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Nonlinear eigenvalue problems for fourth order ordinary differential equations

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Abstract. This paper was inspired by the works of Chiappinelli ([3]) and Schmitt and Smith ([7]). We study the problem $\mathcal{L}u = \lambda au + f(\cdot, u, u', u'', u''')$ with separated boundary conditions on $[0, \pi]$, where $\mathcal{L}$ is a composition of two operators of Sturm–Liouville type. We assume that the nonlinear perturbation f satisfies the inequality $|f(x, u, u', u'', u''')| \leq M|u|$. Because of the presence of f the considered equation does not in general have a linearization about 0. For this reason the global bifurcation theorem of Rabinowitz ([5], [6]) is not applicable here. We use the properties of Leray–Schauder degree to establish the existence of nontrivial solutions and describe their location. The results obtained are similar to those proved by Chiappinelli for Sturm–Liouville operators.

Let $\mathcal{L}$ be a differential operator of the form $\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_0$, where $\mathcal{L}_i$ denotes the Sturm–Liouville operator defined by $\mathcal{L}_i u = -(p_i u')' + q_i u$, $i = 0, 1$. As usual we assume $p_i \in C^{3-2i}[0, \pi]$, $q_i \in C^{2-2i}[0, \pi]$ and $p_i > 0$, $q_i \geq 0$ on $[0, \pi]$. We denote by (B.C.) either the boundary conditions

$$u(0) = u(\pi) = \mathcal{L}_0 u(0) = \mathcal{L}_0 u(\pi) = 0$$

or the boundary conditions

$$u(0) = u(\pi) = u'(0) = u'(\pi) = 0.$$

Let a be a strictly positive continuous function on $[0, \pi]$. We assume that the operator $\mathcal{L}$ is symmetric and positive definite (which is satisfied in particular when $\mathcal{L}_0 = \mathcal{L}_1$). Then the linear problem $\mathcal{L}v = \mu av$ in $(0, \pi)$ together with the boundary conditions (B.C.) has an increasing sequence of eigenvalues $0 < \mu_1 < \mu_2 < \ldots$ with $\lim_{k\to\infty} \mu_k = \infty$. Each μ_k is simple (Bochenek [2]).

Now consider the equation

$$\mathcal{L}u = \lambda au + f(\cdot, u, u', u'', u''') \quad \text{in } (0, \pi) \tag{1}$$

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together with the boundary conditions (B.C.). Assume that the nonlinear function f is continuous on $[0,\pi]\times\mathbb{R}^4$ and satisfies

$$\exists_{M>0}\forall_{(x,\xi,\eta,\gamma,\zeta)\in[0,\pi]\times\mathbb{R}^4}\quad |f(x,\xi,\eta,\gamma,\zeta)|\le M|\xi|. \tag{2}$$

By a solution of (1) we understand a pair $(\lambda,u)\in\mathbb{R}\times(C^4[0,\pi]\cap(\text{B.C.}))$ satisfying (1).

Let $E=C^3[0,\pi]\cap(\text{B.C.})$. It is a Banach space equipped with its usual norm, $\|u\|_3:=\|u\|_0+\|u'\|_0+\|u''\|_0+\|u'''\|_0$, where $\|u\|_0=\sup_{x\in[0,\pi]}|u(x)|$.

Notice that it is sufficient to search for solutions of (1) in $\mathbb{R}\times E$.

Namely, by using the Green function g of $\mathcal{L}$ together with the boundary conditions (B.C.), equation (1) can be converted into an equivalent integral equation in $\mathbb{R}\times E$:

$$u(\cdot)=\int_0^\pi g(\cdot,y)[\lambda a(y)u(y)+f_u(y)]\,dy=\lambda Lu+F(u), \tag{3}$$

where

$$Lu=\int_0^\pi g(\cdot,y)a(y)u(y)\,dy,\qquad F(u)=\int_0^\pi g(\cdot,y)f_u(y)\,dy,$$

$$f_u(y)=f(y,u(y),u'(y),u''(y),u'''(y)).$$

Clearly $F:E\to E$ is continuous, and $L:E\to E$ is compact and linear.

