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# On the uniqueness of continuous solutions of functional equations

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**Abstract.** We consider the problem of the vanishing of non-negative continuous solutions $\psi$ of the functional inequalities

$$\psi(f(x)) \le \beta(x, \psi(x)) \tag{1}$$

and

$$\alpha(x, \psi(x)) \le \psi(f(x)) \le \beta(x, \psi(x)), \tag{2}$$

where $x$ varies in a fixed real interval $I$. As a consequence we obtain some results on the uniqueness of continuous solutions $\varphi : I \to Y$ of the equation

$$\varphi(f(x)) = g(x, \varphi(x)), \tag{3}$$

where $Y$ denotes an arbitrary metric space.

It is well known that the iterative properties of the given function $f$ occurring in (3) play a fundamental role in the theory of continuous solutions of this equation. For the most part, the assumptions imposed on $f$ in the literature imply very simple dynamics of $f$; it is usually assumed that $f$ has exactly one fixed point which is, moreover, attractive (cf. [5] or [6]). Papers in which the dynamical behaviour of $f$ plays a role and this assumption is not imposed appear quite seldom. (The author can only quote [1]–[4].)

In [2] one can find results on the vanishing of non-negative continuous solutions of

$$\alpha(x, \psi(x)) \le \psi(f(x))$$

as well as on the uniqueness of continuous solutions of (3). Now we want to investigate (1), (2) and (3) in the spirit of [2] but under complementary assumptions on the given functions $\alpha$ and $g$.

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We use the following notations. If $f : I \to I$ and $n \in \mathbb{N}$ then the set of all *periodic* points of $f$ with *period* $n$ is denoted by $\mathrm{Per}(f, n)$, i.e.,

$$\mathrm{Per}(f, n) = \{x \in I : f^n(x) = x,\ f^i(x) \neq x \text{ for } i = 1, \ldots, n-1\}.$$

The trajectory $\{f^k(x) : k \in \mathbb{N}_0\}$ of any point $x \in \bigcup_{n=1}^{\infty} \mathrm{Per}\,(f, n)$ is called a *cycle*. Of course any cycle is a finite set. Its cardinality will be called the *order* of the cycle. Clearly, if $C$ is a cycle of order $n$ and $x \in C$ then $x \in \mathrm{Per}(f, n)$ and $C = \{x, f(x), \ldots, f^{n-1}(x)\}$. Furthermore, we put

$$\mathrm{Per}\, f = \bigcup_{n=1}^{\infty} \mathrm{Per}(f, n)$$

and (if $\mathrm{Per}\, f \neq \emptyset$)

$$Z_f = [\inf \mathrm{Per}\, f, \sup \mathrm{Per}\, f].$$

Given a real interval $I$ (not necessarily compact) consider the following hypotheses concerning the functions $\alpha$ and $\beta$.

($\mathrm{H}_1$) $\beta$ *maps* $I \times [0, \infty)$ *into* $[0, \infty)$ *and*

$$\beta(x, 0) = 0 \quad \text{for } x \in I,$$
$$\beta(x, y) < y \quad \text{for } x \in I,\ y \in (0, \infty).$$

($\mathrm{H}_2$) $\alpha$ *maps* $I \times [0, \infty)$ *into* $[0, \infty)$ *and*

$$\alpha(x, 0) = 0 \quad \text{for } x \in I,$$
$$\alpha(x, y) > 0 \quad \text{for } x \in I,\ y \in (0, \infty).$$

Below we list some immediate observations.

Remark 1. *Assume* $f : I \to I$. *If* ($\mathrm{H}_1$) *is satisfied and* $\psi : I \to [0, \infty)$ *is a solution of* (1) *then*

$$\psi(f(x)) \le \psi(x) \quad \text{for } x \in I, \tag{4}$$

*and*, *for every* $x \in I$,

$$\text{if } \psi(x) > 0 \text{ then } \psi(f(x)) < \psi(x). \tag{5}$$

In particular, we have the following simple statement.

