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Existence of local solutions for free boundary problems for viscous compressible barotropic fluids

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Abstract. We prove the local existence of solutions for equations of motion of a viscous compressible barotropic fluid in a domain bounded by a free surface. The solutions are shown to exist in exactly those function spaces where global solutions were found in our previous papers [14, 15].

1. Introduction. We consider the motion of a viscous compressible barotropic fluid in a domain $\Omega_t \subset \mathbb{R}^3$ bounded by a free surface $S_t = \partial\Omega_t$. Let $v = v(x,t)$ be the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $f = f(x,t)$ the external force field per unit mass, $p = p(\varrho)$ the pressure, μ and ν the viscosity coefficients, σ the surface tension coefficient and p_0 the external (constant) pressure. Then the problem is described by the following system (see [4], Chs. 1, 2, 5):

$$
\begin{aligned}
&\varrho(v_t + v\cdot\nabla v) + \nabla p(\varrho) - \mu\Delta v - \nu\nabla\operatorname{div} v = \varrho f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0 && \text{in } \Omega,\\
&\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\phi_t/|\nabla\phi| && \text{on } \widetilde{S}^T,
\end{aligned}
\tag{1.1}
$$

where $\phi(x,t) = 0$ describes S_t, $\widetilde{\Omega}^T = \bigcup_{t\in(0,T)}\Omega_t\times\{t\}$, Ω_t is the domain of the drop at time $t\in(0,T)$, $\Omega = \Omega_0$ is its initial domain, $\widetilde{S}^T = \bigcup_{t\in(0,T)} S_t\times\{t\}$, $\overline{n}$ is the unit outward vector normal to the boundary ($\overline{n} = \nabla\phi/|\nabla\phi|$), and μ, ν, σ are constant coefficients. Moreover, thermodynamic considerations imply $\nu \ge 1/(3\mu) > 0$, $\sigma > 0$. The last condition $(1.1)_5$ means that the free boundary S_t is built up of moving fluid particles.

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Finally, $\mathbb{T} = \mathbb{T}(v,p)$ denotes the stress tensor of the form

$$\begin{aligned}(1.2)\quad \mathbb{T} &= \{T_{ij}(v,p)\} = \{-p\delta_{ij} + \mu(\partial_{x_i} v_j + \partial_{x_j} v_i) + (\nu - \mu)\delta_{ij} \operatorname{div} v\} \\ &\equiv \{-p\delta_{ij}\} + \{D_{ij}(v)\},\end{aligned}$$

where $i,j = 1,2,3$, $\mathbb{D} = \mathbb{D}(v) = \{D_{ij}(v)\}$ is the deformation tensor and H is the double mean curvature of S_t, which is negative for convex domains and can be expressed in the form

$$H\overline{n} = \Delta_{S_t}(t)x, \qquad x = (x_1, x_2, x_3), \tag{1.3}$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t. Let S_t be determined by $x = x(s_1, s_2, t)$, $(s_1, s_2) \in U \subset \mathbb{R}^2$, where U is an open set. Then

$$\Delta_{S_t}(t) = \frac{1}{\sqrt{g}}\partial_{s_\alpha}\frac{1}{\sqrt{g}}\widehat{g}_{\alpha\beta}\partial_{s_\beta} = \frac{1}{\sqrt{g}}\partial_{s_\alpha}\sqrt{g}\; g^{\alpha\beta}\partial_{s_\beta}, \qquad \alpha,\beta = 1,2, \tag{1.4}$$

where the summation convention over repeated indices is assumed, $g = \det\{g_{\alpha\beta}\}_{\alpha,\beta=1,2}$, $g_{\alpha\beta} = x_\alpha \cdot x_\beta$, where $x_\alpha = \partial_{s_\alpha} x$ and the dot denotes the scalar product in the Euclidean space, $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$ and $\{\widehat{g}_{\alpha\beta}\}$ is the matrix of algebraic complements for $\{g_{\alpha\beta}\}$.

Let the domain Ω be given. Then by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi,t),\ \xi \in \Omega\}$, where $x = x(\xi,t)$ is the solution of the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \qquad x|_{t=0} = \xi \in \Omega,\ \xi = (\xi_1, \xi_2, \xi_3). \tag{1.5}$$

Therefore the transformation $x = x(\xi,t)$ connects the Eulerian x and the Lagrangian ξ coordinates of the same fluid particle. Hence

$$x = \xi + \int_0^t u(\xi,s)\,ds \equiv x(\xi,t), \tag{1.6}$$

where $u(\xi,t) = v(x(\xi,t),t)$. Moreover, the kinematic boundary condition $(1.1)_5$ implies that the boundary S_t is a material surface, so if $\xi \in S = S_0$, then $x(\xi,t) \in S_t$ and

$$S_t = \{x : x = x(\xi,t),\ \xi \in S\}.$$

In view of the continuity equation $(1.1)_2$ and $(1.1)_5$ the total mass M is conserved and

$$\int_{\Omega_t} \varrho(x,t)\,dx = M, \qquad t \in [0,T],$$

which is also a relation between ϱ and Ω_t.

We consider simultaneously two cases: $\sigma > 0$ and $\sigma = 0$. The aim of this paper is to prove local existence of solutions to problem (1.1). To prove the existence we use the Lagrangian coordinates. Therefore, we write problem

(1.1) in the form

$$
\begin{aligned}
&\eta u_t - \mu\nabla_u^2 u - \nu\nabla_u\nabla_u\cdot u + \nabla_u q = \eta g && \text{in } \Omega^T = \Omega\times(0,T),\\
&\eta_t + \eta\nabla_u\cdot u = 0 && \text{in } \Omega^T,\\
&\mathbb{T}_u(u,q)\overline{n} - \sigma\Delta_{S_t}(t)x(\xi,t) = -p_0\overline{n} && \text{on } S^T = S\times(0,T),\\
&u|_{t=0} = v_0(\xi) && \text{in } \Omega,\\
&\eta|_{t=0} = \varrho(\xi) && \text{in } \Omega,
\end{aligned}
\tag{1.8}
$$

where $\eta(\xi,t) = \varrho(x(\xi,t),t)$, $q(\xi,t) = p(x(\xi,t),t)$, $g(\xi,t) = f(x(\xi,t),t)$, $\nabla_u = \partial_x\xi_i\nabla_{\xi_i}$, $\partial_{\xi_i} = \nabla_{\xi_i}$, $\mathbb{T}_u(u,q) = -q\delta + \mathbb{D}_u(u)$, $\delta = \{\delta_{ij}\}_{i,j=1,2,3}$ is the unit matrix and $\mathbb{D}(u) = \{\mu(\partial_{x_i}\xi_k\nabla_{\xi_k}u_j + \partial_{x_j}\xi_k\nabla_{\xi_k}u_i) + (\nu-\mu)\delta_{ij}\nabla_u\cdot u\}$, with $\nabla_u\cdot u = \partial_{x_i}\xi_j\nabla_{\xi_j}u_i$, with summation over repeated indices.

Let A be the Jacobi matrix of the transformation $x = x(\xi,t)$ with elements $a_{ij} = \delta_{ij} + \int_0^t \partial_{\xi_j}u_i(\xi,\tau)\,d\tau$. Let $0 < M_0 = \text{const}$ be given. Assuming $|\nabla_\xi u|_{\infty,\Omega^T} \le M_0$ we obtain

$$
0 < c_1(1-M_0t)^3 \le \det\{\partial_{\xi_i}x_j\} \le c_2(1+M_0t)^3, \qquad t\le T,
\tag{1.9}
$$

where c_1, c_2 are constants and T is sufficiently small. Moreover, $\det A = \exp(\int_0^t \nabla_u\cdot u\,d\tau) = \varrho/\eta$.

Since S_t is determined (at least locally) by the equation $\phi(x,t) = 0$, S is described by $\phi(x(\xi,t),t)|_{t=0} \equiv \widetilde{\phi}(\xi) = 0$. Moreover, we have

$$
\overline{n} = \overline{n}(x(\xi,t),t) = \frac{\nabla_x\phi(x,t)}{|\nabla_x\phi(x,t)|}\bigg|_{x=x(\xi,t)}, \qquad \overline{n}_0 = \overline{n}_0(\xi) = \frac{\nabla_\xi\widetilde{\phi}(\xi)}{|\nabla_\xi\widetilde{\phi}(\xi)|}.
$$

The proof of existence of solutions of problem (1.8) is divided into the following steps. First we prove existence of solutions to the problem (see Section 4)

$$
\begin{aligned}
&u_t - \mu\Delta_\xi u - \nu\nabla_\xi\nabla_\xi\cdot u = f_1 && \text{in } \Omega^T,\\
&\Pi_0\mathbb{D}_\xi(u)\overline{n}_0 = g_1 && \text{on } S^T,\\
&\overline{n}_0\mathbb{D}_\xi(u)\overline{n}_0 - \sigma\overline{n}_0\Delta_S(0)\int_0^t u(\tau)\,d\tau = g_2 + \sigma\int_0^t h_1(\tau)\,d\tau && \text{on } S^T,\\
&u|_{t=0} = u_0 && \text{in } \Omega,
\end{aligned}
\tag{1.10}
$$

where Π_0 is the projection defined by $\Pi_0 g = g - (g\cdot\overline{n}_0)\overline{n}_0$ and $\mathbb{D}_\xi(u) = \{\mu(\partial_{\xi_i}u_j + \partial_{\xi_j}u_i) + (\nu-\mu)\delta_{ij}\partial_{\xi_k}u_k\}$.

Next we prove existence of solutions to the problem (see Section 5)

$$
(1.11)\quad
\begin{aligned}
&\eta u_t - \mu\nabla_\omega^2 u - \nu\nabla_\omega\nabla_\omega\cdot u = F &&\text{in } \Omega^T,\\
&\mathbb{T}_\omega(u,q)\overline{n} - \sigma\Delta_{S_t}(t)\Big(\xi + \int_0^t \omega(\xi,\tau)\,d\tau\Big)\\
&\qquad\qquad = G + \sigma\int_0^t H(\tau)\,d\tau &&\text{on } S^T,\\
&u|_{t=0} = v_0 &&\text{in } \Omega,
\end{aligned}
$$

where η and ω are given functions.

Finally, by the method of successive approximations we show existence of solutions of problem (1.8) (see Section 6).

In Section 2 we introduce the necessary notation and present some auxiliary results.

In this paper we prove existence of solutions to problem (1.1) exactly in those classes in which global existence for this problem is shown (see [14, 15]). In [13] the local existence of solutions to (1.1) is proved in totally different anisotropic Sobolev spaces. Therefore the proofs from this paper and [13] are different in many details although the general idea is the same.

In this paper we tried to present numerous details of the proof because the result is fundamental for the considerations in [9, 10, 11, 14, 15], where the local existence has already been assumed.

Local existence to problem (1.1) is also shown in [5] but in a different way and in different spaces.

2. Notation and auxiliary results. We use the anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}(Q^T)$, $l\in\mathbb{R}_+$, $Q^T = Q\times(0,T)$, where Q is either Ω (a domain in $\mathbb{R}^3$) or S (the boundary of Ω), with the norm

$$
\begin{aligned}
\|u\|^2_{W_2^{l,l/2}(Q^T)} &= \sum_{|\overline{\alpha}|\le[l]} \|D^{\overline{\alpha}}_{x,t}u\|^2_{L_2(Q^T)}\\
&\quad+ \sum_{|\overline{\alpha}|\le[l]}\bigg(\int_0^T\int_Q\int_Q \frac{|D^{\overline{\alpha}}_{x,t}u(x,t) - D^{\overline{\alpha}}_{x',t}u(x',t)|^2}{|x-x'|^{s+2(l-[l])}}\,dx\,dx'\,dt\\
&\quad+ \int_Q\int_0^T\int_0^T \frac{|D^{\overline{\alpha}}_{x,t}(x,t) - D^{\overline{\alpha}}_{x,t'}u(x,t')|^2}{|t-t'|^{1+2(l/2-[l/2])}}\,dx\,dt'\,dt\bigg)\\
&\equiv \sum_{|\overline{\alpha}|\le[l]} |D^{\overline{\alpha}}_{x,t}u|^2_{2,Q^T}\\
&\quad+ \sum_{|\overline{\alpha}|=[l]} ([D^{\overline{\alpha}}_{x,t}u]^2_{l-[l],Q^T,x} + [D^{\overline{\alpha}}_{x,t}u]^2_{l/2-[l/2],Q^T,t})\\
&\equiv \|u\|^2_{l,Q^T},
\end{aligned}
$$

where $s = \dim Q$, $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D_{x,t}^{\overline{\alpha}} = D_x^\alpha \partial_t^{\alpha_0}$, $\overline{\alpha} = (\alpha_0, \alpha)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\overline{\alpha}| = 2\alpha_0 + |\alpha|$ and $[l]$ is the integer part of l. For $Q = S$ the above norm is introduced by using local mappings and a partition of unity.

