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The automorphism groups of Zariski open affine subsets of the affine plane

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Abstract. We study some properties of the affine plane. First we describe the set of fixed points of a polynomial automorphism of $\mathbb{C}^2$. Next we classify completely so-called identity sets for polynomial automorphisms of $\mathbb{C}^2$. Finally, we show that a sufficiently general Zariski open affine subset of the affine plane has a finite group of automorphisms.

1. Introduction. The automorphism group of an affine (or more generally non-complete) algebraic variety X is rather difficult to study and only partial results on its structure are known. In [Iit1], [Sak] sufficient conditions for the finiteness of $\mathrm{Aut}(X)$ are given (in terms of logarithmic Kodaira dimension).

Another approach is given in [Jel2] and [Jel3], where we showed that $\mathrm{Aut}(X)$ is finite provided the divisor at infinity of some projective compactification of X is very ample and it does not have uniruled components. Moreover, in [Jel2] we started the study of the automorphism groups of affine Zariski open subvarieties of $\mathbb{C}^n$.

This note is a continuation of this study, as well as a continuation of our work on identity sets for polynomial automorphisms (see [Jel1], [Jel3], [Jel5]).

We concentrate on the first non-trivial case, that of the affine plane $\mathbb{C}^2$. Our first aim is to give a description of the fixed point set of a non-trivial polynomial automorphism of $\mathbb{C}^2$.

We show (Theorem 3.3) that this set is either finite, or a finite union of disjoint plane $\mathbb{C}$-curves. Conversely, for any finite subset S of the plane, or for any finite family S of disjoint plane $\mathbb{C}$-curves we construct an automorphism of the plane for which the set of fixed points is exactly S.

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In particular, we obtain an "if and only if" condition for an affine curve Γ to be an identity set for polynomial automorphisms of $\mathbb{C}^2$. This generalizes some results of [M-W], [Jel1], [Jel5].

The approach is based on the study of curves $\Gamma \subset \mathbb{C}^2$ with infinite group $\mathrm{Stab}_\Gamma = \{f \in \mathrm{Aut}(\mathbb{C}^2) : f(\Gamma) = \Gamma\}$. We classify such curves completely (Theorem 3.8).

Finally, we show that for any finite family $\{\Gamma_1, \ldots, \Gamma_s\}$ of non-rational curves on the plane, the variety $X := \mathbb{C}^2 \setminus \bigcup_{i=1}^s \Gamma_i$ has a finite automorphism group (Theorem 3.9).

2. Preliminaries. Let us recall some properties of fibres of a primitive polynomial p in two complex variables (see [Suz1], [Suz2], [Z-L], [Zai]).

It is well known that all but finitely many fibres of p are pairwise homeomorphic. Such fibres are called *generic*. A generic fibre is smooth and irreducible. All other fibres are called *degenerate*. A point $s \in \mathbb{C}$ for which the fibre $\Gamma_s := p^{-1}(s)$ is generic is called a *generic point*, otherwise it is *degenerate*. The set of fibres of a primitive polynomial p will be called a *family of curves*.

The following proposition was proved in [Suz2] and is crucial to our study:

PROPOSITION 2.1. *Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a primitive polynomial and let χ_p denote the Euler characteristic of a generic fibre. Let S be the set of all degenerate points. For every $s \in S$ we have $\chi(\Gamma_s) > \chi_p$. Moreover,*

$$\sum_{s \in S} \{\chi(\Gamma_s) - \chi_p\} = 1 - \chi_p.$$

We get at once the following interesting

COROLLARY 2.2. *If the generic fibre of a family p is a $\mathbb{C}$-curve (i.e., it is isomorphic to $\mathbb{C}$) then all fibres of p are generic (and isomorphic to $\mathbb{C}$). Conversely, if all fibres of a family p are generic then it is a family of $\mathbb{C}$-curves.*

If a generic fibre is a $\mathbb{C}^*$-curve then the family p has exactly one degenerate fibre Γ_s. Moreover, $\chi(\Gamma_s) = 1$.

In the sequel the topological characterization of an irreducible algebraic curve with Euler characteristic 1 will be useful. Let us begin with the following simple observation.

