The automorphism groups of Zariski open affine subsets of the affine plane

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Abstract. We study some properties of the affine plane. First we describe the set of fixed points of a polynomial automorphism of $\mathbb{C}^2$. Next we classify completely so-called identity sets for polynomial automorphisms of $\mathbb{C}^2$. Finally, we show that a sufficiently general Zariski open affine subset of the affine plane has a finite group of automorphisms.

1. Introduction. The automorphism group of an affine (or more generally non-complete) algebraic variety $X$ is rather difficult to study and only partial results on its structure are known. In [Iit1], [Sak] sufficient conditions for the finiteness of $\text{Aut}(X)$ are given (in terms of logarithmic Kodaira dimension).

Another approach is given in [Jel2] and [Jel3], where we showed that $\text{Aut}(X)$ is finite provided the divisor at infinity of some projective compactification of $X$ is very ample and it does not have uniruled components. Moreover, in [Jel2] we started the study of the automorphism groups of affine Zariski open subvarieties of $\mathbb{C}^n$.

This note is a continuation of this study, as well as a continuation of our work on identity sets for polynomial automorphisms (see [Jel1], [Jel3], [Jel5]).

We concentrate on the first non-trivial case, that of the affine plane $\mathbb{C}^2$. Our first aim is to give a description of the fixed point set of a non-trivial polynomial automorphism of $\mathbb{C}^2$.

We show (Theorem 3.3) that this set is either finite, or a finite union of disjoint plane $\mathbb{C}$-curves. Conversely, for any finite subset $S$ of the plane, or for any finite family $\mathcal{S}$ of disjoint plane $\mathbb{C}$-curves we construct an automorphism of the plane for which the set of fixed points is exactly $S$.

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In particular, we obtain an “if and only if” condition for an affine curve $\Gamma$ to be an identity set for polynomial automorphisms of $\mathbb{C}^2$. This generalizes some results of [M-W], [Jel1], [Jel5].

The approach is based on the study of curves $\Gamma \subset \mathbb{C}^2$ with infinite group $\text{Stab}_\Gamma = \{ f \in \text{Aut}(\mathbb{C}^2) : f(\Gamma) = \Gamma \}$. We classify such curves completely (Theorem 3.8).

Finally, we show that for any finite family $\{\Gamma_1, \ldots, \Gamma_s\}$ of non-rational curves on the plane, the variety $X := \mathbb{C}^2 \setminus \bigcup_{i=1}^s \Gamma_i$ has a finite automorphism group (Theorem 3.9).

2. Preliminaries. Let us recall some properties of fibres of a primitive polynomial $p$ in two complex variables (see [Suz1], [Suz2], [Z-L], [Zai]).

It is well known that all but finitely many fibres of $p$ are pairwise homeomorphic. Such fibres are called generic. A generic fibre is smooth and irreducible. All other fibres are called degenerate. A point $s \in \mathbb{C}$ for which the fibre $\Gamma_s := p^{-1}(s)$ is generic is called a generic point, otherwise it is degenerate. The set of fibres of a primitive polynomial $p$ will be called a family of curves.

The following proposition was proved in [Suz2] and is crucial to our study:

**Proposition 2.1.** Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a primitive polynomial and let $\chi_p$ denote the Euler characteristic of a generic fibre. Let $S$ be the set of all degenerate points. For every $s \in S$ we have $\chi(\Gamma_s) > \chi_p$. Moreover,

$$\sum_{s \in S} \{ \chi(\Gamma_s) - \chi_p \} = 1 - \chi_p.$$

We get at once the following interesting

**Corollary 2.2.** If the generic fibre of a family $p$ is a $\mathbb{C}$-curve (i.e., it is isomorphic to $\mathbb{C}$) then all fibres of $p$ are generic (and isomorphic to $\mathbb{C}$). Conversely, if all fibres of a family $p$ are generic then it is a family of $\mathbb{C}$-curves.

If a generic fibre is a $\mathbb{C}^*$-curve then the family $p$ has exactly one degenerate fibre $\Gamma_s$. Moreover, $\chi(\Gamma_s) = 1$.

In the sequel the topological characterization of an irreducible algebraic curve with Euler characteristic 1 will be useful. Let us begin with the following simple observation.