To consider problems with vanishing initial conditions we need a space of functions which admit a zero extension to $t < 0$. Therefore for every $\gamma \geq 0$ we introduce the space $H_\gamma^{l,l/2}(Q^T)$ with the norm

$$\|u\|^2_{H_\gamma^{l,l/2}(Q^T)} = \int_0^T e^{-2\gamma t}\|u\|^2_{l,Q}\,dt + \|u\|^2_{H_\gamma^{0,l/2}(Q^T)}.$$

For $l/2 \notin \mathbb{Z}$,

$$\|u\|^2_{H_\gamma^{0,l/2}(Q^T)} = \gamma^l \int_0^T e^{-2\gamma t}\|u\|^2_{0,Q}\,dt + \int_0^T e^{-2\gamma t}\,dt \int_0^\infty \frac{\|\partial_t^k u_0(\cdot, t-\tau) - \partial_t^k u_0(\cdot, t)\|^2_{0,Q}}{\tau^{1+2(l/2-k)}}\,d\tau,$$

where $k = [l/2] < l/2$, and $u_0(x,t) = u(x,t)$ for $t > 0$, $u_0(x,t) = 0$ for $t < 0$. For $l/2 \in \mathbb{Z}$,

$$\|u\|^2_{H_\gamma^{0,l/2}(Q^T)} = \int_0^T e^{-2\gamma t}(\gamma^l\|u\|^2_{0,Q} + \|\partial_t^{l/2}u\|^2_{0,Q})\,dt,$$

and we assume that $\partial_t^j u|_{t=0} = 0$, $j = 0, \ldots, l/2 - 1$, so $u_0(x,t)$ has a generalized derivative $\partial_t^{l/2}u_0$ in $Q \times (-\infty, T)$. For simplicity we write $\|u\|_{l,\gamma,Q^T} = \|u\|_{H_\gamma^{l,l/2}(Q^T)}$. In the above definition we used the notation

$$\|u\|_{l,Q} = \Big(\sum_{|\alpha|\leq[l]} |D_x^\alpha u|^2_{2,Q} + \sum_{|\alpha|=[l]} [D_x^\alpha u]^2_{l-[l],Q,x}\Big)^{1/2}.$$

Set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, $\mathbb{R}^{n+1}_T = \mathbb{R}^n \times (0,T)$, $\mathbb{D}^{n+1}_T = \mathbb{R}^n_+ \times (0,T)$, $n = 2, 3$. For functions defined in $\mathbb{R}^{n+1}_\infty$ and vanishing sufficiently fast at infinity we define the Fourier transform with respect to x and the Laplace transform with respect to t by the formula

$$\widetilde{f}(\xi, s) = \int_0^\infty e^{-st}\,dt \int_{\mathbb{R}^n} f(x,t)e^{-ix\cdot\xi}\,dx.$$

Hence we define the norm

$$|||u|||^2_{l,\gamma,\mathbb{R}^{n+1}_\infty} = \int_{\mathbb{R}^n} d\xi \int_{-\infty}^\infty |\widetilde{u}(\xi,s)|^2(|s| + |\xi|^2)^l\,d\xi_0, \qquad s = \gamma + i\xi_0, \gamma \in \mathbb{R}_+.$$

Similarly for functions defined in $\mathbb{D}_\infty^{n+1}$ we have

$$\widetilde{f}(\xi,s,x_n)=\int_0^\infty e^{-st}dt\int_{\mathbb{R}^{n-1}} f(x,t)e^{-ix'\cdot\xi'}\,dx',\qquad x'=(x_1,\dots,x_{n-1}),$$

and introduce the norm

$$|||u|||^2_{l,\gamma,\mathbb{D}_\infty^{n+1}}=\sum_{j\le[l]}\int_{\mathbb{R}^{n-1}}d\xi'\int_{-\infty}^{\infty}\|\partial^j_{x_3}\widetilde{u}(\xi',s,x_3)\|^2_{0,\mathbb{R}^1_+}(|s|+|\xi'|^2)^{l-j}\,d\xi_0$$
$$+\int_{\mathbb{R}^{n-1}}d\xi'\int_{-\infty}^{\infty}\|\widetilde{u}(\xi',s,\cdot)\|^2_{l,\mathbb{R}^1_+}\,d\xi_0,\qquad s=\gamma+i\xi_0,\ \gamma\in\mathbb{R}_+.$$

We introduce

$$\mathring{W}_2^{l,l/2}(Q^T)=\{u\in W_2^{l,l/2}(Q^T):\partial_t^iu|_{t=0}=0,\ i\le[l/2-1/2]\}$$

and $W_{2,\chi}^{l,l/2}(Q^T)$ to be the space with the norm $\|u\|_{l,Q^T}+\|u\|_{[l]+\chi,Q^T}$, where

$$\|u\|_{[l]+\chi,Q^T}=\Big(\sum_{|\alpha|=[l]}\int_0^T\frac{|D^\alpha_{x,t}u|^2_{2,Q}}{t^{2\chi}}\,dt\Big)^{1/2}.$$

For functions defined in Ω we introduce $\|u\|_{l,\Omega}=\|u\|_{H^l(\Omega)}$, $|u|_{p,\Omega}=\|u\|_{L_p(\Omega)}$, $l\in\mathbb{N}\cup\{0\}$, $1\le p\in\mathbb{R}$, and we define $\Gamma_k^l(\Omega)$ to be the space with the norm

$$\|u\|_{\Gamma_k^l(\Omega)}=\sum_{2i\le l-k}\|\partial_t^iu\|_{H^{l-2i}(\Omega)}\equiv|u|_{l,k,\Omega}.$$

Similarly we define $\Gamma_k^l(S)$.

We introduce a partition of unity. Let us define two collections of open subsets $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k\in\mathfrak{M}\cup\mathfrak{N}$, such that $\overline{\omega}^{(k)}\subset\Omega^{(k)}\subset\Omega$, $\bigcup_k\omega^{(k)}=\bigcup_k\Omega^{(k)}=\Omega$, $\overline{\Omega}^{(k)}\cap S=\emptyset$ for $k\in\mathfrak{M}$ and $\overline{\Omega}^{(k)}\cap S\ne\emptyset$ for $k\in\mathfrak{N}$. Assume that at most N_0 of the $\Omega^{(k)}$ have nonempty intersection. Suppose $\sup_k\operatorname{diam}\Omega^{(k)}\le2\lambda$ for some $\lambda>0$. Let $\zeta^{(k)}(x)$ be a smooth function such that $0\le\zeta^{(k)}(x)\le1$, $\zeta^{(k)}(x)=1$ for $x\in\omega^{(k)}$, $\zeta^{(k)}(x)=0$ for $\Omega\setminus\Omega^{(k)}$ and $|D^\nu_x\zeta^{(k)}(x)|\le c/\lambda^{|\nu|}$. Then $1\le\sum(\zeta^{(k)}(x))^2\le N_0$. Introduce the function

$$\eta^{(k)}(x)=\frac{\zeta^{(k)}(x)}{\sum_l(\zeta^{(l)}(x))^2}.$$

We have $\eta^{(k)}(x)=0$ for $x\in\Omega\setminus\Omega^{(k)}$, $\sum_k\eta^{(k)}(x)\zeta^{(k)}(x)=1$ and $|D^\nu\eta^{(k)}(x)|\le c/\lambda^{|\nu|}$. By $\xi^{(k)}$ we denote the center of $\omega^{(k)}$ and $\Omega^{(k)}$ for $k\in\mathfrak{M}$ and the center of $\overline{\omega}^{(k)}\cap S$ and $\overline{\Omega}^{(k)}\cap S$ for $k\in\mathfrak{N}$.

Considering problems invariant with respect to translations and rotations we can introduce a local coordinate system $y=(y_1,y_2,y_3)$ with center at

$\xi^{(k)}$ such that the part $\widetilde{S}^{(k)} = S \cap \overline{\Omega}^{(k)}$ of the boundary is described by $y_3 = F(y_1, y_2)$. Then we consider new coordinates defined by

$$z_i = y_i, \quad i = 1, 2, \quad z_3 = y_3 - F(y_1, y_2).$$

We will denote this transformation by $z = \Phi_k(y)$, where $y \in \omega^{(k)} \subset \Omega^{(k)}$; we assume that the latter sets are described in local coordinates at $\xi^{(k)}$ by the inequalities

$$|y_i| \le \lambda, \quad i = 1, 2, \quad 0 < y_3 - F(y_1, y_2) \le \lambda,$$
$$|y_i| \le 2\lambda, \quad i = 1, 2, \quad 0 < y_3 - F(y_1, y_2) \le 2\lambda,$$

respectively.

Assume $S \in H^{4-1/2}$. Then $\|F\|_{4-1/2,\widetilde{S}^{(k)}} \le M$, where M can be chosen independently of $\xi \in S$. We extend F to a function $\widetilde{F}$ on $\mathbb{R}^3_+$ in such a way that $\|\widetilde{F}\|_{4,\mathbb{R}^3_+} \le cM$. Moreover, $\widetilde{F}$ satisfies $\widetilde{F}(0) = 0$, $\nabla \widetilde{F}(0) = 0$. Therefore, the following inequalities hold:

$$|\widetilde{F}(z)| \le c\lambda M, \quad |\nabla \widetilde{F}(z)| \le c\lambda^a M, \quad a > 0.$$

Let $y = Y_k(t)$ be a transformation from coordinates x to local coordinates y which is the composition of a translation and a rotation. Then we set

$$\widehat{u}^{(k)}(z,t) = u(\Phi_k^{-1} \circ Y_k^{-1}(z), t), \qquad \widetilde{u}^{(k)}(z,t) = \widehat{u}^{(k)}(z,t)\widehat{\zeta}^{(k)}(z,t).$$

Now we recall some results.

LEMMA 2.1 (see [5]). *Let $u \in H^{r,r/2}_\gamma(\Omega^T)$. Then for every $\varepsilon \in (0,1)$ and $0 \le q < r - |\alpha|$,*

$$\begin{aligned}(2.1)\qquad \|D^\alpha_x u\|_{q,\gamma,\Omega^T} &\le \varepsilon^{r-|\alpha|-q}\|u\|_{r,\gamma,\Omega^T} + c\varepsilon^{-q-|\alpha|}\|e^{-\gamma t}u\|_{0,\Omega^T}\\ &\le (\varepsilon^{r-|\alpha|-q} + c\gamma^{-r/2}\varepsilon^{-q-|\alpha|})\|u\|_{r,\gamma,\Omega^T}.\end{aligned}$$

LEMMA 2.2 (see [5]). *There exist constants c_1 and c_2, which do not depend on u and γ, such that*

$$(2.2)\qquad c_1|||u|||_{l,\gamma,\mathbb{R}^{n+1}_\infty} \le \|u\|_{l,\gamma,\mathbb{R}^{n+1}_\gamma} \le c_2|||u|||_{l,\gamma,\mathbb{R}^{n+1}_\gamma}.$$

LEMMA 2.3 (see [5]). *There exist constants c_3 and c_4, which do not depend on u and γ, such that*

$$(2.3)\qquad c_3|||u|||_{l,\gamma,\mathbb{D}^{n+1}_\infty} \le \|u\|_{l,\gamma,\mathbb{D}^{n+1}_\gamma} \le c_4|||u|||_{l,\gamma,\mathbb{D}^{n+1}_\gamma}.$$

We also need

LEMMA 2.4 (see [5]). *Let $u \in H^{l,l/2}_\gamma(\mathbb{R}^{n+1}_T)$ and $0 < 2m + |\alpha| < l$. Then $\partial^m_t D^\alpha_x u \in H^{l_1,l_1/2}_\gamma(\mathbb{R}^{n+1}_T)$, where $l_1 = l - 2m - |\alpha|$ and*

$$(2.4)\qquad \|\partial^m_t D^\alpha_x u\|_{l_1,\gamma,\mathbb{R}^{n+1}_T} \le c\|u\|_{l,\gamma,\mathbb{R}^{n+1}_T}.$$

Moreover, for $\varrho \in (0, l_1)$ *and* $\varepsilon > 0$,

$$\|\partial_t^m D_x^\alpha u\|_{\varrho,\gamma,\mathbb{R}_T^{n+1}} \le \varepsilon^{l_1-\varrho}\|u\|_{l,\gamma,\mathbb{R}_T^{n+1}} + c\varepsilon^{-h}\|e^{-\gamma t}u\|_{0,\mathbb{R}_T^{n+1}}, \tag{2.5}$$

where $h = \varrho + 2m + |\alpha|$.

Let $u \in H_\gamma^{l,l/2}(\mathbb{D}_T^{n+1})$ *and* $0 \le 2m + |\alpha| < l - 1/2$. *Then* $\partial_t^m D_x^\alpha u|_{x_n=0} \in H_\gamma^{l_2,l_2/2}(\mathbb{R}_T^n)$, *where* $l_2 = l - 2m - |\alpha| - 1/2$, *and*

$$\|\partial_t^m D_x^\alpha u|_{x_n=0}\|_{l_2,\gamma,\mathbb{R}_T^n} \le c\|u\|_{l,\gamma,\mathbb{D}_T^{n+1}}. \tag{2.6}$$

3. Existence of solutions to problem (1.10) with vanishing initial data in the half-space. Now we consider problem (1.10) in the half-space $x_3 > 0$. First we examine the following problem:

$$\begin{aligned}
&u_t - \mu\Delta u - \nu\nabla \operatorname{div} u = 0, && x_3 > 0,\\
&\mu\Big(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\Big) = b_i, \quad i = 1,2, && x_3 = 0,\\
&(\mu+\nu)\frac{\partial u_3}{\partial x_3} + (\nu-\mu)\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)\\
&\qquad + \sigma\Delta'\int_0^t u_3(\tau)\,d\tau = b_3, && x_3 = 0,\\
&u|_{t=0} = 0, && x_3 > 0,
\end{aligned} \tag{3.1}$$

where $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$. By applying the Laplace–Fourier transformation

$$\widetilde{f}(\xi', s, x_3) = \int_0^\infty e^{-st}dt \int_{\mathbb{R}^2} f(x,t)e^{-ix'\cdot\xi'}\,dx', \quad \operatorname{Re} s > 0, \tag{3.2}$$

where $\xi' = (\xi_1, \xi_2)$, $x' = (x_1, x_2)$, $\xi' \cdot x' = \xi_1 x_1 + \xi_2 x_2$, problem (3.1) takes the form

$$\begin{aligned}
&\mu\frac{d^2\widetilde{u}_k}{dx_3^2} + \nu i\xi_k\frac{d\widetilde{u}_3}{dx_3} - (s + \mu\xi^2)\widetilde{u}_k - \nu\xi_k\xi_j\widetilde{u}_j = 0, \quad k = 1,2, \quad x_3 > 0,\\
&(\mu+\nu)\frac{d^2\widetilde{u}_3}{dx_3^2} + \nu i\xi_j\frac{d\widetilde{u}_j}{dx_3} - (s + \mu\xi^2)\widetilde{u}_3 = 0, \qquad x_3 = 0,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\mu\frac{d\widetilde{u}_k}{dx_3} + i\xi_k\widetilde{u}_3 = \widetilde{b}_k, \quad k = 1,2, && x_3 = 0,\\
&(\mu+\nu)\frac{d\widetilde{u}_3}{dx_3} + (\nu-\mu)(i\xi_j\widetilde{u}_j) - \frac{\sigma}{s}\xi^2\widetilde{u}_3 = \widetilde{b}_3, && x_3 = 0,
\end{aligned} \tag{3.4}$$

$\widetilde{u} \to 0$ as $x_3 \to \infty$, where $\xi = (\xi_1, \xi_2)$, $\xi^2 = \xi_1^2 + \xi_2^2$.

Every solution to (3.3) vanishing at infinity has the form

$$\widetilde{u} = \Phi(\xi, s)e^{-\tau_1 x_3} + \Psi(\xi, s)(\xi_1, \xi_2, i\tau_2)e^{-\tau_2 x_3}, \tag{3.5}$$

where $\Phi(\xi,s)=(\phi_1,\phi_2,(i/\tau_1)\xi\cdot\phi)$, $\phi_j=\phi_j(\xi,s)$, $j=1,2$, $\tau_1=\sqrt{s/\mu+\xi^2}$, $\tau_2=\sqrt{s/(\mu+\nu)+\xi^2}$, $\arg\tau_j\in(-\pi/4,\pi/4)$, $j=1,2$, $\xi\cdot\phi=\xi_1\phi_1+\xi_2\phi_2$, $\phi=(\phi_1,\phi_2)$.

