + i\beta)| \leq C(\beta)^s(e^{-\varrho t})^{-\alpha} \quad \text{for } \alpha > -b, \]

with some \( s = s(b) \in \mathbb{N}_0 \) and \( C = C(b, \varrho) < \infty \). Moreover, assume that the following limits exist in \( S'(\mathbb{R}^n) \) and are equal:

\[ \lim_{\alpha \to 0^-} f(\alpha + i\beta) = \lim_{\alpha \to 0^+} f(\alpha + i\beta). \]
Then there exists a unique \( u \in D'_[t^-, t^+] \) such that \( Mu = f \). Furthermore, \( f \) is an entire function on \( C^n \) and for every \( b \in \mathbb{R}^n \) and \( g \in \mathbb{R}_+ \) there exist \( C = C(b, g) < \infty \) and \( s = s(b) \in \mathbb{N}_0 \) such that for any \( \sigma \in \{-, +\}^n \),

\[
(8) \quad |f(\alpha + i\beta)| \leq C(\beta)^{\sigma}\left(e^{\sigma_1 \beta_1} \cdots \left(e^{\sigma_n \beta_n} - \alpha_n \right)^{-\alpha_n}ight)
\]

for \( \sigma, \alpha, \beta \leq \sigma_j \beta_j, \ j = 1, \ldots, n \).

**Proof.** By assumption (5), which is the sufficient condition in Theorem 3, there exists a unique distribution \( u \in M'_{[0]}((0, t^+)) \) such that \( Mu = f \). Thus \( \text{supp } u \subset (0, t^+) \) and \( u \in \mathfrak{M}'_{[0]}((0, t^+)) \) for \( \alpha < 0 \). Denote by \( w \) the tempered distribution defined by (7). Hence

\[
\lim_{\alpha \to 0^-} \int_{\mathbb{R}^n} f(\alpha + i\beta)\psi(\beta) \, d\beta = w[\psi] \quad \text{for } \psi \in S(\mathbb{R}^n)
\]

and by (3) and (1) we get

\[
w[\psi] = \lim_{\alpha \to 0^-} \int_{\mathbb{R}^n} (Mu)(\alpha + i\beta)\psi(\beta) \, d\beta
\]

\[
= (2\pi)^{n/2} \lim_{\alpha \to 0^-} \int_{\mathbb{R}^n} F^{-1}(e^{\alpha \beta}(u \circ \mu))(\beta)\psi(\beta) \, d\beta
\]

\[
= (2\pi)^{n/2} \lim_{\alpha \to 0^-} F^{-1}(e^{\alpha \beta}(u \circ \mu))[\psi]
\]

\[
= (2\pi)^{n/2} \lim_{\alpha \to 0^-} (u \circ \mu)[e^{\alpha \beta}F^{-1}\psi]
\]

For \( \psi = F\varphi \) with \( \varphi \in D(\mathbb{R}^n) \) the last formula yields

\[
(9) \quad Fw[\varphi] = (2\pi)^{n/2}(u \circ \mu)[\varphi] \quad \text{for } \varphi \in D(\mathbb{R}^n).
\]

Now observe that by assumption (6),

\[
|f(-\alpha - i\beta)| < C(\beta)^{\sigma}\left(e^{\sigma_1 \beta_1} \cdots \left(e^{\sigma_n \beta_n} - \alpha_n \right)^{-\alpha_n}ight)
\]

for \( \alpha < b \),

where \( 1/x := (1/x_1, \ldots, 1/x_n) \) for \( x \in \mathbb{R}^n_+ \). As before, by Theorem 3, there exists a unique distribution \( \tilde{u} \in M'_{[0]}((0, 1/t^-)) \) such that

\[
f(-\alpha - i\beta) = M\tilde{u}(\alpha + i\beta).
\]

Note that \( \tilde{u} \in \mathfrak{M}'_{[0]}((0, 1/t^-)) \) for \( \alpha < 0 \) and \( f(-\alpha - i\beta) = M_\alpha \tilde{u}(\beta) = (2\pi)^{n/2}F^{-1}(e^{\alpha \beta}(\tilde{u} \circ \mu)) \).