**Lemma 2.3.** Let $X$ be an algebraic complex curve and let $a \in X$. Then

$$\chi(X \setminus \{a\}) = \chi(X) - 1.$$

**Proof.** Consider the triple $\{X \setminus \{a\}, V, V \setminus \{a\}\}$, where $V$ is a small neighbourhood of $a$. More precisely, we can assume that $V$ is a bouquet of $r$
discs, where \( r \) is the number of components in the germ \( X_a \). In this situation the point \( a \) is a retract of \( V \), and \( V \setminus \{a\} \) can be retracted to the disjoint sum \( \bigcup_{i=1}^{r} S^1_i \) of \( r \) circles. Hence \( \chi(V) = 1 \) and \( \chi(V \setminus \{a\}) = r\chi(S^1) = 0 \). Using the Mayer–Vietoris sequence we have
\[
\chi(X \setminus \{a\}) + \chi(V) = \chi(X) + \chi(V \setminus \{a\}).
\]
Thus \( \chi(X \setminus \{a\}) + 1 = \chi(X) \).

The simple but useful consequence of the above result is the following

**Proposition 2.4.** Let \( X \) be an irreducible affine curve of genus \( g \). (This means that a smooth model \( X_1 \) of a compactification of \( X \) has genus \( g \).) Suppose that \( X \) has \( n \) branches at infinity (i.e., \( n \) is the number of points in \( X_1 \setminus X_0 \), where \( X_0 \) is the normalization of \( X \)). Let \( \text{Sing}(X) = \{a_1, \ldots, a_r\} \) be the singular locus. Further, suppose that the germ \( X_a \) has \( k_i \) irreducible components. Then
\[
\chi(X) = 2(1 - g) - n - \sum_{i=1}^{r} (k_i - 1).
\]

In particular, if \( \chi(X) = 1 \) then \( X \) is homeomorphic to \( \mathbb{C} \).

**Proof.** Let \( \pi : X_0 \to X \) be the normalization. Then \( \pi \) is an isomorphism outside \( A := \text{Sing}(X) \), and \( \pi^{-1}(A) \) has \( \sum_{i=1}^{r} k_i \) points. Hence by the lemma
\[
\chi(X) = \chi(X \setminus A) + r = \chi(X_0 \setminus \pi^{-1}(A)) + r = \chi(X_0) - \sum_{i=1}^{r} k_i + r
\]
\[
= \chi(X_1) - n - \sum_{i=1}^{r} (k_i - 1) = 2(1 - g) - n - \sum_{i=1}^{r} (k_i - 1).
\]

Since \( X \) is affine we have \( n > 0 \) and the case \( \chi(X) = 1 \) is possible only if \( g = 0 \) and \( n = 1 \) and all \( k_i = 1 \), i.e., if \( X \) is homeomorphic to \( \mathbb{C} \).

Plane curves homeomorphic to the complex line have a very nice description due to Zaidenberg and Lin (see [Z-L]):

**Proposition 2.5.** Let \( X \subset \mathbb{C}^2 \) be an affine algebraic curve homeomorphic to the complex line. Then in suitable coordinates \( X \) can be written as
\[
X = \{ (x, y) \in \mathbb{C}^2 : x^k = y^l, \ (k, l) = 1 \}.
\]

**Remark 2.6.** The proposition above is a generalization of the famous Abhyankar–Moh–Suzuki theorem (see [A-M], [Suz1]):

If \( \Gamma \subset \mathbb{C}^2 \) is a curve isomorphic to \( \mathbb{C} \) then in some coordinates we have \( \Gamma = \{ (x, y) \in \mathbb{C}^2 : x = 0 \} \).

There is the following useful consequence of the above considerations.
Proposition 2.7. Let \( p : \mathbb{C}^2 \to \mathbb{C} \) be a family of curves. If there is only one degenerate fibre \( \Gamma \) in \( p \) and \( \Gamma \) is irreducible then in some coordinates
\[
\Gamma = \{ (x, y) \in \mathbb{C}^2 : x^k = y^l, \ (k, l) = 1 \}.
\]

Proof. Indeed, by Proposition 2.1 the curve \( \Gamma \) has Euler characteristic equal to 1. Hence by Proposition 2.4 it is homeomorphic to \( \mathbb{C} \) and finally the proof is finished by Proposition 2.5.