Remark 2. *Assume* ($\mathrm{H}_1$) *and let* $f : I \to I$. *If* $\psi : I \to [0, \infty)$ *is a solution of* (1) *then*

$$\psi(x) = 0 \quad \text{for } x \in \mathrm{Per}\, f. \tag{6}$$

In a sense, a converse of Remark 1 holds true:

R e m a r k 3. *Assume* $f : I \to I$. *If* $\psi : I \to [0,\infty)$ *satisfies* (4) *and* (5) *then* $\beta : I \times [0,\infty) \to [0,\infty)$ *defined by*

$$\beta(x,y) = \begin{cases} \psi(f(x)) & \text{if } y = \psi(x), \\ 0 & \text{if } y \neq \psi(x), \end{cases}$$

*satisfies* ($H_1$) *and* $\psi$ *is a solution of* (1).

R e m a r k 4. *Assume* ($H_1$) *and* ($H_2$) *and let* $f : I \to I$. *If* $\psi : I \to [0,\infty)$ *is a solution of* (2) *then, for every* $x \in I$,

$$\psi(x) = 0 \quad \text{if and only if} \quad \psi(f(x)) = 0. \tag{7}$$

Our first aim is to prove the following result:

THEOREM 1. *Assume* ($H_1$) *and let* $f : I \to I$ *be continuous. If* $\psi : I \to [0,\infty)$ *is a continuous solution of* (1) *then*

$$\psi(x) = 0 \quad \text{for } x \in I \cap Z_f.$$

The proof will easily follow from the following lemma. I owe this proof to the referee (the original proof was much longer). In the lemma below we do not need the assumption that $I$ is an interval. It can be an arbitrary topological space.

LEMMA 1. *Assume* ($H_1$), *let* $f : I \to I$ *and let* $A$ *be a compact subset of* $I$ *such that* $A \subset f(A)$. *If* $\psi : I \to [0,\infty)$ *is a continuous solution of* (1) *then* $\psi(x) = 0$ *for* $x \in A$.

P r o o f. Let $x_0 \in A$ be such that $\psi(x_0) = \sup \psi(A)$, and choose an $x_1 \in A$ with $f(x_1) = x_0$. If $\psi(x_0) > 0$ then, by (5), $\psi(x_0) = \psi(f(x_1)) < \psi(x_1)$, which contradicts the choice of $x_0$.

P r o o f o f T h e o r e m 1. Let $a$ and $b$, $a \le b$, be periodic points of $f$ with periods $k$ and $l$, respectively. To complete the proof it is enough to apply Lemma 1 to $f^{kl}$ (in place of $f$; cf. also Remark 1) and $A = [a,b]$.

Now we apply Theorem 1 to the problem of uniqueness of continuous solutions of (3). To this end fix a metric space $(Y,\sigma)$ and consider the following hypothesis:

($H_3$) *$g$ maps a subset $\Omega$ of $I \times Y$ into $Y$ and there exists a function $\beta$ satisfying* ($H_1$) *and such that*

$$\sigma(g(x,y_1), g(x,y_2)) \le \beta(x, \sigma(y_1,y_2))$$

*for every* $(x,y_1),(x,y_2) \in \Omega$.

COROLLARY 1. *Assume* ($H_3$) *and let* $f : I \to I$ *be continuous. If* $\varphi_1,\varphi_2 : I \to Y$ *are continuous solutions of equation* (3) *then* $\varphi_1(x) = \varphi_2(x)$ *for* $x \in I \cap Z_f$.

P r o o f. It is enough to observe that the function $\psi : I \to [0,\infty)$ given by

$$\psi(x) = \sigma(\varphi_1(x), \varphi_2(x)) \tag{8}$$

is a continuous solution of (1), and use Theorem 1.

Now we pass to the study of non-negative continuous solutions of (2). Let us start with the following lemma, important in the proof of Theorem 2.

LEMMA 2. *Assume* ($H_1$) *and* ($H_2$), *let* $f : I \to I$ *be continuous and let* $J \subset I$ *be an interval containing a fixed point of* $f$. *Then there exists a subinterval* $K$ *of* $I$ *containing* $J$ *and such that any continuous solution* $\psi : I \to [0,\infty)$ *of* (2) *vanishing on* $J$ *vanishes also on* $K$ *and*, *moreover*, *either*

- $\{\inf K, \sup K\}$ *contains a fixed point of* $f$, *or*
- $\{\inf K, \sup K\}$ *is a cycle of* $f$ *of order* 2, *or*
- $K = I$.

