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Integral representations of bounded starlike functions

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Abstract. For $\alpha \geq 0$ let $\mathcal{F}_\alpha$ denote the class of functions defined for $|z| < 1$ by integrating $1/(1-xz)^\alpha$ if $\alpha > 0$, and $\log(1/(1-xz))$ if $\alpha = 0$, against a complex measure on $|x| = 1$. We study families of starlike functions where $zf'(z)/f(z)$ ranges over a parabola with given focus and vertex. We prove a number of properties of these functions, among others that they are bounded and that they belong to $\mathcal{F}_0$. In general, it is only known that bounded starlike functions belong to $\mathcal{F}_\alpha$ for $\alpha > 0$.

1. Introduction. Let $U = \{z : |z| < 1\}$, $\Gamma = \{z : |z| = 1\}$ and let $\mathcal{M}$ denote the set of complex-valued Borel measures on Γ. For $\alpha > 0$ let $\mathcal{F}_\alpha$ denote the set of functions f for which there is $\mu \in \mathcal{M}$ such that

$$f(z) = \int_\Gamma \frac{1}{(1-xz)^\alpha}\, d\mu(x) \tag{1.1}$$

for $|z| < 1$, and let $\mathcal{F}_0$ denote the set of functions f for which there is $\mu \in \mathcal{M}$ such that

$$f(z) = \int_\Gamma \log\frac{1}{1-xz}\, d\mu(x) + f(0)$$

for $|z| < 1$. The classes $\mathcal{F}_\alpha$ for $\alpha > 0$ were introduced in [9] and $\mathcal{F}_0$ was introduced in [5]. Denote by $\mathcal{H}$ the class of functions analytic and univalent in U, and by $\mathcal{S}$ the subset of $\mathcal{H}$ with the normalization $f(0) = f'(0) - 1 = 0$. The study of $\mathcal{F}_\alpha$ was mainly motivated by the question whether $\mathcal{H} \subset \mathcal{F}_2$, and MacGregor showed in [9] that this is not true, but that $\mathcal{H} \subset \mathcal{F}_\alpha$ for every $\alpha > 2$. We could also mention the well known fact that every starlike function in $\mathcal{S}$ has the representation

$$f(z) = \int_\Gamma \frac{z}{(1-xz)^2}\, d\mu(x)$$

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with a probability measure μ. This was proved by Brickman, MacGregor and Wilken in [2]. In [6] Hibschweiler and MacGregor investigated membership in $\mathcal{F}_\alpha$ for univalent functions, in particular starlike and convex functions, with restricted growth. The following result was obtained in [6].

Theorem A. (a) *Let $f \in \mathcal{H}$ and assume that $f(U)$ is starlike with respect to $f(0)$ and that for some $A>0$ and $0<\beta<2$, $|f(z)| \le A/(1-|z|)^\beta$. Then $f \in \mathcal{F}_\alpha$ for every $\alpha > \beta$.*

(b) *If $f \in \mathcal{H}$ and $f(U)$ is a bounded convex domain then $f \in \mathcal{F}_0$.*

It is not known whether every bounded starlike function is in $\mathcal{F}_0$. In this paper we introduce some families of starlike functions which turn out to consist only of bounded functions and we prove that all these classes are contained in $\mathcal{F}_0$. Define the class $\mathcal{SP}(\alpha,\beta)$ to be the set of functions $f \in \mathcal{S}$ with the property that

$$\left|\frac{zf'(z)}{f(z)} - (\alpha+\beta)\right| \le \operatorname{Re}\frac{zf'(z)}{f(z)} + \alpha - \beta, \qquad z \in U,$$