3. Main result. First we want to describe the set of fixed points of a polynomial automorphism of \( \mathbb{C}^2 \). To do this, the following lemmas will be useful:

Lemma 3.1. Let \( X \) be an irreducible affine curve with infinite automorphism group. Then \( X \) is either isomorphic to \( \mathbb{C}^* \) or it is homeomorphic to \( \mathbb{C} \). Moreover, in the last case \( X \) can have at most one (necessarily irreducible) singularity.

Proof. Let \( X_0 \) be a normalization of \( X \) and \( X_1 \) be a smooth completion of \( X_0 \). Since \( \text{Aut}(X) \) is infinite, so is \( \text{Aut}(X_0) \). The latter group is a subgroup of \( \text{Aut}(X_1) \) which stabilizes the divisor \( D := X_1 \setminus X_0 \). Since \( D \) is ample, \( \text{Aut}(X_0) \) must be linear (for details see [Jel3], 3.7). Moreover, since \( \text{Aut}(X_0) \) is infinite, \( X_1 \) is rational (see ibid., 3.12), i.e., \( X_1 = \mathbb{P}^1(\mathbb{C}) \). Further, the singular points are permuted by polynomial automorphisms, hence the common number of points at infinity of \( X_0 \) and of points of \( X_1 \) which lie over the singular locus of \( X \) must be at most two (in \( X_1 \)). If there exist two such points we get either \( X = X_0 = \mathbb{C}^* \), or \( X \neq X_0 = \mathbb{C} \), and in the latter case \( X \) has one irreducible singular point (which means that \( X \) is homeomorphic to \( \mathbb{C} \)). If there is only one such point then \( X = X_0 = \mathbb{C} \).

Lemma 3.2. Let \( \Gamma \subset \mathbb{C}^2 \) be a curve with an irreducible equation \( p(x, y) = 0 \). Let \( f \in \text{Stab}_\Gamma \) be an element of infinite order. Then only two possibilities can occur:

1) There exists \( s \in \mathbb{N} \) such that \( f^s \) stabilizes all fibres of \( p \). Moreover, \( p \) is a \( \mathbb{C} \)- or \( \mathbb{C}^* \)-family and \( \Gamma \) is either homeomorphic to \( \mathbb{C} \), or it is isomorphic to \( \mathbb{C}^* \).

2) \( \Gamma \) is the unique degenerate fibre of the family \( p \) and it is homeomorphic to \( \mathbb{C} \).

Proof. By the Hilbert Nullstellensatz we have \( p \circ f = cp \), for some \( c \in \mathbb{C}^* \). There are two cases possible:

1) \( c \) has a finite order, i.e., \( c^s = 1 \) for some \( s \geq 1 \),

2) \( c \) has an infinite order.

1) We can assume \( c = 1 \) and then we have \( p - \lambda = (p - \lambda) \circ f \) for every \( \lambda \in \mathbb{C} \). This means that all fibres of \( p \) are stable under \( f \). Since \( f \) is of infinite
order, a generic fibre has an infinite automorphism group. Since the generic fibre is smooth and irreducible it must be isomorphic either to $\mathbb{C}$ or to $\mathbb{C}^\ast$ (see Lemma 3.1).

In the first case, by Corollary 2.2, $p$ has no degenerate fibres, and in particular $\Gamma \cong \mathbb{C}$.

In the second case, by the same corollary, $p$ has exactly one degenerate fibre. If it is the fibre $p^{-1}(0) = \Gamma$ then by Proposition 2.1 we obtain $\Gamma = \{(x, y) \in \mathbb{C}^2 : x^k = y^l, (k, l) = 1\}$ in some coordinate system. If this fibre is not $\Gamma$ then $\Gamma$ is generic and isomorphic to $\mathbb{C}^\ast$.  

2) In this case the fibre $\Gamma_\lambda := p^{-1}(\lambda)$ is transformed under $f$ onto the fibre $p = c\lambda$. More generally, under $f^r$ this fibre goes onto the fibre $p = c^r\lambda$ for $r \in \mathbb{Z}$. Since $c$ is of infinite order, for $\lambda \neq 0$ the fibre $\Gamma_\lambda$ is isomorphic to an infinite set of other fibres. This means that $\Gamma_\lambda$ is a generic fibre for all $\lambda \neq 0$. Hence $p$ has at most one degenerate fibre, $\Gamma_0$. If $\Gamma_0$ is degenerate, by Proposition 2.7 we have $\Gamma_0 = \{(x, y) \in \mathbb{C}^2 : x^k = y^l, (k, l) = 1\}$ in some coordinate system.