P r o o f. Clearly we can assume that $J$ is not a singleton. Put

$$K_0 = \bigcup_{n=0}^{\infty} f^n(J).$$

By Remark 1, any continuous solution $\psi : I \to [0,\infty)$ of (2) vanishing on $J$ vanishes also on $K_0$. Since $J$ contains a fixed point of $f$, the set $K_0$ is an interval. Moreover, $J \subset K_0 \subset f^{-1}(K_0)$.

By induction we construct a sequence $(K_n : n \in \mathbb{N})$ of intervals such that each $K_n$ is a component of $\mathrm{cl}_I\, f^{-1}(K_{n-1})$ containing $K_{n-1}$. Making use of Remark 4 it is easy to observe that any continuous solution $\psi : I \to [0,\infty)$ of (2) vanishing on $J$ vanishes also on each $K_n$, i.e. on $\bigcup_{n=0}^{\infty} K_n$. Let

$$K = \bigcup_{n=0}^{\infty} K_n, \quad a_n = \inf K_n, \quad b_n = \sup K_n, \quad n \in \mathbb{N}_0.$$

Clearly $K$ is an interval containing $J$ and $K_n = [a_n, b_n] \cap I$ for $n \in \mathbb{N}_0$.

We now prove that for every $n \in \mathbb{N}_0$,

- either $a_{n+1} = \inf I$ or $f(a_{n+1}) \in \{a_n, b_n\}$, and
- either $b_{n+1} = \sup I$ or $f(b_{n+1}) \in \{a_n, b_n\}$.

For suppose that one of the above conditions is not satisfied, say $a_{n+1} > \inf I$ and $f(a_{n+1}) \in (a_n, b_n)$ for some $n \in \mathbb{N}_0$. By the continuity of $f$ there exists a $\delta > 0$ such that $(a_{n+1} - \delta, a_{n+1}] \subset I$ and

$$f((a_{n+1} - \delta, a_{n+1}]) \subset (a_n, b_n).$$

Therefore $(a_{n+1} - \delta, a_{n+1}] \cup K_{n+1}$ is a connected set containing $K_n$ and contained in $\mathrm{cl}_I\, f^{-1}(K_n)$, which contradicts the definition of $K_{n+1}$.

Now, since $(a_n : n \in \mathbb{N})$ decreases and $(b_n : n \in \mathbb{N})$ increases, we infer that

(9) either $a_n = \inf I$ for $n$ sufficiently large or $f(a_{n+1}) \in \{a_n, b_n\}$ for every $n \in \mathbb{N}$, and

(10) either $b_n = \sup I$ for $n$ sufficiently large or $f(b_{n+1}) \in \{a_n, b_n\}$ for every $n \in \mathbb{N}$.

Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then $a = \inf K$, $b = \sup K$ and, by (9) and (10),

- either $a = \inf I$ or $f(a) \in \{a, b\}$, and
- either $b = \sup I$ or $f(b) \in \{a, b\}$.

Assume that $\{a, b\}$ does not contain any fixed point of $f$ and is not a cycle of $f$ of order 2. To finish the proof it is enough to prove that neither

- $\inf I = a = f(b)$ and $b < \sup I$, nor
- $\sup I = b = f(a)$ and $a > \inf I$.