Let $B=\{u\in E:\|u\|_3\le\delta\}$. To verify that F is compact we prove that $F(B)$ is relatively compact. For $u\in B$ we obtain the estimate

$$\|F(u)\|_3=\sum_{i=0}^{3}\sup_{x\in[0,\pi]}\left|\int_0^\pi\frac{\partial^i g}{\partial x^i}(x,y)f_u(y)\,dy\right|\le c\pi M\|u\|_3\le c\pi M\delta,$$

where c depends on bounds for g and $\partial^i g/\partial x^i$. Hence $F(B)$ is bounded in E. Moreover, $w=F(u)$ satisfies

$$\mathcal{L}w=f(\cdot,u,u',u'',u''').$$

Hence, solving the above equation for $w^{(4)}$ we obtain uniform bounds for the fourth derivatives of $F(u)$ in B. Applying the Arzelà–Ascoli theorem we deduce at once the compactness of F.

Notice that the eigenvalues μ_k of $\mathcal{L}$ are equal to the characteristic values of L (i.e. there exist $v_k\in E$, $v_k\ne0$, such that $v_k=\mu_k Lv_k$) . We denote by $r(L)$ the set of characteristic values of L.

Now we give a description of the spectrum for (1).

THEOREM 1. *If* (λ,u) *is a nontrivial solution of* (1), *then*

$$\lambda\in\bigcup_{k=1}^{\infty}[\mu_k-M/a_0,\mu_k+M/a_0],$$

where $a_0=\min_{x\in[0,\pi]}a(x)$.

Proof. The pair (λ, u) satisfies (3). Multiplying both sides of (3) by $a^{1/2}$ we obtain

$$\widehat{u} = \lambda H\widehat{u} + a^{1/2}F(u), \tag{4}$$

where

$$\widehat{u} = a^{1/2}u, \qquad H\widehat{u} = \int_0^\pi a(\cdot)^{1/2}g(\cdot, y)a(y)^{1/2}\widehat{u}(y)\, dy.$$

It is clear that H is a selfadjoint operator on $L^2[0,\pi]$ and the set of characteristic values of H, $r(H)$, equals $r(L)$. For $\lambda \neq \mu_k$, $I - \lambda H$ is invertible, so that (4) is equivalent to

$$\widehat{u} = (I - \lambda H)^{-1}(a^{1/2}F(u)).$$

We have

$$a^{1/2}F(u) = \int_0^\pi a(\cdot)^{1/2}g(\cdot, y)a(y)^{1/2}\frac{f_u(y)}{a(y)^{1/2}}\, dy = H(a^{-1/2}f_u)$$

and

$$\|a^{-1/2}f_u\|_{L^2}^2 \le \int_0^\pi \frac{M^2}{a(y)}u(y)^2\, dy = \int_0^\pi \frac{M^2}{a(y)^2}\widehat{u}(y)^2\, dy \le \frac{M^2}{a_0^2}\|\widehat{u}\|_{L^2}^2.$$

Hence

$$\|\widehat{u}\|_{L^2} \le \|(I - \lambda H)^{-1}H\| \cdot \|a^{-1/2}f_u\|_{L^2} \le \|(I - \lambda H)^{-1}H\|\frac{M}{a_0}\|\widehat{u}\|_{L^2}.$$

Since $\|(I - \lambda H)^{-1}H\|^{-1} = \mathrm{dist}(\lambda, r(H))$ (Kato [4], p. 273) we conclude that $\mathrm{dist}(\lambda, r(L)) \le M/a_0$ and the proof is complete.

From now on we assume additionally that

$$\mu_k - \mu_{k-1} \to \infty \quad \text{as } k \to \infty. \tag{5}$$

This condition seems to be not particularly restrictive. Look at some examples.

Example 1. If $\mathcal{L}_0, \mathcal{L}_1$ commute then the eigenvalues of the problem $\mathcal{L}u = \mu u$ subject to $u(0) = u(\pi) = \mathcal{L}_0u(0) = \mathcal{L}_0u(\pi) = 0$ are of the form $\mu_k = \mu_k^1\mu_k^0$, where μ_k^i denotes the eigenvalue of $\mathcal{L}_i$. It is simple to show that (5) is satisfied since $\mu_k^i - \mu_{k-1}^i \to \infty$ as $k \to \infty, i = 0, 1$.

Example 2. Consider the equation $u^{(4)} = \mu u$ with the boundary conditions $u(0) = u(\pi) = u'(0) = u'(\pi) = 0$. It is easy to compute that the eigenvalues μ_k satisfy the condition $\sqrt[4]{\mu_k} - \sqrt[4]{\mu_{k-1}} \to 1$ as $k \to \infty$ and consequently we have (5).

The assumption (5) implies that, given any $c > 0$, the intervals $[\mu_k - c, \mu_k + c]$ are disjoint for k large enough.

We can now formulate the main result.