If $p$ has no degenerate fibres at all, then Proposition 2.1 shows that $p$ is a family of $\mathbb{C}$-curves, and in particular $\Gamma \cong \mathbb{C}$.

Using the lemma above we describe the set $\text{Fix} f$ of fixed points of a polynomial automorphism $f$ of the affine plane. We have:

**Theorem 3.3.** Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a non-trivial polynomial automorphism. Then the set $S = \text{Fix} f$ is either finite, or a finite union of disjoint $\mathbb{C}$-curves.

Conversely, if $S$ is a finite subset of the plane, or a finite union of disjoint plane $\mathbb{C}$-curves, then there is $f \in \text{Aut}(\mathbb{C}^2)$ such that $S = \text{Fix} f$.

**Proof.** Let $\Gamma$ be a one-dimensional component of $S$. We show that in this case $\Gamma \cong \mathbb{C}$.

Since any automorphism of the affine plane of a finite order is conjugate to a linear one (see e.g. [Kam]), in which case there is nothing to prove, we can assume that $f$ is of infinite order. By Lemma 3.2 and Proposition 2.5 there are only two cases possible:

1) in some coordinates $\Gamma = \{(x, y) \in \mathbb{C}^2 : x^k = y^l, (k, l) = 1\}$,

2) $\Gamma$ is isomorphic to $\mathbb{C}^\ast$.

Now we will show that the first case is possible only if $k = 1$ or $l = 1$, and the second case is excluded. Indeed, the following lemma is true:

**Lemma 3.4.** Let $f$ be a polynomial automorphism of the plane which stabilizes the curve $R = \{(x, y) : x^k = y^l, (k, l) = 1\}$. Then each one-dimensional component $\Gamma$ of $\text{Fix} f$ is isomorphic to $\mathbb{C}$.

**Proof.** Let $f = (f_1, f_2)$. Since $R$ is $f$-stable we have $f_1^k - f_2^l = c(x^k - y^l)$
for some non-zero constant $c$. Hence if $a := \deg f_1$ and $b := \deg f_2$ are both greater than 1 we have $\deg f_1^k = \deg f_2^l$. But it is well known (see [Kul]) that if $f$ is a polynomial automorphism then $\deg f_1$ divides $\deg f_2$ or vice versa, thus we can assume that $b = \lambda a$ for some non-zero integer $\lambda$. Hence $ka = l\lambda a$ and $k = l\lambda$. If $k \neq 1$ and $l \neq 1$ this is a contradiction.

Thus either “$a = 1$ or $b = 1$”, or “$k = 1$ or $l = 1$”.

Assume that $a = 1$ or $b = 1$. This means that one of the polynomials $f_1$, $f_2$, say $f_1$, is linear. We have $\Gamma \subset \{ f_1 - x = 0 \}$. Hence if $f_1 \neq x$ then $\Gamma$ is a line. In the other case $f$ is a triangular automorphism, $f(x, y) = (x, cy + p(x))$, and then Fix $f$ is known to be a finite union of $\mathbb{C}$-curves.

If $k = 1$ or $l = 1$ then $R$ is isomorphic to $\mathbb{C}$ and by the Abhyankar–Moh–Suzuki theorem we can assume that $R = \{ (x, y) : x = 0 \}$. But in this case $f(x, y) = (cx, by + p(x))$ and it is easy to see that a one-dimensional component of Fix $f$ is either a straight line or it has an equation $(b - 1)y + p(x) = 0$ (if $b \neq 1$). In both cases $\Gamma$ is isomorphic to $\mathbb{C}$.

We proceed now with the proof of Theorem 3.3.

1) It follows immediately from the lemma that $\Gamma \cong \mathbb{C}$, i.e., $k = 1$ or $l = 1$.

2) We will show that the case $\Gamma \cong \mathbb{C}^*$ is impossible.