$0<\alpha<\infty$ and $0 \le \beta < 1$. This means that $zf'(z)/f(z)$ for $f \in \mathcal{SP}(\alpha,\beta)$ and $z \in U$ lies in that portion of the plane which contains $w=1$ and is bounded by the parabola $y^2 = 4\alpha(x-\beta)$. The classes $\mathcal{SP}(\alpha,\beta)$ are generalizations of classes that previously have been studied by the author. In [12] the class $\mathcal{S}_p$ was introduced in connection with uniformly convex functions. In the new notation $\mathcal{S}_p = \mathcal{SP}\left(\frac{1}{2},\frac{1}{2}\right)$. In [11] a generalization of $\mathcal{S}_p$ was done, along with the introduction of the concept of *order of uniform convexity*. In the new notation this generalization amounts to the classes $\mathcal{SP}\left(\frac{1-\gamma}{2},\frac{1+\gamma}{2}\right)$, $-1 \le \gamma < 1$. Since $\mathcal{SP}(\alpha,\beta) \subset \mathcal{SP}(\alpha,0)$, it seems to be most interesting in this context to study the classes where $\beta = 0$. For simplicity of notation we define $\mathcal{SP}(\alpha) := \mathcal{SP}(\alpha,0)$, and hence we have

$$\mathcal{SP}(\alpha) = \left\{ f \in \mathcal{S} : \left|\frac{zf'(z)}{f(z)} - \alpha\right| \le \operatorname{Re}\frac{zf'(z)}{f(z)} + \alpha,\ z \in U,\ 0<\alpha<\infty \right\}.$$

Before we proceed, one important fact about $\mathcal{F}_\alpha$ should be mentioned.

Theorem B. *For $\alpha \ge 0$, $f \in \mathcal{F}_\alpha$ if and only if $f' \in \mathcal{F}_{\alpha+1}$.*

The proof for the case $\alpha > 0$ can be found in [9], and the case $\alpha = 0$ is treated in [5].

2. The Carathéodory function associated with $\mathcal{SP}(\alpha)$. Many of the special classes of normalized starlike functions that have been studied over the years are characterized by the range of the functional $zf'(z)/f(z)$. This will be a domain Ω in the right half plane, $1 \in \Omega$, and it is of interest to determine an analytic, univalent function (Carathéodory function) mapping U onto Ω and 0 to 1. In the case of $\mathcal{SP}(\alpha)$ the domain Ω is bounded by a parabola with vertex at the origin, axis along the positive real axis and focus in α.

THEOREM 2.1. *Let* $\Omega_\alpha = \{w : |w-\alpha| \le \operatorname{Re} w + \alpha\}$. *Define* $P_\alpha(z)$ *to be the analytic and univalent function with the properties* $P_\alpha(0) = 1$, $P'_\alpha(0) > 0$ *and* $P_\alpha(U) = \Omega_\alpha$. *Then*

$$P_\alpha(z) = \alpha\left(1 + \frac{4}{\pi^2}\left(\log\frac{1+\sqrt{w_\alpha(z)}}{1-\sqrt{w_\alpha(z)}}\right)^2\right) \tag{2.1}$$

where

$$w_\alpha(z) = \begin{cases} \dfrac{z - \tan^2\big(\pi\sqrt{1-1/\alpha}/4\big)}{1 - z\tan^2\big(\pi\sqrt{1-1/\alpha}/4\big)} & \text{if } \alpha \ge 1, \\[2ex] \dfrac{z + \tanh^2\big(\pi\sqrt{1/\alpha-1}/4\big)}{1 + z\tanh^2\big(\pi\sqrt{1/\alpha-1}/4\big)} & \text{if } 0 < \alpha < 1. \end{cases}$$

Proof. It is a simple exercise in conformal mappings to see that the function

$$Q_\alpha(z) = \alpha\left(1 + \frac{4}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right) \tag{2.2}$$

is analytic and univalent in U and has the properties $Q_\alpha(U) = \Omega_\alpha$ and $Q_\alpha(0) = \alpha$. (The branch of the square root is chosen so that $\operatorname{Im}\sqrt{z} \ge 0$.) Next we find a suitable self-mapping of U, $w(z)$, such that $P_\alpha(z) = Q_\alpha(w(z))$ and $P_\alpha(0) = 1$. Solving the equation $Q_\alpha(\zeta) = 1$ we get

$$\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2 = \frac{\pi^2}{4}\left(\frac{1}{\alpha} - 1\right),$$

which in the case $\alpha > 1$ gives

$$\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}} = e^{(i\pi/2)\sqrt{1-1/\alpha}}$$

and further

$$\begin{aligned}\zeta_\alpha &= \left(\frac{\sin\big(\pi\sqrt{1-1/\alpha}/2\big)}{1+\cos\big(\pi\sqrt{1-1/\alpha}/2\big)}i\right)^2 = -\frac{1-\cos\big(\pi\sqrt{1-1/\alpha}/2\big)}{1+\cos\big(\pi\sqrt{1-1/\alpha}/2\big)} \\ &= -\tan^2\left(\frac{\pi}{4}\sqrt{1-\frac{1}{\alpha}}\right).\end{aligned}$$