Putting (3.5) into (3.4) yields

$$\begin{aligned}&\xi_j\xi\cdot\phi+\tau_1^2\phi_j+2\xi_j\tau_1\tau_2\psi=-\mu^{-1}\widetilde{b}_j\tau_1,\qquad j=1,2,\\&\Big(2\mu+\frac{\sigma}{s}\frac{\xi^2}{\tau_1}\Big)\xi\cdot\phi+\Big((\mu+\nu)\tau_2^2+(\mu-\nu)\xi^2+\frac{\sigma}{s}\xi^2\tau_2\Big)\psi=i\widetilde{b}_3.\end{aligned}\tag{3.6}$$

Solving (3.6) we have

$$\begin{aligned}\xi\cdot\phi&=-\frac{1}{D}\Big[\frac{\tau_1}{\mu}\Big(s+2\mu\xi^2+\frac{\sigma}{s}\xi^2\tau_2\Big)\widetilde{b}\cdot\xi+2\xi^2\tau_1\tau_2 i\widetilde{b}_3\Big],\\\psi&=\frac{1}{D}\Big[\frac{\tau_1}{\mu}\Big(2\mu+\frac{\sigma}{s}\frac{\xi^2}{\tau_1}\Big)\widetilde{b}\cdot\xi+\Big(\frac{s}{\mu}+2\xi^2\Big)i\widetilde{b}_3\Big],\end{aligned}\tag{3.7}$$

where

$$D=\mu\Big[\Big(\frac{s}{\mu}+2\xi^2\Big)^2+\frac{\sigma}{\mu^2}\xi^2\tau_2-4\xi^2\tau_1\tau_2\Big].\tag{3.8}$$

Using (3.6) and (3.7) in (3.5) gives

$$\begin{aligned}\widetilde{u}_k&=\frac{\xi_k}{D}\frac{1}{\tau_1}(\tau_1^2+\xi^2-2\tau_1\tau_2)\widetilde{b}\cdot\xi e_1+\frac{\xi_k}{D}2(\tau_1e_2-\tau_2e_1)\widetilde{b}\cdot\xi\\&\quad+\frac{\xi_k}{D}i\widetilde{b}_3\Big[\Big(\frac{s}{\mu}+2\xi^2-2\tau_1\tau_2\Big)e_2+2\tau_1\tau_2(e_2-e_1)\Big]\\&\quad+\frac{\xi_k}{D}\frac{\sigma}{\mu s\tau_1}(\tau_1e_2-\tau_2e_1)\xi^2\widetilde{b}\cdot\xi-\frac{1}{\mu\tau_1}\widetilde{b}_ke_1,\qquad k=1,2,\\\widetilde{u}_3&=\frac{i}{D}\Big[2\tau_1\tau_2(\tau_2-\tau_1)e_0+\Big(2\tau_1\tau_2-\Big(\frac{s}{\mu}+2\xi^2\Big)\Big)e_1\Big]\widetilde{b}\cdot\xi\\&\quad-\frac{i}{D}\frac{\sigma}{\mu s}\xi^2\tau_2(\tau_1-\tau_2)\widetilde{b}\cdot\xi e_0\\&\quad-\frac{\tau_2}{D}\Big[\Big(\frac{s}{\mu}+2\xi^2\Big)(\tau_2-\tau_1)e_0+\frac{s}{\mu}e_1\Big]\widetilde{b}_3,\end{aligned}\tag{3.9}$$

where $e_i=e^{-\tau_ix_3}$, $i=1,2$, $e_0=(e_1-e_2)/(\tau_1-\tau_2)$.

Using the expressions

$$\tau_1^2-\tau_2^2=\frac{s}{\mu}-\frac{s}{\mu+\nu}=\frac{\nu}{\mu(\mu+\nu)}s\equiv c_0s,$$

$$\tau_1^2+\xi^2-2\tau_1\tau_2=(\tau_1-\tau_2)^2-\frac{s}{\mu+\nu}=\frac{c_0s}{(\tau_1+\tau_2)^2}-\frac{s}{\mu+\nu}\equiv c_1s,$$

$$\tau_1e_2-\tau_2e_1=(\tau_1-\tau_2)e_2+\tau_2(e_2-e_1)=\frac{c_0s}{\tau_1+\tau_2}e_2-\tau_2\frac{c_0s}{\tau_1+\tau_2}e_0,$$

we write (3.9) in the form

$$\begin{aligned}
\widetilde{u}_k &= \bigg(\frac{\xi_k}{D}\frac{c_1 s}{\tau_1}\widetilde{b}\cdot\xi - \frac{1}{\mu\tau_1}\widetilde{b}_k\bigg)e_1\\
&\quad + 2\frac{\xi_k}{D}\bigg(\frac{c_0 s}{\tau_1+\tau_2}\widetilde{b}\cdot\xi + c_1 s i\widetilde{b}_3 + \frac{\sigma}{\mu\tau_1}\frac{c_0\xi^2}{\tau_1+\tau_2}\widetilde{b}\cdot\xi\bigg)e_2\\
&\quad - \frac{\xi_k}{D}\bigg(2\tau_2\frac{c_0 s}{\tau_1+\tau_2}\widetilde{b}\cdot\xi + 2\tau_1\tau_2\frac{c_0 s}{\tau_1+\tau_2}i\widetilde{b}_3\\
&\quad + \frac{\sigma}{\mu\tau_1}\frac{\tau_2 c_0}{\tau_1+\tau_2}\xi^2\widetilde{b}\cdot\xi\bigg)e_0 \qquad (3.10)\\
&\equiv E_{k1}e_1 + E_{k2}e_2 + E_{k0}e_0, \quad k=1,2,\\
\widetilde{u}_3 &= -\frac{1}{D}\bigg[\bigg(2\tau_1\tau_2 i\frac{c_0 s}{\tau_1+\tau_2} + \frac{i\sigma}{\mu}\frac{\xi^2\tau_2 c_0}{\tau_1+\tau_2}\bigg)\widetilde{b}\cdot\xi\\
&\quad + \frac{\tau_2 c_0 s}{\tau_1+\tau_2}\bigg(\frac{s}{\mu}+2\xi^2\bigg)\widetilde{b}_3\bigg]e_0\\
&\quad + \frac{1}{D}\bigg(ic_1 s\widetilde{b}\cdot\xi - \frac{\tau_2 s}{\mu}\widetilde{b}_3\bigg)e_1 \equiv E_{30}e_0 + E_{31}e_1.
\end{aligned}$$

From [8, 12] we have

Lemma 3.1. *For all* $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$ *and* $s=\gamma+i\xi_0$ *with* $\gamma>0$, $\gamma\in\mathbb{R}$, $\xi_0\in\mathbb{R}^1$,

$$|D|\ge c_2|s|\xi^2,\qquad |D|\ge c_3(|s|^2+|\xi|^3). \tag{3.11}$$

Using Lemma 3.1 we obtain

Lemma 3.2. *For* $\xi\in\mathbb{R}^2$ *and* $\gamma=\operatorname{Re}s>0$,

$$|\overline{E}_0|\le c_4|\widetilde{b}|,\qquad |\overline{E}_1|+|\overline{E}_2|\le\frac{c_5}{\sqrt{|s|+\xi^2}}|\widetilde{b}|, \tag{3.12}$$

where $\overline{E}_i=(E_{1i},E_{2i},E_{3i})$, $i=0,1,2$.

Moreover, from [12] (see also [5]) we have

Lemma 3.3. *For* $\xi\in\mathbb{R}^2$, $s=\gamma+i\gamma_0$, $\gamma,\xi_0\in\mathbb{R}$, $\gamma>0$, *and for every nonnegative integer* j,

$$\int_0^\infty\bigg|\frac{d^j e_i(x_3)}{dx_3^j}\bigg|^2dx_3\le\frac{1}{\sqrt2}|\tau_i|^{2j-1},\qquad i=1,2,$$

$$\int_0^\infty\bigg|\frac{d^j e_0(x_3)}{dx_3^j}\bigg|^2dx_3\le c\frac{|\tau_1|^{2j-1}+|\tau_2|^{2j-1}}{|\tau_1|^2}.$$

Lemmas 3.1–3.3 and [12] imply

THEOREM 3.4. *Let* $b_1, b_2 \in H_\gamma^{2+1/2,1+1/4}(\mathbb{R}^3_\infty)$, $b_3 = d_1 + \sigma \int_0^t d_2(\tau)\, d\tau$, $d_1 \in H_\gamma^{2+1/2,1+1/4}(\mathbb{R}^3_\infty)$, *and* $d_2 \in H_\gamma^{2-1/2,1-1/4}(\mathbb{R}^3_\infty)$. *Then solutions of problem* (3.1) *satisfy the estimate*

$$(3.13)\qquad \sum_{i=1}^{3} \|u_i\|_{4,\gamma,\mathbb{D}^4_\infty} \le c(\gamma)\Big(\sum_{\alpha=1}^{2} \|b_\alpha\|_{2+1/2,\gamma,\mathbb{R}^3_\infty} + \|d_1\|_{2+1/2,\gamma,\mathbb{R}^3_\infty} + \|d_2\|_{2-1/2,\gamma,\mathbb{R}^3_\infty}\Big),$$

where $c(\gamma)$ *remains bounded for* $\gamma > \gamma_0 > 0$.

Now we consider the problem

$$(3.14)\qquad \begin{aligned} &u_t - \mu\Delta u - \nu\nabla \operatorname{div} u = f, && x_3 > 0,\\ &\mu\Big(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\Big) = b_i, \quad i = 1,2, && x_3 = 0,\\ &(\mu+\nu)\frac{\partial u_3}{\partial x_3} + (\nu-\mu)\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big) + \sigma\Delta' \int_0^t u_3(\tau)\, d\tau = b_3, && x_3 = 0,\\ &u|_{t=0} = 0, && x_3 > 0. \end{aligned}$$

In view of the considerations in [12] and Theorem 3.4 we have

THEOREM 3.5. *Let the assumptions of Theorem* 3.4 *be satisfied. Let* $f \in H_\gamma^{2,1}(\mathbb{D}^4_\infty)$. *Then there exists a solution to* (3.14) *such that* $u \in H_\gamma^{4,2}(\mathbb{D}^4_\infty)$ *and*

$$(3.15)\qquad \sum_{i=1}^{3} \|u_i\|_{4,\gamma,\mathbb{D}^4_\infty} \le c(\gamma)\Big(\sum_{\alpha=1}^{2} \|b_\alpha\|_{2+1/2,\gamma,\mathbb{R}^3_\infty} + \|d_1\|_{2+1/2,\gamma,\mathbb{R}^3_\infty} + \|d_2\|_{2-1/2,\gamma,\mathbb{R}^3_\infty} + \|f\|_{2,\gamma,\mathbb{D}^4_\infty}\Big).$$

4. Existence of solutions to problem (1.10). First we consider problem (1.10) with vanishing initial data:

$$(4.1)\qquad \begin{aligned} &L(\partial_x, \partial_t) \equiv u_t - \mu\Delta u - \nu\nabla \operatorname{div} u = f && \text{in } \Omega \times (-\infty, T),\\ &B_1(x, \partial_x)u \equiv \Pi_0 \mathbb{T}(u)\overline{n}_0 = \Pi_0 b \equiv b' && \text{on } S \times (-\infty, T),\\ &B_2(x, \partial_x)u \equiv \overline{n}_0 \mathbb{T}(u)\overline{n}_0 - \sigma\overline{n}_0\Delta_S \int_0^t u(\tau)\, d\tau = b\cdot\overline{n} \equiv d_1 + \sigma\int_0^t d_2(\tau)\, d\tau && \text{on } S \times (-\infty, T), \end{aligned}$$

where $\Pi_0 b = b - (b\cdot\overline{n}_0)\overline{n}_0 \equiv b'$. We write $B(x,\partial_x)u = (B_1(x,\partial_x)u, B_2(x,\partial_x)u)$.

Let $f^{(k)}(x,t)=\zeta^{(k)}(x,t)f(x,t)$. We denote by $R^{(k)}$, $k\in\mathfrak{M}$, the operator

$$u^{(k)}(x,t)=R^{(k)}f^{(k)}(x,t), \tag{4.2}$$

where $u^{(k)}(x,t)$ is the solution of the Cauchy problem

$$L(\partial_x,\partial_t)u^{(k)}(x,t)=f^{(k)}(x,t). \tag{4.3}$$

For $k\in\mathfrak{N}$ we define $R^{(k)}$ to be the operator

$$\widehat{u}^{(k)}(z,t)=R^{(k)}(\widehat{f}^{(k)}(z,t),\widehat{b}^{(k)}(z,t)), \tag{4.4}$$

where $\widehat{u}^{(k)}(z,t)$ is the solution to the boundary value problem

$$L(\partial_z,\partial_t)\widehat{u}^{(k)}(z,t)=\widehat{f}^{(k)}(z,t),\qquad B(z,\partial_z)\widehat{u}^{(k)}(z,t)=\widehat{b}^{(k)}(z,t), \tag{4.5}$$

where $\widehat{u}^{(k)}(z,t)=Z_k^{-1}u^{(k)}(x,t)$ and Z_k is the operator which represents the relation between $\widehat{u}^{(k)}(z,t)$ and $u^{(k)}(x,t)$.

Then we define an operator R (called a *regularizer*) by the formula (see [3, 6])

$$Rh=\sum_k \eta^{(k)}(x)u^{(k)}(x,t), \tag{4.6}$$

where

$$h^{(k)}(x,t)=\begin{cases} f^{(k)}(x,t), & k\in\mathfrak{M},\\ \{\widehat{f}^{(k)}(z,t),\widehat{b}^{(k)}(z,t)\}, & k\in\mathfrak{N},\end{cases}$$

and

$$u^{(k)}(x,t)=\begin{cases} R^{(k)}f^{(k)}(x,t), & k\in\mathfrak{M},\\ Z_kR^{(k)}(Z_k^{-1}f^{(k)}(x,t),Z_k^{-1}b^{(k)}(x,t)), & k\in\mathfrak{N}.\end{cases}$$

Theorem 3.5 implies existence of solutions of problems (4.3), (4.5) and the estimates

$$\|u^{(k)}\|_{4,\gamma,\mathbb{R}^4_\infty}\le c\|f^{(k)}\|_{2,\gamma,\mathbb{D}^4_\infty},\qquad k\in\mathfrak{M}, \tag{4.7}$$

and

$$\begin{aligned}\|\widehat{u}^{(k)}\|_{4,\gamma,\mathbb{R}^4_\infty}\le c\Big(\|\widehat{f}^{(k)}\|_{2,\gamma,\mathbb{D}^4_\infty}+\sum_{i=1}^2\|\widehat{b}_i^{(k)}\|_{2+1/2,\gamma,\mathbb{R}^3_\infty}\\ +\|\widehat{d}_1^{(k)}\|_{2+1/2,\gamma,\mathbb{R}^3_\infty}+\|\widehat{d}_2^{(k)}\|_{2-1/2,\gamma,\mathbb{R}^3_\infty}\Big),\qquad k\in\mathfrak{N}.\end{aligned} \tag{4.8}$$

Let $h=(f,b_1,b_2,d_1,d_2)\in H_\gamma^{l,l/2}(\Omega^T)\times H_\gamma^{l+1/2,l/2+1/4}(S^T)\times H_\gamma^{l+1/2,l/2+1/4}(S^T)\times H_\gamma^{l+1/2,l/2+1/4}(S^T)\times H_\gamma^{l-1/2,l/2-1/4}(S^T)\equiv H_\gamma^l$ and let $V_\gamma^l=H_\gamma^{l+2,l/2+1}(\Omega^T)$.