LEMMA 2.3. *Let X be an algebraic complex curve and let $a \in X$. Then*

$$\chi(X \setminus \{a\}) = \chi(X) - 1.$$

Proof. Consider the triple $\{X \setminus \{a\}, V, V \setminus \{a\}\}$, where V is a small neighbourhood of a. More precisely, we can assume that V is a bouquet of r

discs, where r is the number of components in the germ X_a. In this situation the point a is a retract of V, and $V \setminus \{a\}$ can be retracted to the disjoint sum $\bigcup_{i=1}^r S_i^1$ of r circles. Hence $\chi(V) = 1$ and $\chi(V \setminus \{a\}) = r\chi(S^1) = 0$. Using the Mayer–Vietoris sequence we have

$$\chi(X \setminus \{a\}) + \chi(V) = \chi(X) + \chi(V \setminus \{a\}).$$

Thus $\chi(X \setminus \{a\}) + 1 = \chi(X)$. ■

The simple but useful consequence of the above result is the following

PROPOSITION 2.4. *Let X be an irreducible affine curve of genus g. (This means that a smooth model X_1 of a compactification of X has genus g.) Suppose that X has n branches at infinity (i.e., n is the number of points in $X_1 \setminus X_0$, where X_0 is the normalization of X). Let* $\mathrm{Sing}(X) = \{a_1, \ldots, a_r\}$ *be the singular locus. Further, suppose that the germ X_{a_i} has k_i irreducible components. Then*

$$\chi(X) = 2(1-g) - n - \sum_{i=1}^{r} (k_i - 1).$$

In particular, if $\chi(X) = 1$ then X is homeomorphic to $\mathbb{C}$.

Proof. Let $\pi : X_0 \to X$ be the normalization. Then π is an isomorphism outside $A := \mathrm{Sing}(X)$, and $\pi^{-1}(A)$ has $\sum_{i=1}^r k_i$ points. Hence by the lemma

$$\chi(X) = \chi(X \setminus A) + r = \chi(X_0 \setminus \pi^{-1}(A)) + r = \chi(X_0) - \sum_{i=1}^{r} k_i + r$$
$$= \chi(X_1) - n - \sum_{i=1}^{r} (k_i - 1) = 2(1-g) - n - \sum_{i=1}^{r} (k_i - 1).$$

Since X is affine we have $n > 0$ and the case $\chi(X) = 1$ is possible only if $g = 0$ and $n = 1$ and all $k_i = 1$, i.e., if X is homeomorphic to $\mathbb{C}$. ■

Plane curves homeomorphic to the complex line have a very nice description due to Zaĭdenberg and Lin (see [Z-L]):

PROPOSITION 2.5. *Let $X \subset \mathbb{C}^2$ be an affine algebraic curve homeomorphic to the complex line. Then in suitable coordinates X can be written as*

$$X = \{(x, y) \in \mathbb{C}^2 : x^k = y^l,\ (k, l) = 1\}.$$

Remark 2.6. The proposition above is a generalization of the famous Abhyankar–Moh–Suzuki theorem (see [A-M], [Suz1]):

If $\Gamma \subset \mathbb{C}^2$ is a curve isomorphic to $\mathbb{C}$ then in some coordinates we have $\Gamma = \{(x, y) \in \mathbb{C}^2 : x = 0\}$.

There is the following useful consequence of the above considerations.

Proposition 2.7. *Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a family of curves. If there is only one degenerate fibre Γ in p and Γ is irreducible then in some coordinates*

$$\Gamma = \{(x, y) \in \mathbb{C}^2 : x^k = y^l,\ (k, l) = 1\}.$$

Proof. Indeed, by Proposition 2.1 the curve Γ has Euler characteristic equal to 1. Hence by Proposition 2.4 it is homeomorphic to $\mathbb{C}$ and finally the proof is finished by Proposition 2.5. ■

3. Main result. First we want to describe the set of fixed points of a polynomial automorphism of $\mathbb{C}^2$. To do this, the following lemmas will be useful:

Lemma 3.1. *Let X be an irreducible affine curve with infinite automorphism group. Then X is either isomorphic to $\mathbb{C}^*$ or it is homeomorphic to $\mathbb{C}$. Moreover, in the last case X can have at most one (necessarily irreducible) singularity.*

Proof. Let X_0 be a normalization of X and X_1 be a smooth completion of X_0. Since $\mathrm{Aut}(X)$ is infinite, so is $\mathrm{Aut}(X_0)$. The latter group is a subgroup of $\mathrm{Aut}(X_1)$ which stabilizes the divisor $D := X_1 \setminus X_0$. Since D is ample, $\mathrm{Aut}(X_0)$ must be linear (for details see [Jel3], 3.7). Moreover, since $\mathrm{Aut}(X_0)$ is infinite, X_1 is rational (see ibid., 3.12), i.e., $X_1 = \mathbb{P}^1(\mathbb{C})$. Further, the singular points are permuted by polynomial automorphisms, hence the common number of points at infinity of X_0 and of points of X_1 which lie over the singular locus of X must be at most two (in X_1). If there exist two such points we get either $X = X_0 = \mathbb{C}^*$, or $X \neq X_0 = \mathbb{C}$, and in the latter case X has one irreducible singular point (which means that X is homeomorphic to $\mathbb{C}$). If there is only one such point then $X = X_0 = \mathbb{C}$. ■