In the case $\alpha < 1$ we similarly get

$$\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}} = e^{(\pi/2)\sqrt{1/\alpha-1}}$$

and then

$$\zeta_\alpha = \left(\frac{e^{(\pi/2)\sqrt{1/\alpha-1}}-1}{e^{(\pi/2)\sqrt{1/\alpha-1}}+1}\right)^2 = \tanh^2\left(\frac{\pi}{4}\sqrt{\frac{1}{\alpha}-1}\right).$$

Taking

$$w_\alpha(z) = \frac{z+\zeta_\alpha}{1+z\zeta_\alpha},$$

where ζ_α is chosen in accordance with the above, we see that $P_\alpha(z) = Q_\alpha(w_\alpha(z))$ has the required properties. ■

If f is analytic in U we define as usual the integral means

$$M_p(r,f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta\right\}^{1/p}, \qquad 0<p<\infty, \tag{2.3}$$

and the Hardy classes H^p $(0<p<\infty)$ to be the classes of analytic functions for which $M_p(r,f)$ remains bounded as $r \to 1$. We have the following result.

Theorem 2.2. *Let $P_\alpha(z)$ be as in* (2.1). *Then for* $0<\alpha<\infty$, $P_\alpha \in H^2$.

Proof. Let $Q_\alpha(z)$ be the function in (2.2) and define A_k such that

$$Q_\alpha(z) = \alpha + \frac{4\alpha}{\pi^2}\sum_{k=1}^{\infty} A_k z^k.$$

Then, from [10], we know that

$$A_k = \frac{4}{k}\sum_{m=1}^{k}\frac{1}{2m-1}.$$

For k large enough (≥ 8) we can easily verify that

$$A_k < \frac{4\log k}{k}.$$

Using the integral test we can verify that the series $\sum_{k=1}^{\infty}(\log k/k)^2$ converges, and hence so does $\sum_{k=1}^{\infty} A_k^2$. This means that $Q_\alpha(z) \in H^2$. Now, $P_\alpha(z) = Q_\alpha(w_\alpha(z))$ where w_α is analytic and $|w_\alpha(z)|<1$ in $|z|<1$. From a result in [3, p. 29] it follows that $P_\alpha \in H^2$. ■

3. Properties of the functions in $\mathcal{SP}(\alpha)$. We first show that the classes $\mathcal{SP}(\alpha)$ consist only of bounded functions.

Theorem 3.1. *If $f \in \mathcal{SP}(\alpha)$ then there is a constant $K(\alpha)$ such that*

$$|f(z)| < |z|K(\alpha), \qquad |z|<1.$$

Proof. If $\alpha_1 < \alpha_2$ then $\mathcal{SP}(\alpha_1) \subset \mathcal{SP}(\alpha_2)$, so it is enough to prove the theorem for $\alpha > 1$. Let k_α be the function in $\mathcal{SP}(\alpha)$ with the property

$zk'_\alpha(z)/k_\alpha(z) = P_\alpha(z)$. Since $\Omega_\alpha = P_\alpha(U)$ is convex and symmetric about the x-axis we can apply a result from Ma and Minda [7] to conclude that

$$|f(z)| \le k_\alpha(r), \qquad |z| = r < 1.$$

It remains to show that $\lim_{r\to 1} k_\alpha(r) < \infty$, which is equivalent to showing that

$$\lim_{r\to 1} \int_0^r \frac{P_\alpha(x) - 1}{x}\, dx$$

exists. Let $\delta = \tan^2\big(\pi\sqrt{1-1/\alpha}/4\big)$. Then $0 < \delta < 1$ and

$$P_\alpha(x) = \alpha\left(1 + \frac{4}{\pi^2}\left(\log \frac{1+\sqrt{(x-\delta)/(1-\delta x)}}{1-\sqrt{(x-\delta)/(1-\delta x)}}\right)^2\right).$$