Inequalities (4.7) and (4.8) imply

LEMMA 4.1 (see [12]). *Let $S \in H^{4-1/2}$ and $h \in H^2_\gamma$ with γ sufficiently large. Then there exists a bounded linear operator $R : H^2_\gamma \to V^2_\gamma$ such that*

$$\|Rh\|_{V^2_\gamma} \le c\|h\|_{H^2_\gamma}, \tag{4.9}$$

where c does not depend on γ and h.

We write problem (4.1) in the following short form:

$$Au = h, \quad A = (L, B). \tag{4.10}$$

LEMMA 4.2. *Let $S \in H^{4-1/2}$ and $h \in H^2_\gamma$ with γ sufficiently large. Then*

$$ARh = h + Th, \tag{4.11}$$

where T is a bounded operator in H^2_γ with small norm for small λ and large γ.

Proof. We have

$$\begin{aligned} LRh = &\sum_{k\in\mathfrak{M}\cup\mathfrak{N}} (L(\partial_x,\partial_t)\eta^{(k)}u^{(k)} - \eta^{(k)}L(\partial_x,\partial_t)u^{(k)}) \\ &+ \sum_{k\in\mathfrak{N}} \eta^{(k)} Z_k (L(\partial_z - \nabla F\partial_{z_3},\partial_t) - L(\partial_z,\partial_t))Z_k^{-1}u^{(k)}(x,t) \\ &+ \sum_{k\in\mathfrak{M}} \eta^{(k)} L(\partial_x,\partial_t)u^{(k)}(x,t) + \sum_{k\in\mathfrak{N}} \eta^{(k)} Z_k L(\partial_z,\partial_t) Z_k^{-1} u^{(k)}(x,t) \\ = &\, f + T_1 h, \end{aligned}$$

and

$$\begin{aligned} BRh = &\sum_{k\in\mathfrak{M}\cup\mathfrak{N}} (B(x,\partial_x)\eta^{(k)}u^{(k)} - \eta^{(k)}B(x,\partial_x)u^{(k)}) \\ &+ \sum_{k\in\mathfrak{M}\cup\mathfrak{N}} \eta^{(k)}(B(x,\partial_x) - B(\xi^{(k)},\partial_x))u^{(k)} + \sum_{k\in\mathfrak{M}} \eta^{(k)} b^{(k)} \\ &+ \sum_{k\in\mathfrak{N}} \eta^{(k)} Z_k (B(\xi^{(k)},\partial_z - \nabla F\partial_{z_3}) - B(\xi^{(k)},\partial_z)) Z^{-1} u^{(k)}(x,t) \\ &+ \sum_{k\in\mathfrak{N}} \eta^{(k)} Z_k B(\xi^{(k)},\partial_z) Z_k^{-1} u^{(k)}(x,t) = b + T_2 h. \end{aligned}$$

Now we estimate operators T_1 and T_2. By using Lemmas 2.1, 2.4 and Theorems 3.4, 3.5 the first term in $T_1 h$ is estimated in the following way:

$$\begin{aligned} \Big\| \sum_{k\in\mathfrak{M}\cup\mathfrak{N}} (L\eta^{(k)}u^{(k)} - \eta^{(k)}Lu^{(k)}) \Big\|_{2,\gamma,\Omega^T} &\le c \sum_{k\in\mathfrak{M}\cup\mathfrak{N}} \|u^{(k)}\|_{3,\gamma,Q^{(k)}} \\ &\le c(\varepsilon^{\delta_1} + c_0(\varepsilon)\gamma^{-\delta_2}) \sum_{k\in\mathfrak{M}\cup\mathfrak{N}} \|u^{(k)}\|_{4,\gamma,Q^{(k)}} \le c(\varepsilon^{\delta_1} + c_0(\varepsilon)\gamma^{-\delta_2})\|h\|_{H^2_\gamma}, \end{aligned}$$

where $\delta_i > 0$, $i = 1, 2$, $Q^{(k)} = \Omega^{(k)} \times (0, T)$, and $c_0(\varepsilon)$ is a decreasing function.

The second term in $T_1 h$ is bounded by

$$
\begin{aligned}
c\sum_{k\in\mathfrak{N}}(\|(\nabla\widetilde{F}\nabla^2\widetilde{F}\nabla\widehat{u}^{(k)})|_{z=\Phi_k(y(x))}\|_{2,\gamma,Q^{(k)}}\\
&+\|(\nabla\widetilde{F}(1+\nabla\widetilde{F})\nabla^2\widehat{u}^{(k)})|_{z=\Phi_k(y(x))}\|_{2,\gamma,Q^{(k)}}\\
&+\|(\nabla^2\widetilde{F}\nabla\widehat{u}^{(k)})|_{z=\Phi_k(y(x))}\|_{2,\gamma,Q^{(k)}})\\
\le c\sum_{k\in\mathfrak{N}}(p(\|\nabla\widetilde{F}\|_{3,Q^{(k)}})\|u^{(k)}\|_{3,\gamma,Q^{(k)}}\\
&+\sup_{Q^{(k)}}|\nabla\widetilde{F}|(1+\sup_{Q^{(k)}}|\nabla\widetilde{F}|)\|u^{(k)}\|_{4,\gamma,Q^{(k)}})\equiv I,
\end{aligned}
$$

where p is a polynomial of degree two. Using $\sup_{\Omega^{(k)}}|\nabla\widetilde{F}|\le c\lambda^{1/2}\|\nabla\widetilde{F}\|_{3,\Omega^{(k)}}$, the interpolation inequalities and Theorems 3.4 and 3.5 we have

$$
I\le c(\varepsilon^{\delta_1}+c_0(\varepsilon)(\lambda^{\delta_2}+\gamma^{-\delta_3}))\|h\|_{H^2_\gamma},\quad \delta_i>0,\ i=1,2,3,
$$

and $c_0(\varepsilon)$ is a decreasing function.

Similar considerations can be applied to the other terms of T_1 and T_2. Summarizing we have

$$
\|Th\|_{H^2_\gamma}\le c[\varepsilon^{\delta_1}+c_0(\varepsilon)(\lambda^{d_2}+\gamma^{-\delta_3})]\|h\|_{H^2_\gamma}. \tag{4.12}
$$

This concludes the proof.

LEMMA 4.2 (see [12]). *Let $S\in H^{4-1/2}$. Then for every $v\in V^2_\gamma$,*

$$
RAv=v+Wv, \tag{4.13}
$$

where W is a bounded operator in V^2_γ whose norm can be made small for small λ and large γ, because

$$
\|Wv\|_{V^2_\gamma}\le c[\varepsilon^{\delta_1}+c_0(\varepsilon)(\lambda^{\delta_2}+\gamma^{-\delta_3})]\|v\|_{V^2_\gamma},\quad \varepsilon\in(0,1), \tag{4.14}
$$

where $c_0(\varepsilon)$ is a decreasing function.

Proof. See the proof of Theorem 3.4 of [12].

For sufficiently large γ and sufficiently small ε and λ the norms of W and T are less than one. Therefore Lemmas 4.1 and 4.2 imply

THEOREM 4.3. *Let $f\in H^{2,1}_\gamma(\Omega^T)$, $b',d_1\in H^{2+1/2,1+1/4}_\gamma(S^T)$, $d_2\in H^{2-1/2,1-1/4}_\gamma(S^T)$ and $S\in H^{4-1/2}_\gamma$. Then for sufficiently large γ there exists a unique solution of problem (4.1) such that $u\in H^{4,2}_\gamma(\Omega^T)$ and*

$$
\begin{aligned}
\|u\|_{4,\gamma,\Omega^T}\le c(\|f\|_{2,\gamma,\Omega^T}+\|b'\|_{2+1/2,\gamma,S^T}\\
+\|d_1\|_{2+1/2,\gamma,S^T}+\|d_2\|_{2-1/2,\gamma,S^T}),
\end{aligned} \tag{4.15}
$$

where c does not depend on u and γ.

Now we consider problem (1.10) with nonvanishing initial data. Then we have

THEOREM 4.4. *Let* $f \in W_2^{2,1}(\Omega^T)$, $g_i \in W_{2,1/4}^{2+1/2,1+1/4}(S^T)$, $i = 1, 2$, $h \in W_2^{1+1/2,1/2+1/4}(S^T)$, $S \in H^{4-1/2}$, $u_0 \in H^3(\Omega)$, *and* $T < \infty$. *Then there exists a solution of problem* (1.10) *such that* $u \in W_2^{4,2}(\Omega^T)$ *and*

$$\|u\|_{4,\Omega^T} \le c(T)(X_1 + X_2), \tag{4.16}$$

where $X_1 = \|f_1\|_{2,\Omega^T} + \sum_{i=1}^2 \|g_i\|_{(2+1/2),S^T,1/4} + \|h\|_{1+1/2,S^T}$, $X_2 = \|f(0)\|_{1,\Omega} + \|u_0\|_{3,\Omega}$, *and* $c(T)$ *is an increasing function of* T.

Proof. Let $\phi^0 = u_0 \in H^3(\Omega)$ and $\phi^1 = \mu\Delta u_0 + \nu\nabla \operatorname{div} u_0 + f(0) \in H^1(\Omega)$. We extend the functions onto $\mathbb{R}^3$ in such a way that the extended functions $\widetilde{\phi}^0$, $\widetilde{\phi}^1$ satisfy $\widetilde{\phi}^0 \in H^3(\mathbb{R}^3)$, $\widetilde{\phi}^1 \in H^1(\mathbb{R}^3)$ and $\|\widetilde{\phi}^0\|_{3,\mathbb{R}^3} \le c\|\phi^0\|_{3,\Omega}$, $\|\widetilde{\phi}^1\|_{1,\mathbb{R}^3} \le c\|\phi^1\|_{1,\Omega}$.

In view of Lemma 4.5 below there exists a function $\widetilde{v} \in W_2^{4,2}(\mathbb{R}^3 \times \mathbb{R}^1)$ such that

$$\partial_t^i \widetilde{v}|_{t=0} = \widetilde{\phi}^i, \quad i = 1, 2, \tag{4.17}$$

and

$$\begin{aligned} \|v\|_{4,\Omega^T} \le \|\widetilde{v}\|_{4,\mathbb{R}^3\times\mathbb{R}^1} &\le c(\|\widetilde{\phi}^0\|_{3,\mathbb{R}^3} + \|\widetilde{\phi}^1\|_{1,\mathbb{R}^3}) \\ &\le c(\|u_0\|_{3,\Omega} + \|f(0)\|_{1,\Omega}), \end{aligned} \tag{4.18}$$

where $v = \widetilde{v}|_{\Omega^T}$. Introducing the function

$$w = u - v \tag{4.19}$$

we see that it is a solution of the problem

$$\begin{aligned} &w_t - \mu\Delta w - \nu\nabla \operatorname{div} w = f' && \text{in } \Omega^T, \\ &\Pi_0 \mathbb{D}(w)\overline{n}_0 = g_1' && \text{on } S^T, \\ &\overline{n}_0 \mathbb{D}(w)\overline{n}_0 - \sigma\overline{n}_0 \Delta_S(0) \int_0^t w(\tau)\,d\tau = g_2' + \sigma \int_0^t h'(\tau)\,d\tau && \text{on } S^T, \\ &w|_{t=0} = 0 && \text{in } \Omega, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} &f' = f - (v_t - \mu\Delta v - \nu\nabla \operatorname{div} v) \in \mathring{W}_2^{2,1}(\Omega^T), \\ &g_1' = g_1 - \Pi_0 \mathbb{D}(v)\overline{n}_0 \in \mathring{W}_2^{2+1/2,1+1/4}(S^T), \\ &g_2' = g_2 - \overline{n}_0 \mathbb{D}(v)\overline{n} \in \mathring{W}_2^{2+1/2,1+1/4}(S^T), \\ &h' = h - \overline{n}_0 \Delta_S(0) v \in \mathring{W}_2^{2-1/2,1-1/4}(S^T). \end{aligned} \tag{4.21}$$

To prove existence of solutions to problem (4.20) we have to extend the right-hand side functions by zero for $t < 0$. The function f' can be extended easily to a function $f''\in H_0^{2,1}(\Omega^T)$ and

$$\|f''\|_{2,0,\Omega^T}\le c\|f'\|_{2,\Omega^T}\le c(\|f\|_{2,\Omega^T}+\|v\|_{4,\Omega^T}). \tag{4.22}$$

Since $1-1/4-[1-1/4]=3/4>1/2$, in view of Lemma 2.5 of [13], h' can be extended by zero to a function $h''\in H_0^{2-1/2,1-1/4}(S^T)$ and

$$\|h''\|_{2-1/2,0,S^T}\le c\|h'\|_{2-1/2,S^T}\le c(\|h\|_{2-1/2,S^T}+\|v\|_{4,\Omega^T}). \tag{4.23}$$

Since $1+1/4-[1+1/4]=1/4<1/2$, to extend the function g_i', $i=1,2$, we have to assume that $g_i'\in W_{2,1/4}^{2+1/2,1+1/4}(S^T)$, $i=1,2$. Hence $g_i\in H_{2,1/4}^{2+1/2,1+1/4}(S^T)$, $i=1,2$, and v must be such that

$$\begin{aligned}\|\Pi_0\mathbb{D}(v)\overline{n}_0\|_{(2+1/2),S^T,1/4}+\|\overline{n}_0\mathbb{D}(v)\overline{n}_0\|_{(2+1/2),S^T,1/4}\\ \le c(\|u_0\|_{3,\Omega}+\|f(0)\|_{1,\Omega}).\end{aligned} \tag{4.24}$$

If we show this, then the extended functions $g_i''\in H_0^{2+1/2,1+1/4}(S^T)$, $i=1,2$, and

$$\begin{aligned}\|g_i''\|_{2+1/2,0,S^T}&\le c\|g_i'\|_{(2+1/2),S^T,1/4}\\ &\le c(\|g_i\|_{(2+1/2),S^T,1/4}+\|u_0\|_{3,\Omega}+\|f(0)\|_{1,\Omega}).\end{aligned} \tag{4.25}$$

To prove (4.24) it is sufficient to estimate the expressions

$$\begin{aligned}&\Big(\int_0^T\frac{|D^2_{\xi,t}(\Pi_0\mathbb{D}_\xi(v)\overline{n}_0)|^2_{2,S}}{t^{1/2}}\,dt\Big)^{1/2}+\Big(\int_0^T\frac{|D^2_{\xi,t}(\overline{n}_0\mathbb{D}(v)\overline{n}_0)|^2_{2,S}}{t^{1/2}}\,dt\Big)^{1/2}\\ &\quad\le\Big(\int_0^T\frac{|D_\xi v|^2_{2,S}+|D^2_\xi v|^2_{2,S}+|D^3_\xi v|^2_{2,S}+|D_\xi\partial_t v|^2_{2,S}}{t^{1/2}}\,dt\Big)^{1/2}\\ &\quad\le\Big(\int_0^\infty\frac{|D_\xi\widetilde v|^2_{2,S}+|D^2_\xi\widetilde v|^2_{2,S}+|D^3_\xi\widetilde v|^2_{2,S}+|D_\xi\partial_t\widetilde v|^2_{2,S}}{t^{1/2}}\,dt\Big)^{1/2}\equiv I,\end{aligned}$$

where we have used the fact that $S\in H^{4-1/2}$. In view of Lemma 2.6 of [13] we have the estimate

$$\begin{aligned}I&\le c\Big(\int_0^\infty dt\int_0^\infty dt'\Big(\frac{|D_\xi\widetilde v(t)-D_\xi\widetilde v(t')|^2_{2,S}}{|t-t'|^{1+1/2}}+\frac{|D^2_\xi\widetilde v(t)-D^2_\xi\widetilde v(t')|^2_{2,S}}{|t-t'|^{1+1/2}}\\ &\quad+\frac{|D^3_\xi\widetilde v(t)-D^3_\xi\widetilde v(t')|^2_{2,S}}{|t-t'|^{1+1/2}}+\frac{|D_\xi\partial_t\widetilde v(t)-D_\xi\partial_{t'}\widetilde v(t')|^2_{2,S}}{|t-t'|^{1+1/2}}\Big)\Big)^{1/2}\\ &\le c\|\widetilde v\|_{4,\Omega\times\mathbb{R}^1}.\end{aligned}$$

Hence in view of (4.18) we have (4.24).