Lemma 3.2. *Let $\Gamma \subset \mathbb{C}^2$ be a curve with an irreducible equation $p(x, y) = 0$. Let $f \in \mathrm{Stab}_\Gamma$ be an element of infinite order. Then only two possibilities can occur:*

1) *There exists $s \in \mathbb{N}$ such that f^s stabilizes all fibres of p. Moreover, p is a $\mathbb{C}$- or $\mathbb{C}^*$-family and Γ is either homeomorphic to $\mathbb{C}$, or it is isomorphic to $\mathbb{C}^*$,*

2) *Γ is the unique degenerate fibre of the family p and it is homeomorphic to $\mathbb{C}$.*

Proof. By the Hilbert Nullstellensatz we have $p \circ f = cp$, for some $c \in \mathbb{C}^*$. There are two cases possible:

1) c has a finite order, i.e., $c^s = 1$ for some $s \geq 1$,

2) c has an infinite order.

1) We can assume $c = 1$ and then we have $p - \lambda = (p - \lambda) \circ f$ for every $\lambda \in \mathbb{C}$. This means that all fibres of p are stable under f. Since f is of infinite

order, a generic fibre has an infinite automorphism group. Since the generic fibre is smooth and irreducible it must be isomorphic either to $\mathbb{C}$ or to $\mathbb{C}^*$ (see Lemma 3.1).

In the first case, by Corollary 2.2, p has no degenerate fibres, and in particular $\Gamma \cong \mathbb{C}$.

In the second case, by the same corollary, p has exactly one degenerate fibre. If it is the fibre $p^{-1}(0) = \Gamma$ then by Proposition 2.1 we obtain $\Gamma = \{(x,y) \in \mathbb{C}^2 : x^k = y^l,\ (k,l) = 1\}$ in some coordinate system. If this fibre is not Γ then Γ is generic and isomorphic to $\mathbb{C}^*$.

2) In this case the fibre $\Gamma_\lambda := p^{-1}(\lambda)$ is transformed under f onto the fibre $p = c\lambda$. More generally, under f^r this fibre goes onto the fibre $p = c^r\lambda$ for $r \in \mathbb{Z}$. Since c is of infinite order, for $\lambda \neq 0$ the fibre Γ_λ is isomorphic to an infinite set of other fibres. This means that Γ_λ is a generic fibre for all $\lambda \neq 0$. Hence p has at most one degenerate fibre, Γ_0. If Γ_0 is degenerate, by Proposition 2.7 we have $\Gamma_0 = \{(x,y) \in \mathbb{C}^2 : x^k = y^l, (k,l) = 1\}$ in some coordinate system.

If p has no degenerate fibres at all, then Proposition 2.1 shows that p is a family of $\mathbb{C}$-curves, and in particular $\Gamma \cong \mathbb{C}$. ∎

Using the lemma above we describe the set $\operatorname{Fix} f$ of fixed points of a polynomial automorphism f of the affine plane. We have:

THEOREM 3.3. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a non-trivial polynomial automorphism. Then the set $S = \operatorname{Fix} f$ is either finite, or a finite union of disjoint $\mathbb{C}$-curves.*

Conversely, if S is a finite subset of the plane, or a finite union of disjoint plane $\mathbb{C}$-curves, then there is $f \in \operatorname{Aut}(\mathbb{C}^2)$ such that $S = \operatorname{Fix} f$.

Proof. Let Γ be a one-dimensional component of S. We show that in this case $\Gamma \cong \mathbb{C}$.

Since any automorphism of the affine plane of a finite order is conjugate to a linear one (see e.g. [Kam]), in which case there is nothing to prove, we can assume that f is of infinite order. By Lemma 3.2 and Proposition 2.5 there are only two cases possible:

1) in some coordinates $\Gamma = \{(x,y) \in \mathbb{C}^2 : x^k = y^l,\ (k,l) = 1\}$,

2) Γ is isomorphic to $\mathbb{C}^*$.