Suppose, for instance, that the first alternative holds true. (In the second case we proceed analogously.) Since $a = f(b)$ we have $a \in I$. If $\inf I < a_n$ for $n \in \mathbb{N}$ then, by (9), $f(a) \in \{a, b\}$, whence either $a = f(a)$ or $\{a, b\}$ is a cycle of $f$ of order 2. Consequently, we may assume that there exists an $n_0 \in \mathbb{N}$ such that $a_n = \inf I$ for $n \ge n_0$. Then, according to (10) and the fact that $f(b) = a$, we can find an $n \ge n_0$ for which $f(b_{n+1}) = a_n = \inf I$. Since $b_{n+1} \le b < \sup I$, from the continuity of $f$ we deduce that there exists a $\delta > 0$ such that $[b_{n+1}, b_{n+1} - \delta] \subset I$ and

$$a_n = \inf I \le f(x) < b_n \quad \text{for } x \in [b_{n+1}, b_{n+1} + \delta).$$

Therefore $K_{n+1} \cup [b_{n+1}, b_{n+1} + \delta)$ is a connected set containing $K_n$ and contained in $\mathrm{cl}_I\, f^{-1}(K_n)$, which contradicts the definition of $K_{n+1}$ and finishes the proof of the lemma.

THEOREM 2. *Assume* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$, *let* $f : I \to I$ *be continuous and let* $J \subset I$ *be an interval containing a fixed point of* $f$ *and such that* $\mathrm{cl}_I(I \setminus J)$ *contains no cycle of* $f$ *of order not greater than two. If* $\psi : I \to [0, \infty)$ *is a continuous solution of* (2) *vanishing on* $J$ *then* $\psi$ *is the zero function.*

Proof. Clearly we can assume that $J = \mathrm{cl}_I\, J$. If $\inf I < \inf J$ and $\sup J < \sup I$ then the assertion follows from Lemma 2. Thus let $\inf J = \inf I$ or $\sup J = \sup I$. Assume, for instance, the first possibility and fix a continuous solution $\psi : I \to [0, \infty)$ of (2) vanishing on $J$. We now prove that $\psi(x_0) = 0$ for each $x_0 \in I$. Of course, we can consider the case $x_0 > \sup J$ only.

First assume that $x_0 < f(x_0)$. Then, by our assumptions,

$$f(x) > x \quad \text{for } x \in I \cap [\sup J, \infty),$$

whence we can construct a sequence $(x_n : n \in \mathbb{N})$ of points of $I$ converging to $\sup \mathrm{Per}(f,1)$ such that $f(x_{n+1}) = x_n$ for $n \in \mathbb{N}_0$. Since $\sup \mathrm{Per}(f,1) < \sup J$ it follows that $x_n \in J$ for an $n \in \mathbb{N}$. Thus $\psi(x_n) = 0$, which means (cf. Remark 4) that $\psi(x_0) = \psi(f^n(x_n)) = 0$.

In the case $f(x_0) < x_0$ we proceed similarly. Then $f(x) < x$ for $x \in I \cap [\sup J, \infty)$, whence we deduce that if $f^n(x_0) > \sup J$ then $f^{n+1}(x_0) < f^n(x_0)$, for every $n \in \mathbb{N}_0$. Therefore either

- $f^n(x_0) \in J$ for some $n \in \mathbb{N}$, or
- $\sup J \le f^{n+1}(x_0) < f^n(x_0)$ for every $n \in \mathbb{N}$.

But in the latter case we would have

$$\lim_{n\to\infty} f^n(x_0) \in \mathrm{Per}(f,1) \cap \mathrm{cl}_I(I \setminus J),$$

which is impossible. Therefore $f^n(x_0) \in J$ for some $n \in \mathbb{N}$. Consequently, $\psi(f^n(x_0)) = 0$, which means (cf. (7)) that $\psi(x_0) = 0$.

As a consequence of Theorems 1 and 2 we get the following fact:

COROLLARY 2. *Assume* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ *and let* $f : I \to I$ *be continuous. If* $\mathrm{Per} f \ne \emptyset$ *and* $\mathrm{cl}_I(I \setminus Z_f)$ *contains no cycle of* $f$ *of order not greater than* 2 *then the zero function is the unique continuous solution* $\psi : I \to [0,\infty)$ *of* (2).

In order to apply Theorem 2 and Corollary 2 to the problem of uniqueness of continuous solutions of (3) fix a metric space $(Y,\sigma)$ and consider the following hypothesis:

$(\mathrm{H}_4)$ *$g$ maps a subset $\Omega$ of $I \times Y$ into $Y$ and there exist $\beta$ and $\alpha$ satisfying* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *respectively, and such that*

$$\alpha(x, \sigma(y_1,y_2)) \le \sigma(g(x,y_1), g(x,y_2)) \le \beta(x, \sigma(y_1,y_2)),$$

*for every* $(x,y_1), (x,y_2) \in \Omega$.