Since $T<\infty$ the norms of $H_\gamma^{l,l/2}(\Omega^T)$ and $H_0^{l,l/2}(\Omega^T)$ are equivalent (and similarly for boundary norms). Therefore, $f''\in H_\gamma^{2,1}(\Omega^T)$, $g_1'',g_2''\in$

$H_\gamma^{2+1/2,1+1/4}(S^T)$, $h'' \in H_\gamma^{1+1/2,1/2+1/4}(S^T)$ and there exists a constant $c(\gamma)$ such that

$$\begin{aligned} &\|f''\|_{2,\gamma,\Omega^T} \le c(\gamma)\|f''\|_{2,0,\Omega^T},\\ &\|g_i''\|_{2+1/2,\gamma,S^T} \le c(\gamma)\|g_i''\|_{2+1/2,0,S^T}, \qquad i=1,2,\\ &\|h''\|_{1+1/2,\gamma,S^T} \le c(\gamma)\|h''\|_{1+1/2,0,S^T}. \end{aligned} \tag{4.26}$$

On using the above extensions, problem (4.20) takes the form

$$\begin{aligned} &\widetilde{w}_t - \mu\nabla^2\widetilde{w} - \nu\nabla\operatorname{div}\widetilde{w} = f'' && \text{in } \Omega\times(-\infty,T),\\ &\Pi_0\mathbb{D}(\widetilde{w})\overline{n}_0 = g_1'' && \text{on } S\times(-\infty,T),\\ &\overline{n}_0\mathbb{D}(\widetilde{w})\overline{n}_0 - \sigma\overline{n}_0\Delta_S(0)\int_0^t \widetilde{w}(\tau)\,d\tau \\ &\qquad = g_2' + \sigma\int_0^t h''(\tau)\,d\tau && \text{on } S\times(-\infty,T), \end{aligned} \tag{4.27}$$

where $\widetilde{w}$ is zero for $t<0$ and $\widetilde{w}=w$ for $t\ge 0$.

In view of Theorem 4.3 and (4.22), (4.23), (4.25), (4.26) there exists a solution of problem (4.27) such that $\widetilde{w}\in H_\gamma^{4,2}(S^T)$ and

$$\|\widetilde{w}\|_{4,\gamma,\Omega^T} \le c(\gamma)(X_1+X_2). \tag{4.28}$$

Now (4.19), (4.18), (4.28) and the equivalence of the norms of $H_0^{l,l/2}(\Omega^T)$ and $H_\gamma^{l,l/2}(\Omega^T)$ for $T<\infty$ imply

$$\begin{aligned} \|u\|_{4,\Omega^T} &\le \|w\|_{4,\Omega^T} + \|v\|_{4,\Omega^T} \le \|w\|_{4,0,\Omega^T} + cX_2\\ &\le c(\gamma)\|w\|_{4,\gamma,\Omega^T} + cX_2 \le c(\gamma)(X_1+X_2). \end{aligned} \tag{4.29}$$

Hence (4.29) implies (4.16). This concludes the proof.

To prove Theorem 4.1 we needed the following result:

LEMMA 4.5 (see also [2], Section 3, Ch. 2, Theorem 21). *Let $\phi_0,\ldots,\phi_k$, $\phi_j\in H^{l-2j-1}(\mathbb{R}^n)$, $l-2k-1\ge 0$, $l,k\in\mathbb{N}\cup\{0\}$, be given. Then there exists a function $u\in W_2^{l,l/2}(\mathbb{R}^n\times\mathbb{R}_+)$ such that*

$$\frac{\partial^j u}{\partial t^j}\bigg|_{t=0} = \phi_j(x), \qquad j=0,\ldots,k, \tag{4.30}$$

and

$$\|u\|_{l,\mathbb{R}^n\times\mathbb{R}_+} \le c\sum_{j=0}^k \|\phi_j\|_{l-2j-1,\mathbb{R}^n}, \tag{4.31}$$

where the constant c does not depend on ϕ_j, $j=0,\ldots,k$.

Proof. Assume that $\phi_j \in C_0^\infty(\mathbb{R}^n)$. Define a Fourier transform of $u(x,t)$ with respect to the variables $x = (x_1, \dots, x_n)$ by

$$\widehat{u}(\xi, t) = \sum_{j=0}^{k} \frac{\Phi_j((1+\xi^2)t)}{(1+\xi^2)^j} \widehat{\phi}_j(\xi), \qquad \xi = (\xi_1, \dots, \xi_n), \tag{4.32}$$

where the $\Phi_j \in C_0^\infty(\mathbb{R}^1)$ satisfy the relations

$$\frac{d^i \Phi_j(s)}{ds^i}\bigg|_{s=0} = \delta_{ij}, \qquad i, j = 0, \dots, k, \tag{4.33}$$

and $\widehat{\phi}_j(\xi)$ is the Fourier transform of ϕ_j.

We can take $\Phi_j(s) = (s^j/j!)\Phi_0(s)$, $\Phi_0 \in C_0^\infty(\mathbb{R}^1)$, $\Phi_0(s) = 1$ for small s.

Taking the Fourier transform of (4.32) with respect to t gives

$$\widehat{u}(\xi, \xi_0) = \sum_{j=0}^{k} \frac{\widehat{\Phi}_j(\xi_0/(1+\xi^2))}{(1+\xi^2)^{j+1}} \widehat{\phi}_j(\xi), \tag{4.34}$$

where $\widehat{\Phi}_j$ is the Fourier transform of Φ_j.

Now we estimate the norm

$$\begin{aligned} \|u\|^2_{l,\mathbb{R}^n \times \mathbb{R}_+} &= \int\limits_{\mathbb{R}^n \times \mathbb{R}} (1 + |\xi|^2 + \xi_0)^l \bigg| \sum_{j=0}^{k} \widehat{\Phi}_j \left(\frac{\xi_0}{1+\xi^2} \right) \frac{\widehat{\phi}_j(\xi)}{(1+\xi^2)^{j+1}} \bigg|^2 d\xi \, d\xi_0 \\ &\le \sum_{j=0}^{k} \int\limits_{\mathbb{R}^n} \frac{|\widehat{\phi}_j(\xi)|^2}{|1+\xi^2|^{2j+2}} \, d\xi \int\limits_0^\infty \bigg| \widehat{\Phi}_j \left(\frac{\xi_0}{1+\xi^2} \right) \bigg|^2 (1 + |\xi|^2 + \xi_0)^l \, d\xi_0. \end{aligned}$$

Introducing a new variable in the inner integral,

$$\eta = \frac{\xi_0}{1+\xi^2},$$

we get

$$\begin{aligned} \|u\|^2_{l,\mathbb{R}^n \times \mathbb{R}_+} &= \sum_{j=0}^{k} \int\limits_{\mathbb{R}^n} |\widehat{\phi}_j(\xi)|^2 (1+|\xi|^2)^{l-2j-1} \, d\xi \int\limits_0^\infty |\widehat{\Phi}_j(\eta)|^2 (1+\eta)^l \, d\eta \\ &\le c \sum_{j=0}^{k} \int\limits_{\mathbb{R}^n} |\widehat{\phi}_j(\xi)|^2 (1+|\xi|^2)^{l-2j-1} \, d\xi. \end{aligned}$$

Hence (4.31) follows. This concludes the proof.

5. Existence of solutions to problem (1.11). First we consider the following problem with $\eta > 0$:

$$
(5.1)\qquad
\begin{aligned}
&\eta u_t - \mu\nabla_\xi^2 u - \nu\nabla_\xi\nabla_\xi\cdot u = F && \text{in } \Omega^T,\\
&\Pi_0\mathbb{D}_\xi(u)\overline{n}_0 = G_1 && \text{on } S^T,\\
&\overline{n}_0\mathbb{D}_\xi(u)\overline{n}_0 - \sigma\overline{n}_0\Delta_S(0)\int_0^t u(\tau)\,d\tau = G_2 + \sigma\int_0^t H(\tau)\,d\tau && \text{on } S^T,\\
&u|_{t=0} = u_0 && \text{in } \Omega.
\end{aligned}
$$

LEMMA 5.1. *Let* $0 < \eta \in C^\alpha(\Omega^T)\cap L_\infty(0,T;\Gamma_1^2(\Omega))$, $1/\eta \in L_\infty(\Omega^T)$, $f \in W_2^{2,1}(\Omega^T)$, $G_i \in W_{2,1/4}^{2+1/2,1+1/4}(S^T)$, $i = 1,2$, $H \in W_2^{2-1/2,1-1/4}(S^T)$, *and* $S \in H^{4-1/2}$. *Then there exists a solution of* (5.1) *such that* $u \in W_2^{4,2}(\Omega^T)$ *and*

$$
(5.2)\qquad
\begin{aligned}
\|u\|_{4,\Omega^T} &\le \phi_1(|1/\eta|_{\infty,\Omega^T}, \sup_t |\eta|_{2,1,\Omega})\|u\|_{2,\Omega^T}\\
&\quad + \phi_2(|1/\eta|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}, |\eta|_{C^\alpha(\Omega^T)})\\
&\quad \times [\|F\|_{2,\Omega^T} + \|G\|_{2+1/2,S^T,1/4} + \|H\|_{2-1/2,S^T} + |u(0)|_{3,0,\Omega}],
\end{aligned}
$$

where ϕ_1, ϕ_2 *are increasing functions of their arguments and* $G = (G_1, G_2)$.

P r o o f. First we consider problem (5.1) with vanishing initial data. Introducing a partition of unity $\zeta^{(k,l)}(\xi,t)$ in Ω^T such that $\operatorname{supp}\zeta^{(k,l)}(\xi,t) \subset \Omega_k \times (T_{l-1}, T_l)$ (see Section 2) and setting $u_{(k,l)} = u\zeta^{(k,l)}$ we write problem (5.1) locally in the form (see [13])

$$
\begin{aligned}
&\eta(\xi_k,t_l)u_{(k,l)t} - \mu\nabla_\xi^2 u_{(k,l)} - \nu\nabla_\xi\nabla_\xi\cdot u_{(k,l)}\\
&\quad = [\eta(\xi_k,t_l) - \eta(\xi,t)]u_{(k,l)t} + \eta u\zeta_t^{(k,l)} - \mu[\nabla_\zeta^2, \zeta^{(k,l)}]u\\
&\qquad - \nu[\nabla_\xi\nabla_\xi\cdot, \zeta^{(k,l)}]u + F_{(k,l)} \equiv F'_{(k,l)} + F_{(k,l)} \equiv \widetilde{F}_{(k,l)},\\
&\Pi_0\mathbb{D}_\xi(u_{(k,l)})\overline{n}_0 = \Pi_0\mathbb{D}_\xi(\zeta^{(k,l)})\overline{n}_0 u + G_{1(k,l)}\\
&\qquad\qquad \equiv G'_{1(k,l)} + G_{1(k,l)} \equiv \widetilde{G}_{1(k,l)},
\end{aligned}
$$

$$
(5.3)\qquad
\begin{aligned}
&\overline{n}_0\mathbb{D}_\xi(u_{(k,l)})\overline{n}_0 - \sigma\overline{n}_0\Delta_S(0)\int_0^t u_{(k,l)}(\tau)\,d\tau\\
&\quad = \overline{n}_0\mathbb{D}_\xi(\zeta^{(k,l)})\overline{n}_0 u + G_{2(k,l)}\\
&\qquad + \int_0^t (-\sigma\overline{n}_0[\Delta_S(0), \zeta^{(k,l)}]u + \overline{n}_0\mathbb{D}_\xi(u)\zeta_{,\tau}^{(k,l)}\overline{n}_0)\,d\tau\\
&\qquad + \int_0^t (-G_2\xi_{,\tau}^{(k,l)} + H_{(k,l)}(\tau))\,d\tau
\end{aligned}
$$

$$\equiv G'_{2(k,l)} + G_{2(k,l)} + \sigma \int_0^t (H'_{(k,l)}(\tau) + H_{(k,l)}(\tau))\, d\tau$$

(5.3) [cont.]
$$\equiv \widetilde{G}_{2(k,l)} + \sigma \int_0^t \widetilde{H}_{(k,l)}(\tau)\, d\tau,$$

$$u_{(k,l)}|_{t=0} = 0,$$

where we have used the notation $K_{(k,l)} = K\zeta^{(k,l)}$, $K \in \{F, G_1, G_2, H\}$.

Introducing a new variable $\tau = \eta_{kl}^{-1} t$, $\eta_{kl} = \eta(\xi_k, t_l)$, where $\xi_k \in \Omega_k$, $t_l \in (T_{l-1}, T_l)$, applying Theorem 4.4 and then going back to the variable t we have

$$\begin{aligned}\|u_{(k,l)}\|_{4,\Omega^T} &\le \phi(1/\eta_{kl}, \eta_{kl}, T) \\ &\quad \times (\|\widetilde{F}_{(k,l)}\|_{2,\Omega^T} + \|\widetilde{G}_{(k,l)}\|_{(2+1/2),S^T,1/4} \\ &\quad + \|\widetilde{H}_{(k,l)}\|_{1+1/2,S^T} + \|F_{(k,l)}(0)\|_{1,\Omega}),\end{aligned} \tag{5.4}$$

where ϕ is a positive increasing function of its arguments.