THEOREM 2. *Let* $k_0 = \min\{\overline{k} \in \mathbb{N} : \mu_k - \mu_{k-1} > 2M/a_0$ *for* $k > \overline{k}\}$. *Then for every* $k > k_0$ *and* $\delta > 0$ *there exists a solution* (λ, u) *of* (1) *with* $\|u\|_3 = \delta$ *and* $\lambda \in [\mu_k - M/a_0, \mu_k + M/a_0]$.

Proof. Let $B = \{u \in E : \|u\|_3 \le \delta\}$. Now fix $k > k_0$ and choose $\varepsilon > 0$ such that

$$\underline{\lambda} := \mu_k - (M/a_0 + \varepsilon) > \mu_{k-1} + (M/a_0 + \varepsilon),$$
$$\overline{\lambda} := \mu_k + (M/a_0 + \varepsilon) > \mu_{k+1} - (M/a_0 + \varepsilon).$$

It is easy to see that

$$\operatorname{dist}(\underline{\lambda}, r(L)) = \operatorname{dist}(\overline{\lambda}, r(L)) = M/a_0 + \varepsilon. \tag{6}$$

We argue by contradiction, so assume that $u \ne \lambda Lu + F(u)$ for all $u \in \partial B$ and $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. Since $\lambda L + F$ is compact on B, the Leray–Schauder degree of $\Phi(\lambda) = I - \lambda L - F$ with respect to B and the point 0 is well defined for all $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. By the homotopy invariance of the degree we get

$$d(\Phi(\lambda), B, 0) = \text{const} \quad \text{for } \lambda \in [\underline{\lambda}, \overline{\lambda}].$$

In particular, we have

$$d(\Phi(\underline{\lambda}), B, 0) = d(\Phi(\overline{\lambda}), B, 0). \tag{7}$$

Consider now the first term in (7). Notice that $u \ne \underline{\lambda} Lu + tF(u)$ for $u \in \partial B$ and $t \in [0, 1]$. If not, proceeding as in the proof of Theorem 1, we obtain $\operatorname{dist}(\underline{\lambda}, r(L)) \le M/a_0$, which contradicts (6). So, using the homotopy invariance again we obtain

$$d(\Phi(\underline{\lambda}), B, 0) = d(I - \underline{\lambda} L, B, 0) = i(\underline{\lambda}) = (-1)^{\underline{\beta}},$$

where $\underline{\beta}$ is the sum of the multiplicities of the characteristic values of $\underline{\lambda} L$ in (0,1). The same argument can be used for $\overline{\lambda}$, so that

$$d(\Phi(\overline{\lambda}), B, 0) = i(\overline{\lambda}) = (-1)^{\overline{\beta}}.$$

The $\overline{\beta}$ sum differs from the $\underline{\beta}$ sum by a term equal to the multiplicity of the characteristic value $\mu_k/\overline{\lambda}$ of $\overline{\lambda} L$. Since this is just the multiplicity of μ_k and μ_k is simple, $i(\underline{\lambda}) = -i(\overline{\lambda}) \ne 0$ contrary to (7). The theorem is proved.

Following Berestycki ([1]), by a *bifurcation interval* we understand an interval which contains at least one bifurcation point.

Let us mention an important consequence of Theorem 2.

Remark 3. For every $k > k_0$, $[\mu_k - M/a_0, \mu_k + M/a_0] \times \{0\}$ is a bifurcation interval for (1).

EXAMPLE 3. Consider

$$u^{(4)} = \lambda u + |u| \tag{$*$}$$

in $(0,\pi)$ with the boundary conditions $u(0)=u(\pi)=u''(0)=u''(\pi)=0$. The equation $(*)$ has the family of solutions $(\lambda_\gamma,u_\gamma)=(k^4-\operatorname{sgn}\gamma,\gamma\sin kx)\in\bigcup_{k=1}^\infty[k^4-1,k^4+1]\times E$. It is clear that all bifurcation points for $(*)$ are of the form $(k^4-1,0)$ or $(k^4+1,0)$.

EXAMPLE 4. Consider

$$u^{(4)}=\lambda u+u\sin(u''^2+u'''^2)^{-1/2} \tag{$**$}$$

in $(0,\pi)$ with $u(0)=u(\pi)=u''(0)=u''(\pi)=0$. Let $k=1$. We have the family of solutions $(\lambda_\gamma,u_\gamma)=(1-\sin(1/|\gamma|),\gamma\sin x)\in[0,2]\times E$. All the points of the interval $[0,2]\times\{0\}$ are bifurcation points for $(**)$.

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