The function $P_\alpha(x)$ is easily seen to be strictly increasing and $P_\alpha(\delta) = \alpha$. Define x_0 to be the value of x where $P_\alpha(x) = 2\alpha$. We then see that for $x \ge x_0$,

$$P_\alpha(x) - 1 \le (2 - 1/\alpha)(P_\alpha(x) - \alpha).$$

Therefore,

$$\int_0^r \frac{P_\alpha(x) - 1}{x}\, dx \le \int_0^{x_0} \frac{P_\alpha(x) - 1}{x}\, dx + \left(2 - \frac{1}{\alpha}\right) \int_{x_0}^r \frac{P_\alpha(x) - \alpha}{x}\, dx.$$

Since $x_0 > \delta$ it suffices to show that the integral

$$I = \int_\delta^1 \frac{1}{x}\left(\log \frac{1+\sqrt{(x-\delta)/(1-\delta x)}}{1-\sqrt{(x-\delta)/(1-\delta x)}}\right)^2 dx$$

exists. We substitute $t = (x-\delta)/(1-\delta x)$ to get

$$I = \int_0^1 \frac{1-\delta^2}{(1+\delta t)(t+\delta)}\left(\log \frac{1+\sqrt{t}}{1-\sqrt{t}}\right)^2 dt.$$

Clearly $I \le (1-\delta^2)\int_0^1 (1/t)(\log((1+\sqrt{t})/(1-\sqrt{t})))^2\, dt$, and the latter integral was examined in [12] and found to have the value $7\zeta(3)$. This ends the proof of the theorem. ∎

THEOREM 3.2. *Let $f \in \mathcal{SP}(\alpha)$, $0 < \alpha < \infty$. Then $f' \in H^2$.*

Proof. It is enough to prove that $zf' \in H^2$, and from Theorem 3.1 it follows that

$$|zf'(z)|^2 < K(\alpha)^2 \left|\frac{zf'(z)}{f(z)}\right|^2.$$

Now, $zf'(z)/f(z) \prec P_\alpha(z)$ and then it follows from Theorem 2.2 and Littlewood's subordination theorem [3, p. 10] that $zf'(z)/f(z) \in H^2$. The proof is complete. ∎

In particular, we have $f' \in H^1$, and from [3, p. 40] we know that f' has the representation

$$f'(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{f'(\zeta)}{\zeta - z}\, d\zeta.$$

Choosing the measure μ by $d\mu(x) = (f'(e^{i\theta})/(2\pi))d\theta$ where $x = e^{-i\theta}$ and $f'(e^{i\theta}) = \lim_{r\to 1} f'(re^{i\theta})$ we see that f' has a Cauchy–Stieltjes representation (1.1) with $\alpha = 1$, and from Theorem B we get

COROLLARY 3.3. *Every function in $\mathcal{SP}(\alpha)$ belongs to $\mathcal{F}_0$.*

Remark. It is natural to compare the classes $\mathcal{SP}(\alpha)$ to the classes of strongly starlike functions, $SS(\alpha)$, studied e.g. in [1]. A function $f \in SS(\alpha)$ if and only if $|\arg(zf'(z)/f(z))| < \pi\alpha/2$, so in this case we have an angular domain instead of a parabola. According to results in [1] the functions in $SS(\alpha)$ share many properties of the functions in $\mathcal{SP}(\alpha)$, e.g. that they are bounded and that $f' \in H^1$. However, we do not get $f' \in H^2$ as in $\mathcal{SP}(\alpha)$, only $f' \in H^p$ for each $p < 1/\alpha$. These classes of functions provide examples of bounded starlike functions belonging to $\mathcal{F}_0$, whereas in general we only know that bounded starlike functions belong to $\mathcal{F}_\alpha$ for every $\alpha > 0$ (Theorem A).