Now we estimate the particular terms on the right-hand side of (5.4). First we consider

$$\begin{aligned}&\|F'_{(k,l)}\|_{2,\Omega^T} \\ &\quad \le c\lambda^\alpha |\eta|_{C^\alpha(\Omega^T)} \|u_{(k,l)}\|_{4,\Omega^T} \\ &\quad + c\Big(\int_0^T \int_\Omega (|\nabla\eta\nabla u_{(k,l)t}|^2 + |\nabla^2\eta u_{(k,l)t}|^2 + |\eta_t u_{(k,l)t}|^2)\, d\xi\, dt\Big)^{1/2} \\ &\quad + c(\|\nabla u\|_{2,\Omega_k\times(T_{l-1},T_l)} + \|u\|_{2,\Omega_k\times(T_{l-1},T_l)}),\end{aligned} \tag{5.5}$$

where the middle term is estimated by

$$c\sup_t |\eta|_{2,1,\Omega}(\varepsilon\|u_{(k,l)}\|_{4,\Omega^T} + c(\varepsilon)\|u_{(k,l)}\|_{2,\Omega^T}), \quad \varepsilon \in (0,1).$$

Next, we consider

$$\begin{aligned}\|G'_{(k,l)}\|_{2+1/2,S^T,1/4} &\le c(\|\overline{n}_0 u\|_{2+1/2,S_k\times(T_{l-1},T_l),1/4} \\ &\quad + \|\overline{n}_0\overline{n}_0 u\|_{2+1/2,S_k\times(T_{l-1},T_l),1/4}),\end{aligned} \tag{5.6}$$

where $S_k = S \cap \overline{\Omega}_k$. To estimate the right-hand side of (5.6) it is sufficient to find a bound for

$$\|\overline{n}_0 u\|_{2+1/2,S_k\times(T_{l-1},T_l)} + \Big(\int_{T_{l-1}}^{T_l} \frac{|u|^2_{2,0,S_k}}{t^{1/4}}\, dt\Big)^{1/2} \equiv I_1 + I_2. \tag{5.7}$$

To estimate I_1 and I_2 we consider only the highest order terms. First we estimate the expression

$$
\begin{aligned}
&[\overline{n}_0 u]_{2+1/2,S_k\times(T_{l-1},T_l),x}\\
&\quad\le \sum_{|\overline{\alpha}|\le 2}\Big(\int\limits_{T_{l-1}}^{T_l}\int\limits_{S_k}\int\limits_{S_k}\frac{|D^{\overline{\alpha}}_{\xi,t}u-D^{\overline{\alpha}}_{\xi',t}u|^2|D^{2-\alpha}_{\xi}\overline{n}_0|^2}{|\xi-\xi'|^3}\,d\xi\,d\xi'\,dt\Big)^{1/2}\\
&\qquad+\sum_{|\overline{\alpha}|=2}\Big(\int\limits_{T_{l-1}}^{T_l}\int\limits_{S_k}\int\limits_{S_k}\frac{|D^{\overline{\alpha}}_{\xi,t}u|^2|D^{2-\alpha}_{\xi}\overline{n}_0-D^{2-\alpha}_{\xi'}\overline{n}_0|^2}{|\xi-\xi'|^3}\,d\xi\,d\xi'\,dt\Big)^{1/2}\\
&\quad\le c\Big(\int\limits_{T_{l-1}}^{T_l}\int\limits_{S_k}\int\limits_{S_k}\frac{|u(\xi)-u(\xi')|^2}{|\xi-\xi'|^3}|D^2_\xi\overline{n}_0(\xi)|^2\,d\xi\,d\xi'dt\Big)^{1/2}\\
&\qquad+\phi(\|S_{k-1}\|_{4-1/2})\\
&\qquad\times\Big(\int\limits_{T_{l-1}}^{T_l}([D^2_{\xi,t}]^2_{1/2,S_k}+[D_\xi u]^2_{1/2,S_k}+|D^2_{\xi,t}u|^2_{0,S_k}+|D_\xi u|^2_{0,S_k})\,dt\Big)^{1/2}\\
&\qquad+c\Big(\int\limits_{T_{l-1}}^{T_l}|u|^2_{\infty,S_k}dt\Big)^{1/2}[D^2_\xi\overline{n}_0]_{1/2,S_k},
\end{aligned}
$$

where ϕ is a positive increasing function, $D^2_{\xi,t}=\sum_{|\bar{\alpha}|=2}D^{\bar{\alpha}}_{\xi,t}$, $D^2_\xi=\sum_{|\alpha|=2}D^\alpha_\xi$, $\overline{\alpha}=(\alpha_0,\alpha)$.

Using the interpolation inequalities (see [1], Secs. 10, 18)

$$
\begin{gathered}
[D^2_\xi u]_{1/2,S_k}+[D_\xi u]_{1/2,S_k}+\|u\|_{2,S_k}\le\varepsilon\|u\|_{4,\Omega_k}+c(\varepsilon)\|u\|_{0,\Omega_k},\\
[\partial_t u]_{1/2,S_k}+|\partial_t u|_{0,S_k}\le\varepsilon\|u_t\|_{2,\Omega_k}+c(\varepsilon)\|u\|_{0,\Omega_k},
\end{gathered}
$$

we obtain

$$
\begin{aligned}
(5.8)\quad&[\overline{n}_0 u]_{2+1/2,S_k\times(T_{l-1},T_l),x}\\
&\le\varepsilon\|u\|_{4,\Omega_k\times(T_{l-1},T_l)}+c(\varepsilon,\|S_k\|_{4-1/2})\|u\|_{2,\Omega_k\times(T_{l-1},T_l)},\qquad\varepsilon\in(0,1).
\end{aligned}
$$

Now we examine

$$
\begin{aligned}
(5.9)\quad&[\overline{n}_0 u]_{2+1/2,S_k\times(T_{l-1},T_l),t}\\
&\le\phi(\|S_k\|_{4-1/2})\Big(\int\limits_{T_{l-1}}^{T_l}\int\limits_{T_{l-1}}^{T_l}\frac{\|u(t)-u(t')\|^2_{2,S_k}+\|\partial_t u-\partial_{t'}u\|^2_{0,S_k}}{|t-t'|^{3/2}}\,dt\,dt'\Big)^{1/2}\\
&\le\varepsilon\|u\|_{4,\Omega_k\times(T_{l-1},T_l)}+c(\varepsilon)\|u\|_{2,\Omega_k\times(T_{l-1},T_l)},
\end{aligned}
$$

where the last inequality follows from interpolation inequalities (see [1], Sec. 18) and $c(\varepsilon)$ depends also on the length of the interval $[T_{l-1},T_l]$, which is fixed.

Finally, the second term in (5.7) is bounded by

$$\Big(\int\limits_{T_{l-1}}^{T_l}\frac{\varepsilon|D^3_{\xi,t}u|^2_{2,\Omega_k}+c(\varepsilon)|D^2_{\xi,t}u|^2_{2,\Omega_k}}{t^{1/2}}\,dt\Big)^{1/2}$$
$$\le\varepsilon(\|u\|_{4,\Omega_k\times(T_{l-1},T_l)}+\sup_{t\in(T_{l-1},T_l)}|u|_{3,0,\Omega_k})+c\|u\|_{2,\Omega_k\times(T_{l-1},T_l)}.$$

Summarizing the above considerations we obtain

$$\begin{aligned}(5.10)\qquad &\|u_{(k,l)}\|_{4,\Omega^T}\\ &\le c\lambda^\alpha|\eta|_{C^\alpha(\Omega^T)}\|u_{(k,l)}\|_{4,\Omega^T}\\ &\quad+\varepsilon(\|u_{(k,l)}\|_{4,\Omega^T}+\|u\|_{4,\Omega_k\times(T_{l-1},T_l)}+\sup_{t\in(T_{l-1},T_l)}|u|_{3,0,\Omega_k})\\ &\quad+\widetilde\phi_1(|1/\eta|_{\infty,\Omega^T},\sup_t|\eta|_{2,1,\Omega})(\|u_{(k,l)}\|_{2,\Omega^T}+\|u\|_{2,\Omega_k\times(T_{l-1},T_l)})\\ &\quad+\widetilde\phi_2(|1/\eta|_{\infty,\Omega^T},|\eta|_{\infty,\Omega^T})\\ &\quad\times(\|F_{(k,l)}\|_{2,\Omega^T}+\|G_{(k,l)}\|_{2+1/2,S^T,1/4}+\|H_{(k,l)}\|_{2-1/2,S^T}),\end{aligned}$$

where $\widetilde\phi_1,\widetilde\phi_2$ are positive increasing functions. Summing (5.10) over all subdomains of the partition of unity and using the fact that λ and ε are sufficiently small we obtain (5.2) for vanishing initial data.

To obtain (5.2) for nonvanishing initial data we write problem (5.1) in the form of two problems

$$\begin{aligned}&\omega_t-\operatorname{div}_\xi\mathbb{D}_\xi(\omega)=F,\\ &\Pi_0\mathbb{D}_\xi(\omega)\overline n_0=0,\\ &\overline n_0\mathbb{D}_\xi(\omega)\overline n_0-\sigma\overline n_0\Delta_S(0)\int\limits_0^t\omega(\tau)\,d\tau=0,\\ &\omega|_{t=0}=u_0,\end{aligned}$$

and

$$\begin{aligned}&\eta v_t-\operatorname{div}_\xi\mathbb{D}_\xi(v)=(1-\eta)\omega_t,\\ &\Pi_0\mathbb{D}_\xi(v)\overline n_0=G_1,\\ &\overline n_0\mathbb{D}_\xi(v)\overline n_0-\sigma\overline n_0\Delta_S(0)\int\limits_0^t v(\tau)\,d\tau=G_2+\sigma\int\limits_0^t H(\tau)\,d\tau,\\ &v|_{t=0}=0,\end{aligned}$$

where $u=v+\omega$.

Applying Theorem 4.4 to the first problem and estimate (5.2) for solutions of the second problem, which has just been shown above, we obtain (5.2) for solutions of (5.1).

We prove existence of solutions to (5.1) by the method of successive approximations. We put u_{m+1} into the left-hand sides of (5.3) and u_m into the right-hand sides. In view of estimate (5.10) the sequence converges for sufficiently small λ and ε. This concludes the proof.

Now we examine the problem

$$
\begin{aligned}
&\eta u_t - \mu\nabla_\omega^2 u - \nu\nabla_\omega\nabla_\omega\cdot u = F,\\
&\Pi_0\mathbb{D}_\omega(u)\overline{n}_0 = G_1,\\
&\overline{n}_0\mathbb{D}_\omega(u)\overline{n}_0 - \sigma\overline{n}_0\Delta_{S_t}(t)\int_0^t u(\tau)\,d\tau = G_2 + \sigma\int_0^t H(\tau)\,d\tau,\\
&u|_{t=0} = u_0.
\end{aligned}
\tag{5.11}
$$

LEMMA 5.2. *Let the assumptions of Lemma 5.1 be satisfied. Let $\omega \in W_2^{4,2}(\Omega^T)$. There exists a function ϕ_3 and T such that if*

$$
\begin{aligned}
T^{1/2}\phi_3(T^{1/4}(\|\omega\|_{4,\Omega^T} + \sup_t|\omega|_{3,0,\Omega}), T)&\\
\times\phi_2(|1/\eta|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}, |\eta|_{C^\alpha(\Omega^T)}) &\le \delta,
\end{aligned}
\tag{5.12}
$$

then there exists a solution to problem (5.11) for δ sufficiently small such that $u \in W_2^{4,2}(\Omega^T)$ and

$$
\begin{aligned}
\|u\|_{4,\Omega^T} \le c\phi_2(\|F\|_{2,\Omega^T} + \|G\|_{(2+1/2),S^T,1/4}&\\
+ \|H\|_{2-1/2,S^T} + |u(0)|_{3,0,\Omega}) + c\phi_1\|u\|_{3,\Omega^T}&.
\end{aligned}
\tag{5.13}
$$

P r o o f. We write problem (5.11) in the form

$$
\begin{aligned}
&\eta u_t - \mu\nabla_\xi^2 u - \nu\nabla_\xi\nabla_\xi\cdot u\\
&\qquad = -\mu(\nabla_\xi^2 - \nabla_\omega^2)u - \nu(\nabla_\xi\nabla_\xi\cdot - \nabla_\omega\nabla_\omega\cdot)u + F \equiv \widetilde{F} + F,\\
&\Pi_0\mathbb{D}_\xi(u)\overline{n}_0 = \Pi_0\mathbb{D}_\xi(u)\overline{n}_0 - \Pi_0\Pi\mathbb{D}_\omega(u)\overline{n}_0 + G_1 \equiv \widetilde{G}_1 + G_1,\\
&\overline{n}_0\mathbb{D}_\xi(u)\overline{n}_0 - \sigma\overline{n}_0\Delta_S(0)\int_0^t u(\tau)\,d\tau\\
&\qquad = \overline{n}_0\mathbb{D}_\xi(u)(\overline{n}_0 - \overline{n}) + \overline{n}_0(\mathbb{D}_\xi(u) - \mathbb{D}_\omega(u))\overline{n}\\
&\qquad\quad - \sigma\overline{n}_0(\Delta_S(0) - \Delta_{S_t}(t))\int_0^t u(\tau)\,d\tau + G_2 + \sigma\int_0^t H(\tau)\,d\tau\\
&\qquad \equiv \widetilde{G} + G_2 + \sigma\int_0^t (\widetilde{H}(\tau) + H(\tau))\,d\tau,\\
&u|_{t=0} = u_0.
\end{aligned}
\tag{5.14}
$$

Using Lemma 5.1 we have the following estimate for a solution of (5.14):

$$\begin{aligned}(5.15)\quad \|u\|_{4,\Omega^T} &\le \phi_1(|1/\eta|_{\infty,\Omega^T}, \sup_t |\eta|_{2,1,\Omega})\|u\|_{2,\Omega^T}\\ &+ \phi_2(|1/\eta|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}, |\eta|_{C^\alpha(\Omega^T)})[\|\widetilde{F}+F\|_{2,\Omega^T}\\ &+ \|\widetilde{G}+G\|_{2+1/2,S^T,1/4} + \|\widetilde{H}+H\|_{2-1/2,S^T} + |u|_{3,0,\Omega}].\end{aligned}$$

Now we have to estimate the particular terms on the right-hand side of (5.15). The functions $\widetilde{F}, \widetilde{G}$ and $\widetilde{H}$ have the following qualitative forms:

$$\begin{aligned}&\widetilde{F} = f_1 \int_0^t \omega_{\xi\xi}\,d\tau\,u_\xi + f_2\int_0^t \omega_\xi\,d\tau\,u_{\xi\xi},\\ (5.16)\quad &\widetilde{G} = f_3\Big(\int_0^t \omega_\xi\,d\tau\,u_\xi\Big)\Big|_S,\\ &\widetilde{H} = f_4\Big(\int_0^t \omega_{\xi\xi}\,d\tau\,u_\xi + \int_0^t u_\xi\,d\tau\,u_{\xi\xi}\Big)\Big|_S,\end{aligned}$$

where f_i, $i=1,\ldots,4$, depend on $\delta + \int \omega_\xi\,d\tau$, where δ is the unit matrix, and f_3, f_4 depend additionally on $\nabla\widetilde{\phi}$, where $\widetilde{\phi}(\xi)=0$ describes S locally.