Since by (7), \( w = \lim_{\alpha \to 0^+} f(\alpha + i\cdot) \) we have \( \lim_{\alpha \to 0^-} f(-\alpha - i\cdot) = w^\vee \) where \( ^\vee \) denotes the reflection \( \beta \rightarrow -\beta \). Take \( \varphi \in D(\mathbb{R}^n) \) and let \( \psi = F\varphi \).
Then $\psi \in S(\mathbb{R}^n)$ and
\[
\psi(\xi) = \lim_{\alpha \to 0} \int_{\mathbb{R}^n} f(-\alpha - i\beta)\psi(\beta) \, d\beta
\]
\[= (2\pi)^{n/2} \lim_{\alpha \to 0} \int_{\mathbb{R}^n} F^{-1}(e^{\alpha \mu}(\tilde{u} \circ \mu))\psi(\beta) \, d\beta = (2\pi)^{n/2}(\tilde{u} \circ \mu)[\varphi].
\]
Hence
\[Fw^\nu[\varphi] = (2\pi)^{n/2}(\tilde{u} \circ \mu)[\varphi] \quad \text{for } \varphi \in D(\mathbb{R}^n).
\]
This together with (9) yields $(u \circ \mu)^\nu = \tilde{u} \circ \mu$. Let $\lambda$ be the mapping $\mathbb{R}^n \ni x \mapsto 1/x$. Since $(u \circ \mu)^\nu = (u \circ \lambda) \circ \mu$ we get $u \circ \lambda = \tilde{u}$. Hence $u \circ \lambda \in M^{(0)}((0,1/\lambda^t))$ and by definition of $\lambda$ we have supp $u \subset \{x : x \geq t\}$, which together with $u \in M^{(0)}((0,1/t^+)\}$ gives the desired assertion $u \in D^{(1/t^+)}$. By Theorem 1, $u \in M^{(n)}$ for every $a \in \mathbb{R}^n$ (i.e. $u \in M^{(n)}$) and hence by Theorem 3 (the necessary condition this time) $f = Mu$ is entire on $\mathbb{C}^n$ and the estimate (5) holds for $\lambda \leq b$ for every $b \in \mathbb{R}^n$. Since $u \circ \lambda \in D^{(1/t^+)}$ and $M(u \circ \lambda)(z) = Mu(-z)$ we get as before, for all $b \in \mathbb{R}^n$ and $\varphi \in \mathbb{R}^+_t$,
\[|Mu(\alpha + i\beta)| \leq C\langle \beta \rangle^s(e^{\theta t^-})^{-\alpha} \quad \text{for } \alpha \geq b
\]
with $s = s(b)$ and $C = C(b, \varphi)$. Thus we have proved (8) for $\sigma = (+, \ldots, +)$ and $\sigma = (-, \ldots, -)$. To get the proof for $\sigma^j = (\sigma_1^j, \ldots, \sigma_n^j)$ with $\sigma_i^j = 1$ if $i \neq j$, and $\sigma_i^j = 0$ if $j = 1, \ldots, n$, take the mapping
\[\lambda_j : \mathbb{R}^n \ni x \mapsto (x_1, \ldots, x_{j-1}, 1/x_j, x_{j+1}, \ldots, x_n).
\]
Then
\[M(u \circ \lambda_j)(z) = Mu(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_n)
\]
\[= f(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_n),
\]
\[\text{supp}(u \circ \lambda_j) \subset \{x : t^-_i \leq x_i \leq t^+_i \text{ for } i \neq j, 1/t^+_j \leq x_j \leq 1/t^-_j \}.
\]
Fix arbitrarily $b \in \mathbb{R}^n$, $\varphi \in \mathbb{R}^+_t$ and $j$ with $1 \leq j \leq n$. Take $\tilde{b} = (b_1, \ldots, b_{j-1}, -b_j, b_{j+1}, \ldots, b_n)$. By Theorem 3,
\[|M(u \circ \lambda_j)(\alpha + i\beta)| \leq C\langle \beta \rangle^s(e^{\theta t^+_j})^{-\alpha^j_1} \ldots (e^{\theta t^+_j})^{-\alpha^j_j} \ldots (e^{\theta t^+_n})^{-\alpha^j_n}
\]
for $\alpha \leq \tilde{b}$ and hence
\[|f(\alpha + i\beta)| \leq C\langle \beta \rangle^s(e^{\theta t^+_j})^{-\alpha^j_1} \ldots (e^{\theta t^-_j})^{-\alpha^j_j} \ldots (e^{\theta t^+_n})^{-\alpha^j_n}
\]
for $\alpha_j = \beta_j$ if $i \neq j$, $-\alpha_j_j \leq \tilde{b}_j = -b_j$, i.e. (8) holds for $\sigma^j (j = 1, \ldots, n)$.

The remaining cases of $\sigma$ are left to the reader.

Remark 1. Note that in contrast to the classical Paley–Wiener theorem the sufficiency part of Theorem 4 does not require that the function
$f$ be entire. Instead we assume holomorphy in two wedges with a common edge and identity of the corresponding boundary values. By the necessity result this gives holomorphy in $C^n$ as well as estimates in the “missing” wedges, which can be regarded as a variant of the Bochner tube theorem.

Remark 2. By applying the techniques of the theory of Fourier hyperfunctions and analytic functionals one can prove a variant of Theorem 4 with $\langle |\beta| \rangle^s$ in the estimates (5), (6) and (8) replaced by $e^{\theta|\beta|}$ for some $\theta > 0$. Then the identity (7) should be understood as the equivalence of pertinent boundary values in the sense of Fourier hyperfunctions.

References