COROLLARY 3. *Assume* $(\mathrm{H}_4)$, *let* $f : I \to I$ *be continuous and let* $J \subset I$ *be an interval containing a fixed point of* $f$ *such that* $\mathrm{cl}_I(I \setminus J)$ *contains no cycle of* $f$ *of order not greater than* 2. *If* $\varphi_1, \varphi_2 : I \to Y$ *are continuous solutions of equation* (3) *and* $\varphi_1(x) = \varphi_2(x)$ *for* $x \in J$, *then* $\varphi_1 = \varphi_2$.

Proof. Since $\psi : I \to [0,\infty)$ defined by (8) is a continuous solution of (2) it suffices to use Theorem 2.

COROLLARY 4. *Assume* $(\mathrm{H}_4)$ *and let* $f : I \to I$ *be continuous. If* $\mathrm{Per} f \ne \emptyset$ *and* $\mathrm{cl}_I(I \setminus Z_f)$ *contains no cycle of* $f$ *of order not greater than* 2 *then* (3) *has at most one continuous solution* $\varphi : I \to Y$.

Modifying a little a classical reasoning from [5] we show in the next two examples that inequalities (2) can allow a lot of non-negative continuous solutions when $\mathrm{cl}_I(I \setminus Z_f)$ contains a cycle of $f$ of order 2 as well as when it contains a fixed point of $f$.

Example 1. Fix an $s \in (0,1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous function satisfying the following conditions:

- $f([0,1]) \subset [0,1]$,
- $f|_{(-\infty,0]}$ is strictly decreasing,
- $f|_{[1,\infty)}$ is decreasing,
- $1 < f(x) < -x+1$ for $x < 0$,
- $-x+1 < f(x) < 0$ for $x > 1$.

Then

$$x < f^2(x) < 0 \quad \text{for } x \in (-\infty,0),$$
$$1 < f^2(x) < x \quad \text{for } x \in (1,\infty).$$

Consequently, $\mathrm{Per}\, f \subset [0,1]$, $\inf \mathrm{Per}\, f = 0$, $\sup \mathrm{Per}\, f = 1$ and these points form a cycle of $f$ of order 2. (They belong to $\mathrm{cl}(\mathbb{R} \setminus Z_f)$.)

We now show that for each $x_0 \in (-\infty,0)$ and for every continuous function $\psi_0 : [x_0, f^2(x_0)] \to [0,\infty)$ with

$$\psi_0(f^2(x_0)) = s^2\psi_0(x_0) \tag{11}$$

there exists a continuous solution $\psi : \mathbb{R} \to [0,\infty)$ of the equation

$$\psi(f(x)) = s\psi(x) \tag{12}$$

such that

$$\psi|_{[x_0, f^2(x_0)]} = \psi_0. \tag{13}$$

To this end define $g : (-\infty,0] \to (-\infty,0]$ by $g(x) = f^2(x)$. Observe that $g$ is strictly increasing and

$$x < g(x) < 0 \quad \text{for } x \in (-\infty,0). \tag{14}$$

Fix an $x_0 \in (-\infty,0)$ and let $\psi_0 : [x_0, f^2(x_0)] \to [0,\infty)$ be a continuous function satisfying (11). According to [5, Theorem 2.10] there exists a (unique) continuous function $\psi_1 : (-\infty,0] \to \mathbb{R}$ such that

$$\psi_1(g(x)) = s^2\psi_1(x) \quad \text{for } x \in (-\infty,0] \tag{15}$$

and

$$\psi_1|_{[x_0, f^2(x_0)]} = \psi_0. \tag{16}$$

We show that $\psi_1$ is non-negative. Notice that, iterating (15), we obtain

$$\psi_1(g^n(x)) = s^{2n}\psi_1(x) \quad \text{for } x \in (-\infty,0] \tag{17}$$

and for every $n \in \mathbb{N}_0$. Fix an $x \in (-\infty,0]$. Taking into account (14) we see that there exists a $y \in [x_0, g(x_0)] = [x_0, f^2(x_0)]$ and an $n \in \mathbb{N}_0$ such that either $g^n(x) = y$ or $g^n(y) = x$. From (16) and (17) we infer that $\psi_1(x) \ge 0$.