In view the Hölder inequality and imbedding theorems we have

$$\begin{aligned}(5.17)\quad &\|\widetilde{F}\|_{2,\Omega^T} + \|\widetilde{G}\|_{2+1/2,S^T,1/4} + \|\widetilde{H}\|_{2-1/2,S^T}\\ &\le cT^{1/4}(\|\omega\|_{4,\Omega^T} + \sup_t|\omega|_{3,0,\Omega})\\ &\quad\times \psi_1(T^{1/2}\|\omega\|_{4,\Omega^T}, T)T^{1/4}(\|u\|_{4,\Omega^T} + \sup_t |u|_{3,0,\Omega})\\ &\equiv c\phi_3(T^{1/4}(\|\omega\|_{4,\Omega^T} + \sup_t|\omega|_{3,0,\Omega}), T)\\ &\quad\times T^{1/2}(\|u\|_{4,\Omega^T} + \sup_t|u|_{3,0,\Omega}),\end{aligned}$$

where ψ_1 and ϕ_3 are increasing functions of their arguments and c does not depend on T.

From (5.15), (5.17) we obtain (5.13) for sufficiently small δ.

Existence can be proved by the method of successive approximations. This concludes the proof.

6. Existence of solutions to problem (1.1). First we consider the continuity equation $(1.8)_2$ with the initial condition $(1.8)_5$. By the method of characteristics we have

$$(6.1)\quad \eta(\xi,t) = \varrho_0(\xi)\exp\Big[-\int_0^t \nabla_u\cdot u(\xi,\tau)\,d\tau\Big].$$

LEMMA 6.1. *Assume that* $u \in W_2^{4,2}(\Omega^T) \cap L_\infty(0,T;\Gamma_0^3(\Omega))$, $\varrho_0 \in H^3(\Omega)$, *and* $T < \infty$. *Then the solution* (6.1) *of problem* $(1.1)_{2,3}$ *is such that* $\eta \in C_0([0,T];H^3(\Omega))$, $\eta_t \in C_0([0,T];H^2(\Omega)) \cap L_2(0,T;H^3(\Omega))$, $\eta_{tt} \in L_2(0,T;H^1(\Omega))$ *and the following estimates hold*:

$$\sup_t \|\eta\|_{3,\Omega} \le c\|\varrho_0\|_{3,\Omega}\psi_1(T^{1/2}\|u\|_{4,\Omega^T})(T^{1/2}\|u\|_{4,\Omega^T}+1), \tag{6.2}$$

$$\sup_t \|\eta_t\|_{2,\Omega} \le c\|\varrho_0\|_{3,\Omega}\psi_2(T^{1/2}\|u\|_{4,\Omega^T})(\|u\|_{4,\Omega^T}+\|u(0)\|_{3,\Omega}), \tag{6.3}$$

$$\|\eta_t\|_{L_2(0,T;H^3(\Omega))} \le c\|\varrho_0\|_{3,\Omega}\psi_3(T^{1/2}\|u\|_{4,\Omega^T})\|u\|_{4,\Omega^T}, \tag{6.4}$$

$$\begin{aligned} &\|\eta_{tt}\|_{L_2(0,T;H^1(\Omega))} \le c\|\varrho_0\|_{3,\Omega}\psi_4(T^{1/2}\|u\|_{4,\Omega^T})\|u\|_{4,\Omega^T},\\ &\sup_t \|\eta_{tt}\|_{0,\Omega}\\ &\qquad \le c\|\varrho_0\|_{3,\Omega}\psi_4'(T^{1/2}\|u\|_{4,\Omega^T},|u(0)|_{3,0,\Omega})(\|u\|_{4,\Omega^T}+|u(0)|_{3,0,\Omega}),\end{aligned} \tag{6.5}$$

$$|1/\eta|_{\infty,\Omega^T}+|\eta|_{\infty,\Omega^T} \le (|1/\varrho_0|_{\infty,\Omega}+|\varrho_0|_{\infty,\Omega})\exp(T^{1/2}\|u\|_{4,\Omega^T}), \tag{6.6}$$

$$\begin{aligned}\|\eta\|_{C^\alpha(\Omega^T)} &\le \|\varrho_0\|_{C^\alpha(\Omega)}\psi_5(T^{1/2}\|u\|_{4,\Omega^T})\\ &\quad\times(\psi_6(T^{1/2}\|u\|_{4,\Omega^T})+T^{1-\alpha}(\|u\|_{4,\Omega^T}+\|u(0)\|_{3,\Omega})),\end{aligned} \tag{6.7}$$

where ψ_i, $i=1,\dots,6$, *and* ψ_4' *are positive increasing functions.*

Proof. First we show (6.2). We calculate

$$\begin{aligned}\|\eta\|_{3,\Omega} &\le c\|\varrho_0\|_{3,\Omega}\Big\|\exp\Big[-\int_0^t \nabla_u\cdot u\,d\tau\Big]\Big\|_{3,\Omega}\\ &\le c\|\varrho_0\|_{3,\Omega}\exp\Big|\int_0^t \nabla_u\cdot u\,d\tau\Big|_{\infty,\Omega}\\ &\quad\times\Big(1+\Big\|\int_0^t \nabla_u\cdot u\,d\tau\Big\|_{1,\Omega}^3\\ &\quad+\Big\|\int_0^t \nabla_u\cdot u\,d\tau\Big\|_{1,\Omega}\Big\|\int_0^t \nabla_u\cdot u\,d\tau\Big\|_{2,\Omega}+\Big\|\int_0^t \nabla_u\cdot u\,d\tau\Big\|_{3,\Omega}\Big),\end{aligned} \tag{6.8}$$

and we have the estimates

$$\begin{aligned}\Big|\int_0^t \nabla_u\cdot u\,d\tau\Big|_{\infty,\Omega} &\le c\Big\|\int_0^t \nabla_u\cdot u\,d\tau\Big\|_{2,\Omega} \le c\Big\|\int_0^t \nabla_u\cdot u\,d\tau\Big\|_{3,\Omega}\\ &\le \widetilde{\psi}_1(T^{1/2}\|u\|_{4,\Omega^T})T^{1/2}\|u\|_{4,\Omega^T},\end{aligned} \tag{6.9}$$

where $\widetilde{\psi}_1$ is an increasing positive function. Using (6.9) in (6.8) implies (6.2).

From (6.1) we have

$$\eta_t = \varrho_0 \exp\Big[-\int_0^t \nabla_u \cdot u\, d\tau\Big](-\nabla_u \cdot u), \tag{6.10}$$

so

$$\|\eta_t\|_{2,\Omega} \le c\|\varrho_0\|_{3,\Omega}\Big\| \exp\Big[-\int_0^t \nabla_u \cdot u\, d\tau\Big]\Big\|_{3,\Omega}\|\nabla_u \cdot u\|_{2,\Omega}. \tag{6.11}$$

In view of the Hölder inequality and imbedding theorems we have

$$\begin{aligned} \|\nabla_u \cdot u\|_{2,\Omega} &\le \widetilde{\psi}_2(T^{1/2}\|u\|_{4,\Omega^T})\|u\|_{3,\Omega} \\ &\le \widetilde{\psi}_2(T^{1/2}\|u\|_{4,\Omega^T})(\|u\|_{4,\Omega^T} + \|u(0)\|_{3,\Omega}), \end{aligned} \tag{6.12}$$

where the last inequality follows from Theorem 2 of [8] and $\widetilde{\psi}_2$ is a positive increasing function.

Using (6.8), (6.9) and (6.12) in (6.11) implies (6.3).

From (6.10) we have

$$\|\eta_t\|_{3,\Omega} \le c\|\varrho_0\|_{3,\Omega}\Big\| \exp\Big[-\int_0^t \nabla_u \cdot u\, d\tau\Big]\Big\|_{3,\Omega}\|\nabla_u \cdot u\|_{3,\Omega}.$$

Hence (6.4) holds.

From (6.10) we obtain

$$\eta_{tt} = \varrho_0 \exp\Big[-\int_0^t \nabla_u \cdot u\, d\tau\Big]\Big[(\nabla_u \cdot u)^2 - \nabla_u \cdot u_t + \widetilde{\psi}_3\Big(\int_0^t u_\xi\, d\tau\Big)(\nabla_\xi u)^2\Big].$$

Therefore, (6.5) is valid. Similarly we show (6.6) and (6.7). This concludes the proof.

Finally, we prove the main result of the paper.

THEOREM 6.2. *Assume that* $v_0, \varrho_0 \in H^3(\Omega)$, $1/\varrho_0 \in L_\infty(\Omega)$, $S \in H^{1-1/2}$, *and* $f \in W_2^{2,1}(\Omega^T)$. *Let* G *be the function from* (6.18) *below and suppose* $A > G(\gamma, 0, 0)$, *where* γ *is defined in* (6.17)$_2$. *Let* $|v(0)|_{3,0,\Omega} < A$. *Let* δ *be sufficiently small. Let* T_* *be so small that*

$$\begin{gathered} T_*^{1/2}\phi_3(T_*A, T_*)\phi_2(A, A, A) \le \delta \quad \text{(see (5.12))}, \\ 0 < c_1(1 - AT_*)^3 \le \det\{\partial x/\partial \xi\} \le c_2(1 + AT_*)^3, \end{gathered}$$

where $x = \xi + \int_0^t \widetilde{v}_0(\xi, \tau)\, d\tau$, $t \le T_*$, $G(\gamma, T_*^{1/2}A, T_*) < A$ *and* $\widetilde{v}$ *is defined below. Moreover, let compatibility conditions be satisfied* (*see proof below*). *Then there exists* T_{**}, $0 < T_{**} \le T_*$, *such that for* $T \le T_{**}$ *there exists a unique solution to problem* (1.1) *such that* $u \in W_2^{4,2}(\Omega^T)$, $\eta \in C([0, T]; \Gamma_0^3(\Omega))$, $\eta_t \in L_2(0, T; H^3(\Omega))$, $\eta_{tt} \in L_2(0, T; H^1(\Omega))$ *and*

$$\|u\|_{4,\Omega^T} \le A,$$

$$\sup_t \|\eta\|_{3,\Omega} + \sup_t \|\eta_t\|_{2,\Omega} + \|\eta_t\|_{L_2(0,T,H^3(\Omega))} + \|\eta_{tt}\|_{L_2(0,T;H^1(\Omega))} \le \Phi_1(T, T^a A)\|\varrho_0\|_{3,\Omega},$$

$$|1/\eta|_{\infty,\Omega^t} \le |1/\varrho_0|_{\infty,\Omega}\Phi_2(T^{1/2}A),$$

where Φ_1, Φ_2 are some increasing positive functions.

Proof. To prove existence of solutions to problem (1.1) we use the following method of successive approximations:

$$(6.13)\quad \begin{aligned} &\eta_m \partial_t u_{m+1} - \mu\nabla^2_{u_m} u_{m+1} - \nu\nabla_{u_m}\nabla_{u_m}\cdot u_{m+1} = -\nabla_{u_m} q(\eta_m) + \eta_m g,\\ &\Pi\mathbb{D}_{u_m}(u_{m+1})\overline{n}(u_m) = 0,\\ &\overline{n}_0\mathbb{D}_{u_m}(u_{m+1})\overline{n}(u_m) - \sigma\overline{n}_0\Delta_m(t)\int_0^t u_{m+1}(\tau)\,d\tau\\ &\qquad = \overline{n}_0\cdot\overline{n}(u_m)(q(\eta_m) - p_0) + \sigma\overline{n}_0\Delta_m(t)\xi,\\ &u_{m+1}|_{t=0} = v_0, \end{aligned}$$

and

$$(6.14)\quad \begin{aligned} &\partial_t\eta_m + \eta_m\nabla_{u_m}\cdot u_m = 0,\\ &\eta_m|_{t=0} = \varrho_0, \end{aligned}$$

where $m = 0, 1, \ldots$ and $u_0 = \widetilde{v}_0$. Now we define $\widetilde{v}_0$. Let us introduce the functions $\phi^i = \partial_t^i u|_{t=0}$, $i = 0, 1$, which are calculated from $(1.8)_1$. The functions ϕ^i satisfy the following compatibility conditions:

$$\partial_t^i(\mathbb{T}_u(u,q)\overline{n}(\xi,t) - \sigma\Delta_{S_t}x(\xi,t) + p_0\overline{n}(\xi,t))|_{t=0} = 0, \quad i = 0, 1,$$

where $\partial_t^i u|_{t=0}$ and $\partial_t^i\eta|_{t=0}$, $i = 0, 1$, have to be calculated from $(1.8)_{1,4}$ and $(1.8)_{2,5}$, respectively. Next, we extend ϕ^i to functions $\widetilde{\phi}^i$ on $\mathbb{R}^3$, and define $\widetilde{v}$ to be the solution of the Cauchy problem

$$(\partial_t - \Delta)^2\widetilde{v} = 0, \quad \partial_t^i\widetilde{v}|_{t=0} = \widetilde{\phi}^i, \quad i = 0, 1.$$

Finally, $\widetilde{v}_0 = \widetilde{v}|_\Omega$. First we obtain a uniform estimate. Applying Lemma 5.1 to (6.13) yields

$$(6.15)\quad \begin{aligned} &\|u_{m+1}\|_{4,\Omega^T}\\ &\quad \le c\phi_1(|1/\eta_m|_{\infty,\Omega^T}, \sup_t |\eta_m|_{2,1,\Omega})\|u_{m+1}\|_{2,\Omega^T}\\ &\qquad + \varphi_2(|1/\eta_m|_{\infty,\Omega^T}, |\eta_m|_{\infty,\Omega^T}, |\eta_m|_{C^\alpha(\Omega^T)})\\ &\qquad \times [\|-\nabla_{u_m}q(\eta_m) + \eta_m g\|_{2,\Omega^T}\\ &\qquad + \|\overline{n}_0\cdot\overline{n}(u_m)(q(\eta_m) - p_0) + \sigma\overline{n}_0\Delta_m(t)\xi\|_{2+1/2,S^T,1/4}\\ &\qquad + |u_{m+1}(0)|_{3,0,\Omega}]. \end{aligned}$$

Now we estimate the particular terms on the right-hand side of (6.15). First we consider

$$\begin{aligned}
&\|-\nabla_{u_m} q(\eta_m)+\eta_m g\|_{2,\Omega^T}\\
&\quad\le \|\xi_x(u_m)q'(\eta_m)\nabla\eta_m\|_{2,\Omega^T}+\|\eta_m g\|_{2,\Omega^T}\\
&\quad\le [T^{1/2}\alpha_1(T^{1/2}\|u_m\|_{4,\Omega^T},\sup_t\|\eta_m\|_{2,\Omega^T})(1+\sup_t\|u_m\|_{3,\Omega})+\|g\|_{2,\Omega^T}]\\
&\qquad\times(\sup_t\|\eta_m\|_{3,\Omega}+\sup_t\|\eta_{mt}\|_{1,\Omega}),
\end{aligned}$$

where $q'(\eta)=dq/d\eta$ and α_1 is an increasing positive function of its arguments.