When $f' \in H^1$ it is well known [3, p. 42] that f is absolutely continous on $|z| = 1$ and furthermore that $w = f(e^{i\theta})$ is a parametrization of the boundary of $f(U)$. Now, the length of the boundary curve will be given by $\int_0^{2\pi} |f'(e^{i\theta})|\, d\theta$ and hence we get

COROLLARY 3.4. *Every function in $\mathcal{SP}(\alpha)$ maps $|z| = 1$ onto a rectifiable Jordan curve.*

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is a function in $\mathcal{SP}(\alpha)$ then there is a $\mu \in \mathcal{M}$ such that $f(z) = \int_\Gamma \log(1/(1 - xz))\, d\mu(x)$, which means that $a_n = (1/n)\int_\Gamma x^n\, d\mu(x)$ and therefore we have

COROLLARY 3.5. *The order of growth of the coefficients in $\mathcal{SP}(\alpha)$ is $\mathcal{O}(1/n)$.*

Remark. Ma and Minda [8] proved that the order of growth of the coefficients for functions in $\mathcal{SP}\big(\frac{1}{2}, \frac{1}{2}\big)$ is $\mathcal{O}(1/n)$. Note that by Corollary 3.5 this order of growth holds in all the classes $\mathcal{SP}(\alpha, \beta)$.

4. Some special cases. We now go back to the more general classes $\mathcal{SP}(\alpha, \beta)$. Because of the inclusion $\mathcal{SP}(\alpha, \beta) \subset \mathcal{SP}(\alpha, 0)$, $0 < \beta < 1$, the results about boundedness and membership in $\mathcal{F}_0$ will also hold for $\mathcal{SP}(\alpha, \beta)$. As mentioned before, the classes $\mathcal{SP}\big(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\big)$, $-1 \le \gamma < 1$, and in particular $\mathcal{SP}\big(\frac{1}{2}, \frac{1}{2}\big)$, play a central role in connection with the so-called *uniformly* *convex functions* (UCV). A function $f \in \mathcal{S}$ is called uniformly convex if it maps every circular arc inside of U with center also inside of U to a convex arc, and according to a result in [12],

$$f \in \mathrm{UCV} \Leftrightarrow zf' \in \mathcal{SP}\left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

The Carathéodory function associated with $\mathcal{SP}\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$ is

$$P_\gamma(z) = 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$$

and the bound on $|f(z)|$ is (see [11])

$$K_\gamma = \exp\left(\frac{14(1-\gamma)}{\pi^2}\zeta(3)\right). \tag{4.1}$$

For these classes we can obtain some results more explicit than the general ones.

THEOREM 4.1. *Let $f \in \mathcal{SP}\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$ and let K_γ be as in* (4.1), $-1 \le \gamma < 1$. *Then $f(z)$ maps $|z| = 1$ onto a rectifiable curve of length at most $2\pi K_\gamma I_\gamma$ where*

$$I_\gamma = \int_0^\infty \frac{\sqrt{(1+\gamma)^2 + 2(3-4\gamma+\gamma^2)v^2 + (1-\gamma)^2 v^4}\, e^{\pi v/2}}{1+e^{\pi v}}\, dv.$$

P r o o f. The length of $f(|z|=1)$ equals $\int_0^{2\pi} |f'(e^{i\theta})|\, d\theta$. Now $zf'(z)/f(z) \prec P_\gamma(z)$ and $P_\gamma \in H^1$ so we have

$$\int_0^{2\pi} |f'(e^{i\theta})|\, d\theta \le K_\gamma \int_0^{2\pi} |P_\gamma(e^{i\theta})|\, d\theta.$$

Computing we get

$$P_\gamma(e^{i\theta}) = \frac{1+\gamma}{2} + \frac{1-\gamma}{2\pi^2}\left(\log\frac{1+\cos(\theta/2)}{1-\cos(\theta/2)}\right)^2 + \frac{i(1-\gamma)}{\pi}\log\frac{1+\cos(\theta/2)}{1-\cos(\theta/2)}.$$

Introducing

$$v = \frac{1}{\pi}\log\frac{1+\cos(\theta/2)}{1-\cos(\theta/2)}$$

we get

$$\int_0^{2\pi} |P_\gamma(e^{i\theta})|\, d\theta = 2\int_0^{\pi} |P_\gamma(e^{i\theta})|\, d\theta = 2\pi \int_0^\infty \frac{2|P_\gamma| e^{\pi v/2}}{1+e^{\pi v}}\, dv$$

with

$$2|P_\gamma| = \sqrt{(1+\gamma)^2 + 2(3-4\gamma+\gamma^2)v^2 + (1-\gamma)^2 v^4}. \quad \blacksquare \tag{4.2}$$