Since $\psi_1(0) = 0$, the function $\psi : \mathbb{R} \to [0, \infty)$ defined by

$$\psi(x) = \begin{cases} \psi_1(x), & x \in (-\infty, 0), \\ 0, & x \in [0, 1], \\ \frac{1}{s}\psi_1(f(x)), & x \in (1, \infty), \end{cases}$$

is continuous. Moreover (cf. (16)), it satisfies (13). We prove that $\psi$ is a solution of (12). Fix an $x \in \mathbb{R}$. If $x \in (-\infty, 0)$ then, by the definitions of $\psi$ and $g$ and property (15), we have

$$\psi(f(x)) = \frac{1}{s}\psi_1(f^2(x)) = \frac{1}{s}\psi_1(g(x)) = \frac{1}{s}s^2\psi_1(x) = s\psi(x).$$

For $x \in [0, 1]$ equality (12) is evident. Finally, assume that $x \in (1, \infty)$. Then $\psi(f(x)) = \psi_1(f(x)) = s\psi(x)$.

Example 2. Fix an $s \in (0, 1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

- $f([0, 1]) \subset [0, 1]$,
- $f|_{(-\infty,0]\cup[1,\infty)}$ is strictly increasing,
- $x < f(x) < 0$ for $x < 0$,
- $1 < f(x) < x$ for $x > 1$.

Then clearly $\inf \mathrm{Per} f = 0$, $\sup \mathrm{Per} f = 1$ and these points are both fixed points of $f$. (They belong to $\mathrm{cl}(\mathbb{R} \setminus Z_f)$.) Moreover, reasoning as in the previous example we infer that for all $x_0 \in (-\infty, 0)$, $y_0 \in (1, \infty)$ and for every non-negative continuous function $\psi_0 : [x_0, f(x_0)] \cup [f(y_0), y_0] \to \mathbb{R}$ such that

$$\psi_0(f(x_0)) = s\psi_0(x_0), \qquad \psi_0(f(y_0)) = s\psi_0(y_0),$$

there exists a non-negative continuous solution $\psi : \mathbb{R} \to \mathbb{R}$ of (12) such that $\psi|_{[x_0,f(x_0)]\cup[f(y_0),y_0]} = \psi_0$.

We end this paper by another two corollaries concerning solutions of (2) and (3).

Corollary 5. *Assume* ($\mathrm{H}_1$), ($\mathrm{H}_2$) *and let* $f : I \to I$ *be continuous. If* $\mathrm{Per} f \neq \emptyset$ *and*

$$I \cap Z_f \subset \mathrm{int}_I f(I \cap Z_f)$$

*then the zero function is the unique continuous solution* $\psi : I \to [0, \infty)$ *of* (2).

Proof. Fix a continuous solution $\psi : I \to [0, \infty)$ of (2). By Theorem 1, $\psi$ vanishes on $I \cap Z_f$. By Remark 4, $\psi$ vanishes on $f(I \cap Z_f)$. Moreover, $f(I \cap Z_f)$ contains a fixed point of $f$ and

$$\mathrm{cl}_I(I \setminus f(I \cap Z_f)) = I \setminus \mathrm{int}_I f(I \cap Z_f) \subset I \setminus Z_f.$$

Now use Theorem 2.

COROLLARY 6. *Assume* ($H_4$) *and let* $f : I \to I$ *be continuous. If* $\operatorname{Per} f \neq \emptyset$ *and*

$$I \cap Z_f \subset \operatorname{int}_I f(I \cap Z_f)$$

*then* (3) *has at most one continuous solution* $\varphi : I \to Y$.

Proof. Use Corollary 5 for $\psi : I \to [0, \infty)$ defined by (8).

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