Next, we have

$$\begin{aligned}
&\|\overline{n}_0\cdot\overline{n}(u_m)(q(\eta_m)-p_0)\|_{2+1/2,S^T}\\
&\qquad\le \alpha_2(T^{1/2}\|u_m\|_{4,\Omega^T},\sup_t\|\eta_m\|_{2,\Omega},\sup_t\|\eta_{mt}\|_{1,\Omega^T},T)\\
&\qquad\quad\times\Big[T^{1/4}\|u_m\|_{4,\Omega^T}+\int_0^T\|\eta_m\|_{3,\Omega}^2\,d\tau+\int_0^T\|\eta_{mt}\|_{1,\Omega}^2\,d\tau\\
&\qquad\quad+\Big(\int_0^T\int_0^T\Big(\frac{\|\eta_m(t)-\eta_m(t')\|_{2,S}^2}{|t-t'|^{3/2}}\\
&\qquad\quad+\frac{\|\eta_m(t)-\eta_m(t')\|_{2,\Omega}^2+\|\eta_{mt}(t)-\eta_{mt'}(t')\|_{0,S}^2}{|t-t'|^{3/2}}\Big)\,dt\,dt'\Big)^{1/2}\Big],
\end{aligned}$$

where

$$\begin{aligned}
&\Big(\int_0^T\|\eta_m\|_{3,\Omega}^2\,d\tau\Big)^{1/2}\le\|\varrho_0\|_{3,\Omega}T^{1/2}\chi_1(T^{1/2}\|u_m\|_{4,\Omega^T}),\\
&\Big(\int_0^T\|\eta_{mt}\|_{1,\Omega}^2\,d\tau\Big)^{1/2}\\
&\qquad\le\|\varrho_0\|_{2,\Omega}T^{1/2}\chi_2(T^{1/2}\|u_m\|_{4,\Omega^T},T)(\|u_m\|_{3,\Omega^T}+\|u(0)\|_{2,\Omega}),\\
&\int_0^T\int_0^T\Big(\frac{\|\eta_m(t)-\eta_m(t')\|_{2,S}^2}{|t-t'|^{3/2}}\\
&\qquad\quad+\frac{\|\eta_m(t)-\eta_m(t')\|_{2,\Omega}^2+\|\eta_{mt}(t)-\eta_{mt'}(t')\|_{0,S}^2}{|t-t'|^{3/2}}\Big)\,dt\,dt'\\
&\qquad\le\|\varrho_0\|_{3,\Omega}\chi_3(T^{1/2}\|u_m\|_{4,\Omega^T},T)T^a\|u_m\|_{4,\Omega^T},
\end{aligned}$$

where χ_i, $i=1,2,3$, are positive increasing functions, $a>0$.

Next, we have

$$
\begin{aligned}
&\|\overline{n}_0\cdot\overline{n}(u_m)(q(\eta_m)-p_0)\|_{2+1/4,S^T}\\
&\le \alpha_3(T^{1/2}\|u_m\|_{4,\Omega^T},\sup_t\|\eta_m\|_{3,\Omega})\\
&\quad\times\bigg(\int\limits_\Omega\int\limits_0^T\frac{|\int_0^t\|u_m\|_{4,\Omega}d\tau|^2+\|\eta_m\|_{3,\Omega}^2+\|u_m\|_{2,\Omega}^2+\|\eta_{mt}\|_{1,\Omega}^2}{t^{1/2}}\,d\xi\,dt\bigg)^{1/2}\\
&\le \alpha_3T^{1/4}(T^{1/2}\|u_m\|_{4,\Omega^T}+\sup_t\|\eta_m\|_{3,\Omega}+\sup_t\|\eta_{mt}\|_{1,\Omega}+\sup_t\|u_m\|_{2,\Omega}).
\end{aligned}
$$

Finally, we consider the expression

$$
\overline{n}_0\cdot\Delta_m(t)\xi=\frac{1}{\sqrt{g}}\overline{n}_0\cdot g^{\alpha\beta}\partial_{s^\alpha s^\beta}\xi.
$$

Then we have

$$
\begin{aligned}
&\|\overline{n}_0\cdot\Delta_m(t)\xi\|_{2+1/2,S^T}\\
&\qquad\le\alpha_4(T^{1/2}\|u_m\|_{4,\Omega^T})\|\xi\|_{4+1/2,S}^2\\
&\qquad\quad\times(T+\|u_m\|_{2,\Omega^T}^2+T^{1/2}\|u_m\|_{4,\Omega^T}^2+T\|u_m\|_{4,\Omega^T}^4),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\overline{n}_0\cdot\Delta_m(t)\xi\|_{2+1/2,S^T}\\
&\quad\le\alpha_5(T^{1/2}\|u_m\|_{4,\Omega^T})\|\xi\|_{4+1/2,S}T^{1/4}(T^{1/2}\|u_m\|_{4,\Omega^T}+\|u_m\|_{2,\Omega^T}).
\end{aligned}
$$

Summarizing the above considerations and using Lemma 6.1 we have

$$
y_{m+1}(t)\le\beta(t^{1/2}y_m(t),t,\gamma)+\alpha(t^{1/2}y_m(t),t,\gamma)\int\limits_0^t y_{m+1}(\tau)\,d\tau, \tag{6.16}
$$

where α, β are positive increasing functions and

$$
\begin{gathered}
y_m(t)=\|u_m\|_{4,\Omega^t}^2+\sup_t|u_m|_{3,0,\Omega}^2,\\
\gamma=\|\varrho_0\|_{3,\Omega}^2+|u_m(0)|_{3,0,\Omega}^2.
\end{gathered}\tag{6.17}
$$

In view of the Gronwall lemma we have

$$
\begin{aligned}
y_{m+1}(t)&\le\exp[t\alpha(t^{1/2}y_m(t),t,\gamma)]\beta(t^{1/2}y_m(t),t,\gamma)\\
&\equiv G(\gamma,t^{1/2}y_m(t),t),
\end{aligned}\tag{6.18}
$$

where $G(\gamma,0,0)=G_0(\gamma)>0$ and G is an increasing positive function of its arguments.

Let $0<A$ be sufficiently large and such that $G_0(\gamma)<A$, $y_m(0)<A$. Then there exists a time T_* such that for $t\le T_*$ we have

$$
y_{m+1}(t)\le G(\gamma,t^{1/2}A,t)\le A.
$$

In this way we have shown that

$$y_m(t) \le A \quad \text{for } m = 0, 1, \ldots \text{ and } t \le T_*. \tag{6.19}$$

Now we prove convergence of the sequence $\{u_m, \eta_m\}$. To show this we consider the system of problems for the differences $U_m = u_m - u_{m-1}$ and $H_m = \eta_m - \eta_{m-1}$:

$$\begin{aligned}
&\eta_m \partial_t U_{m+1} - \mu \nabla_{u_m}^2 U_{m+1} - \nu \nabla_{u_m} \nabla_{u_m} \cdot U_{m+1} \\
&\qquad = -H_m \partial_t u_m - \mu(\nabla_{u_m}^2 - \nabla_{u_{m-1}}^2) u_m \\
&\qquad\quad - \nu(\nabla_{u_m} \nabla_{u_m} \cdot - \nabla_{u_{m-1}} \nabla_{u_{m-1}} \cdot) u_m \\
&\qquad\quad + \nabla_{u_m} q(\eta_m) - \nabla_{u_{m-1}} q(\eta_{m-1}) + H_m g \equiv F_1 + F_2, \\
&\Pi_0 \mathbb{D}_{u_m}(U_{m+1}) \overline{n}(u_m) \\
&\qquad = \Pi_0 [\mathbb{D}_{u_m}(u_m) \overline{n}(u_m) - \mathbb{D}_{u_{m-1}}(u_m) \overline{n}(u_{m-1})] \equiv G_1, \\
&\overline{n}_0 \mathbb{D}_{u_m}(U_{m+1}) \overline{n}(u_m) - \sigma \overline{n}_0 \Delta_m(t) \int_0^t U_{m+1}(\tau)\, d\tau \\
&\qquad = \overline{n}_0 [\mathbb{D}_{u_m}(u_m) \overline{n}(u_m) - \mathbb{D}_{u_{m-1}}(u_m) \overline{n}(u_{m-1})] \\
&\qquad\quad - \sigma \overline{n}_0 (\Delta_m(t) - \Delta_{m-1}(t)) \int_0^t u_m(\tau)\, d\tau \\
&\qquad\quad + \overline{n}_0 \cdot [\overline{n}(u_m) q(\eta_m) - \overline{n}(u_{m-1}) q(\eta_{m-1})] \\
&\qquad\quad - p_0 \overline{n_0} \cdot (\overline{n}(u_m) - \overline{n}(u_{m-1})) \\
&\qquad\quad + \sigma \overline{n}_0 (\Delta_m(t) - \Delta_{m-1}(t)) \xi \equiv G_2 + G_3, \\
&U_{m+1}|_t = 0,
\end{aligned} \tag{6.20}$$

where

$$\begin{aligned}
F_2 &= -H_m \partial_t u_m + H_m g + q'(\widetilde{\eta}_m) \nabla_{u_m} H_m, \\
G_3 &= \overline{n}(u_m)(q(\eta_m) - q(\eta_{m-1}))
\end{aligned} \tag{6.21}$$

and F_1, G_2 are determined by the remaining terms on the right-hand sides.

To estimate the right-hand sides of (6.20) we shall restrict to their qualitative forms:

$$\begin{aligned}
F_1 &= f_1 \int_0^t U_{m\xi}\, d\tau\, u_{m\xi\xi} + f_2 \int_0^t U_{m\xi\xi}\, d\tau\, u_{m\xi} \\
&\quad + f_3 \int_0^t U_{m\xi}\, d\tau\, f_1' \eta_{m-1,\xi}, \\
G_1 &= f_4 \int_0^t U_{m\xi}\, d\tau\, u_{m\xi}, \\
G_2 &= f_5 \int_0^t U_{m\xi}\, d\tau\, (1 + u_{m\xi}) + f_2' \int_0^t U_{m\xi}\, d\tau,
\end{aligned} \tag{6.22}$$

where $f_i = f_i(\delta + \int_0^t u_{m\xi}\,d\tau, \delta + \int_0^t u_{m-1\xi}\,d\tau)$, $i = 1,\dots,6$, $f'_j = f'_j(\eta_m)$, $j = 1,2$, are C^∞ functions of their arguments. Moreover, we have

$$K = -\overline{n}_0(\Delta_m(t) - \Delta_{m-1}(t))u_m = f_6\Big(\int_0^t U_{m\xi}\,d\tau u_{m\xi\xi} + \int_0^t U_{m\xi\xi}\,d\tau u_{m\xi}\Big). \tag{6.23}$$

Now we have to estimate the functions (6.21)–(6.23):

$$\begin{aligned}
&\|F_2\|_{2,\Omega^T} \le C(A)(\sup_t \|H_m\|_{3,\Omega} + \sup_t \|H_{mt}\|_{1,\Omega}),\\
&\|F_1\|_{2,\Omega^T} \le C(A)T^{1/2}(\|U_m\|_{4,\Omega^T} + \sup_t |U_m|_{3,0,\Omega}),\\
&\|G_i\|_{2+1/2,S^T} \le C(A)T^{1/2}(\|U_m\|_{4,\Omega^T} + \sup_t |U_m|_{3,0,\Omega}) \quad i=1,2,\\
&\|G_i\|_{2+1/4,S^T} \le C(A,T)T^{1/4}(\|U_m\|_{4,\Omega^T} + \sup_t |U_m|_{3,0,\Omega}),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,\\
&\|K\|_{2-1/2,S^T} \le C(A)T^{1/2}(\|U_m\|_{4,\Omega^T} + \sup_t |U_m|_{3,0,\Omega}),\\
&\|G_3\|_{2+1/2,S^T,1/4} \le C(A,T)T^{1/4}(\sup_t \|H_m\|_{3,\Omega} + \sup_t \|H_{mt}\|_{1,\Omega}).
\end{aligned}\tag{6.24}$$

Summarizing the above considerations we have shown

$$\|U_{m+1}\|_{4,\Omega^t} \le \beta_1(T,A)T^{1/4}(\|U_m\|_{4,\Omega^T} + \sup_t |U_m|_{3,0,\Omega}) + \beta_2(T,A)\sup_t |H_m|_{3,0,\Omega}. \tag{6.25}$$

Next, we consider the problem

$$\begin{aligned}
&\partial_t H_m + H_m \operatorname{div}_{u_m} u_m = -\eta_{m-1}(\operatorname{div}_{u_m} u_m - \operatorname{div}_{u_{m-1}} u_{m-1}),\\
&H_m|_{t=0} = 0.
\end{aligned}\tag{6.26}$$

Integrating (6.26) with respect to time one obtains

$$\begin{aligned}
H_m(\xi,t) &= -\exp\Big[-\int_0^t \operatorname{div}_{u_m} u_m\,d\tau\Big]\\
&\quad\times \int_0^t \Big[\eta_{m-1}(\operatorname{div}_{u_m} u_m - \operatorname{div}_{u_{m-1}} u_{m-1}) \exp \int_0^{t'} \operatorname{div}_{u_m} u_m\,dt''\Big]\,dt',
\end{aligned}\tag{6.27}$$

so one has

$$\sup_t |H_m|_{3,0,\Omega} \le \beta_3(T,A)T^{1/2}\|U_m\|_{4,\Omega^t}. \tag{6.28}$$

To obtain (6.28) the most difficult point is when we differentiate the middle term in (6.27) with respect to t. Then we have to estimate the expression

$$\begin{aligned}
&\sup_t \|\xi_x(u_{m-1})U_{m\xi}\exp[\ ]\|_{1,\Omega} \\
&\quad \le \phi(A)\sup_t \|U_{m\xi}\|_{1,\Omega} \le c\phi(A)\|U_{m\xi}\|_{2,\Omega^t} \\
&\quad = c\phi(A)\Big(\int_0^t (\|U_{m\xi}\|_{2,\Omega}^2 + \|U_{m\xi t}\|_{0,\Omega}^2)\,dt\Big)^{1/2} \\
&\quad \le c\phi(A)t^{1/2}\sup_t(\|U_{m\xi}\|_{2,\Omega} + \|U_{mt\xi}\|_{0,\Omega}) \le c\phi(A)t^{1/2}\|U_m\|_{4,\Omega^t}.
\end{aligned}$$

Now (6.25) and (6.28) imply that the sequence $\{u_m,\eta_m\}$ converges to a limit $\{u,\eta\}\in W_2^{4,2}(\Omega^t)\times L_\infty(0,t;\Gamma_0^3(\Omega))$ for $t\le T_{**}$, where T_{**} is sufficiently small. Uniqueness can be proved in the standard way. This concludes the proof.

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