R e m a r k. The class $\mathcal{SP}\left(\frac{1}{2}, \frac{1}{2}\right)$ is contained in $SS\left(\frac{1}{2}\right)$, and this inclusion is sharp [12]. Denote the upper bounds on the length of $f(|z|=1)$ in these two classes by L_1 and L_2. Then from (4.1) and Theorem 4.1 we have

$$L_1 \le 2\pi e^{(14/\pi^2)\zeta(3)} \int_0^\infty \frac{\sqrt{1+6v^2+v^4}\,e^{\pi v/2}}{1+e^{\pi v}}\,dv \approx 43.66.$$

Using results from [1] we get (here γ denotes Euler's constant)

$$L_2 \le 2\pi \cdot \frac{1}{4} e^{-\Gamma'(1/4)/\Gamma(1/4)-\gamma} \frac{1}{\cos(\pi/4)} \approx 85.48.$$

For the classes $\mathcal{SP}\big(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\big)$ we can also give an explicit upper bound on the integral means $M_2(r, f)$ for the derivative.

THEOREM 4.2. *Let* $f \in \mathcal{SP}\big(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\big)$. *Then*

$$M_2(r, f') \le K_\gamma \sqrt{3 - 4\gamma + 2\gamma^2},$$

where K_γ *is as in* (4.1).

Proof. As in the proof of Theorem 4.1 we have

$$\int_0^{2\pi} |f'(e^{i\theta})|^2\,d\theta \le K_\gamma^2 \int_0^{2\pi} |P_\gamma(e^{i\theta})|^2\,d\theta$$

and further, also as in the previous proof, we get

$$\int_0^{2\pi} |P_\gamma(e^{i\theta})|^2\,d\theta = \pi \int_0^\infty \frac{4|P_\gamma|^2 e^{\pi v/2}}{1+e^{\pi v}}\,dv.$$

A formula in [4, p. 60] states that

$$\int_0^\infty \frac{v^{2n} e^{\pi v/2}}{1+e^{\pi v}}\,dv = \frac{1}{2}|E_{2n}|, \qquad n = 0, 1, \ldots,$$

where E_n is the nth Euler number. Introducing $4|P_\gamma|^2$ from (4.2) and the Euler numbers $E_0 = 1$, $E_2 = -1$ and $E_4 = 5$ we get

$$\int_0^{2\pi} |P_\gamma(e^{i\theta})|^2 d\theta = \frac{\pi}{2}((1+\gamma)^2 + 2(3-4\gamma+\gamma^2) + 5(1-\gamma)^2) = \pi(6-8\gamma+4\gamma^2).$$

Using the definition of $M_2(r, f')$ in (2.3), the result follows. ∎

References

[1] D. A. Brannan and W. E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc. (2) 1 (1969), 431–443.

[2] L. Brickman, T. H. MacGregor and D. R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. 156 (1971), 91–107.

[3] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[4] W. Gröbner und N. Hofreiter, *Integraltafel,* 2ter Teil*, Bestimmte Integrale*, Springer, Wien, 1958.

[5] R. A. Hibschweiler and T. H. MacGregor, *Closure properties of families of Cauchy–Stieltjes transforms*, Proc. Amer. Math. Soc. 105 (1989), 615–621.

[6] —, —, *Univalent functions with restricted growth and Cauchy–Stieltjes integrals*, Complex Variables Theory Appl. 15 (1990), 53–63.

[7] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in: Proc. Internat. Conf. of Complex Analysis, to appear.

[8] —, —, *Uniformly convex functions*, Ann. Polon. Math. 57 (1992), 165–175.

[9] T. H. MacGregor, *Analytic and univalent functions with integral representation involving complex measures*, Indiana Univ. Math. J. 36 (1987), 109–130.

[10] F. Rønning, *A survey on uniformly convex and uniformly starlike functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, to appear.

[11] —, *On starlike functions associated with parabolic regions*, ibid. 45 (14) (1991), 117–122.

[12] —, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. 118 (1993), 189–196.

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