Let $p$ be an irreducible equation of $\Gamma$. Since $\Gamma$ is not homeomorphic to $\mathbb{C}$, Lemma 3.2 implies that $f^s$ stabilizes all fibres of $p$ for some $s \in \mathbb{N}$. We can assume that $s = 1$. This means that all fibres of $p$ are stable under $f$ and the generic fibre is $\mathbb{C}$ or $\mathbb{C}^*$. By Corollary 2.2 the first case is impossible, hence the generic fibre must be $\mathbb{C}^*$. Hence $p$ has only one degenerate fibre. By Proposition 2.7 it cannot be the fibre over 0. Let $w$ be the unique degenerate point and let $R$ be some irreducible component of $\Gamma_w$. Since $\Gamma_w$ has only a finite number of irreducible components the curve $R$ is stable under some iteration $F := f^r$ of $f$. We have again two cases to consider:

- $(*)$ $R = \Gamma_w$.
- $(**)$ $R$ is a proper component of $\Gamma_w$.

$(*)$ In this case $R$ is homeomorphic to $\mathbb{C}$ and stable under $F$ and by Lemma 3.4 we get $\Gamma \cong \mathbb{C}$, which is a contradiction.

$(**)$ Let $R = \{ q = 0 \}$ for some irreducible polynomial $q$. Since $R \neq \{ p = 0 \}$ we see that $\deg q < \deg p$, which shows that the families $p$ and $q$ are different. By Lemma 3.2 only two possibilities can occur: either all fibres of $q$ are stable under some iteration of $F$, or $R$ is homeomorphic to $\mathbb{C}$. The second possibility cannot occur, by Lemma 3.4.

Hence we can assume that $F$ stabilizes all fibres of $q$. But $F$ is an iteration of $f$, hence it also stabilizes all fibres of $p$. Since a generic fibre of $q$ intersects a generic fibre of $p$ in at most $N = (\deg q)(\deg p)$ points, the order of $F$ is at most $N! = 1 \cdot \ldots \cdot N$. This is a contradiction again.
Hence case 2) is excluded and we have proved that if a curve $\Gamma$ is a component of $\text{Fix} f$ for a polynomial automorphism $f$ then $\Gamma$ is isomorphic to $C$. Further in this case we can assume by the Abhyankar–Moh–Suzuki theorem (see Remark 2.6) that $\Gamma = \{(x, y) \in C^2 : x = 0\}$. Since $f$ is the identity on $\Gamma$ we have $f = (cx, y + p(x))$ with $p(0) = 0$. This means that $\text{Fix} f$ consists of one or more (disjoint) straight lines, in particular, it is of pure dimension. Hence $\text{Fix} f$ is either finite, or a union of disjoint $C$-curves.

Now we prove the converse: if $S$ is a finite subset of the plane, or a finite union of disjoint plane $C$-curves, then there is $f \in \text{Aut}(C^2)$ such that $S = \{x \in C^2 : f(x) = x\}$. Of course we can assume that $S$ is non-empty.

First assume that $S$ is finite. The following is proved in [Jel4]:

**Lemma 3.5.** Let $n \geq 2$ and $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots, b_r\} \subset \mathbb{C}^n$, where $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. Then there is a polynomial automorphism $F$ of $\mathbb{C}^n$ such that $F(a_i) = b_i$, $i = 1, \ldots, r$.

Let $S = \{a_1, \ldots, a_r\}$ and suppose $F \in \text{Aut}(C^2)$ has the property that $F(a_i) = (i, 0)$, $i = 1, \ldots, r$. Let $B = \{(1,0), (2,0), \ldots, (r,0)\}$. If we construct an automorphism $G$ with $B = \text{Fix} G$ then $f = F^{-1} \circ G \circ F$ has $\text{Fix} f = S$.

It is easy to check that we can take for $G$ the automorphism

$$G(x, y) = \left(x + y + \prod_{i=1}^{r} (x - i), y + \prod_{i=1}^{r} (x - i)\right).$$

Now let $S$ be a finite union of disjoint plane curves, i.e., $S = \bigcup_{i=1}^{r} \Gamma_i$, where $\Gamma_i \cong C$ and $\Gamma_i \cap \Gamma_j = \emptyset$. By the Abhyankar–Moh–Suzuki theorem (see Remark 2.6) we can reduce the problem (as above) to the case when $\Gamma_1 = \{(x,y) : x = 0\}$. Then necessarily $\Gamma_i = \{(x, y) : x = a_i\}$ for some non-zero distinct complex numbers $a_i$, $i = 2, \ldots, r$. Indeed, the polynomial $x$ restricted to $\Gamma_i$ is a non-zero function on $\Gamma_i$, hence it is some constant $a_i$ and if $h_i$ is an irreducible equation of $\Gamma_i$ then $h_i$ divides $x - a_i$, i.e. $h_i = \text{const}(x - a_i)$.

Thus $\Gamma_i = \{x = a_i\}$, $i = 1, \ldots, r$ (here $a_1 = 0$). Now it is easy to check that the automorphism

$$G(x, y) = \left(x, y + \prod_{i=1}^{r} (x - a_i)\right)$$

has $\text{Fix} G = S$. ■

Now we use the above theorem to determine the one-dimensional identity sets in $C^2$. Let us recall the definition:

**Definition 3.6** (see [Jel1], [Jel2]). Let $\Gamma$ be an affine curve in $C^2$. We say that $\Gamma$ is an identity set for polynomial automorphisms of $C^2$ if any two polynomial automorphisms that coincide on $\Gamma$ must be equal.
The following corollary generalizes some results from [M–W], [Jel1], [Jel2]:

**Corollary 3.7.** An affine curve $\Gamma \subset \mathbb{C}^2$ is an identity set for polynomial automorphisms of $\mathbb{C}^2$ if and only if it is not isomorphic to a union of disjoint $\mathbb{C}$-curves.

**Proof.** The condition is necessary by the last part of Theorem 3.3. It is also sufficient. Indeed, suppose $\Gamma$ is not isomorphic to a union of disjoint $\mathbb{C}$-curves and let $f, g \in \text{Aut}(\mathbb{C}^2)$ be two automorphisms that coincide on $\Gamma$. Then $\Gamma$ is a one-dimensional subset of $\text{Fix} F$ for the automorphism $F := f \circ g^{-1}$, and by Theorem 3.3, $F$ must be trivial. Hence $f = g$ and we have proved that $\Gamma$ is an identity set.

Now we are in a position to describe irreducible affine curves with infinite group $\text{Stab}_\Gamma$.

**Theorem 3.8.** Let $\Gamma \subset \mathbb{C}^2$ be an irreducible affine curve with $\text{Stab}_\Gamma = \{f \in \text{Aut}(\mathbb{C}^2) : f(\Gamma) = \Gamma\}$ infinite. Then only two cases are possible:

1) $\Gamma \cong \mathbb{C}^*$,

2) $\Gamma$ is homeomorphic to $\mathbb{C}$, i.e., in some coordinates $\Gamma = \{(x, y) \in \mathbb{C}^2 : x^k = y^l, (k, l) = 1\}$.

**Proof.** We can assume that $\Gamma$ is not isomorphic to $\mathbb{C}$. Hence $\Gamma$ is an identity set and consequently the restriction to $\Gamma$ gives the inclusion $\text{Stab}_\Gamma \subset \text{Aut}(\Gamma)$. Since $\text{Stab}_\Gamma$ is infinite, so is $\text{Aut}(\Gamma)$. By Lemma 3.1 and Proposition 2.5 the proof is finished.

We conclude this paper with the following theorem:

**Theorem 3.9.** Let $\{\Gamma_1, \ldots, \Gamma_s\}$ be a finite family of non-rational curves in $\mathbb{C}^2$. Then the automorphism group of the variety $X := \mathbb{C}^2 \setminus \bigcup_{i=1}^s \Gamma_i$ is finite.

**Proof.** Let $f \in \text{Aut}(X)$. By Corollary 54 in [Jel2] we can extend $f$ to the whole of $\mathbb{C}^2$ and consequently $\text{Aut}(X) = \text{Stab}_\Gamma$, where $\Gamma := \bigcup_{i=1}^s \Gamma_i$. Moreover, if $H := \text{Stab}_\Gamma \cap \text{Stab}_{\Gamma_1}$, then $\langle \text{Aut}(X) : H \rangle \leq s$ and it is enough to show that $H$ is finite. But since $\Gamma_1$ is non-rational, $\text{Stab}_{\Gamma_1}$ is finite according to Theorem 3.8.

**Remark 3.10.** Wakabayashi (see [Iit2], pp. 15–16) gave conditions for the complement of a finite family of straight lines in $\mathbb{C}^2$ to have a finite automorphism group.

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