Now we will show that the first case is possible only if $k = 1$ or $l = 1$, and the second case is excluded. Indeed, the following lemma is true:

LEMMA 3.4. *Let f be a polynomial automorphism of the plane which stabilizes the curve $R = \{(x,y) : x^k = y^l,\ (k,l) = 1\}$. Then each one-dimensional component Γ of $\operatorname{Fix} f$ is isomorphic to $\mathbb{C}$.*

Proof. Let $f = (f_1, f_2)$. Since R is f-stable we have $f_1^k - f_2^l = c(x^k - y^l)$

for some non-zero constant c. Hence if $a := \deg f_1$ and $b := \deg f_2$ are both greater than 1 we have $\deg f_1^k = \deg f_2^l$. But it is well known (see [Kul]) that if f is a polynomial automorphism then $\deg f_1$ divides $\deg f_2$ or vice versa, thus we can assume that $b = \lambda a$ for some non-zero integer λ. Hence $ka = l\lambda a$ and $k = l\lambda$. If $k \neq 1$ and $l \neq 1$ this is a contradiction.

Thus either "$a = 1$ or $b = 1$", or "$k = 1$ or $l = 1$".

Assume that $a = 1$ or $b = 1$. This means that one of the polynomials f_1, f_2, say f_1, is linear. We have $\Gamma \subset \{f_1 - x = 0\}$. Hence if $f_1 \neq x$ then Γ is a line. In the other case f is a triangular automorphism, $f(x,y) = (x, cy + p(x))$, and then $\operatorname{Fix} f$ is known to be a finite union of $\mathbb{C}$-curves.

If $k = 1$ or $l = 1$ then R is isomorphic to $\mathbb{C}$ and by the Abhyankar–Moh–Suzuki theorem we can assume that $R = \{(x,y) : x = 0\}$. But in this case $f(x,y) = (cx, by + p(x))$ and it is easy to see that a one-dimensional component of $\operatorname{Fix} f$ is either a straight line or it has an equation $(b-1)y + p(x) = 0$ (if $b \neq 1$). In both cases Γ is isomorphic to $\mathbb{C}$. ■

We proceed now with the proof of Theorem 3.3.

1) It follows immediately from the lemma that $\Gamma \cong \mathbb{C}$, i.e., $k = 1$ or $l = 1$.

2) We will show that the case $\Gamma \cong \mathbb{C}^*$ is impossible.

Let p be an irreducible equation of Γ. Since Γ is not homeomorphic to $\mathbb{C}$, Lemma 3.2 implies that f^s stabilizes all fibres of p for some $s \in \mathbb{N}$. We can assume that $s = 1$. This means that all fibres of p are stable under f and the generic fibre is $\mathbb{C}$ or $\mathbb{C}^*$. By Corollary 2.2 the first case is impossible, hence the generic fibre must be $\mathbb{C}^*$. Hence p has only one degenerate fibre. By Proposition 2.7 it cannot be the fibre over 0. Let w be the unique degenerate point and let R be some irreducible component of Γ_w. Since Γ_w has only a finite number of irreducible components the curve R is stable under some iteration $F := f^r$ of f. We have again two cases to consider:

$(*)$ $R = \Gamma_w$,

$(**)$ R is a proper component of Γ_w.

$(*)$ In this case R is homeomorphic to $\mathbb{C}$ and stable under F and by Lemma 3.4 we get $\Gamma \cong \mathbb{C}$, which is a contradiction.

$(**)$ Let $R = \{q = 0\}$ for some irreducible polynomial q. Since $R \neq \{p = 0\}$ we see that $\deg q < \deg p$, which shows that the families p and q are different. By Lemma 3.2 only two possibilites can occur: either all fibres of q are stable under some iteration of F, or R is homeomorphic to $\mathbb{C}$. The second possibility cannot occur, by Lemma 3.4.

Hence we can assume that F stabilizes all fibres of q. But F is an iteration of f, hence it also stabilizes all fibres of p. Since a generic fibre of q intersects a generic fibre of p in at most $N = (\deg q)(\deg p)$ points, the order of F is at most $N! = 1 \cdot \ldots \cdot N$. This is a contradiction again.

Hence case 2) is excluded and we have proved that if a curve Γ is a component of $\operatorname{Fix} f$ for a polynomial automorphism f then Γ is isomorphic to $\mathbb{C}$. Further in this case we can assume by the Abhyankar–Moh–Suzuki theorem (see Remark 2.6) that $\Gamma = \{(x,y) \in \mathbb{C}^2 : x = 0\}$. Since f is the identity on Γ we have $f = (cx, y + p(x))$ with $p(0) = 0$. This means that $\operatorname{Fix} f$ consists of one or more (disjoint) straight lines, in particular, it is of pure dimension. Hence $\operatorname{Fix} f$ is either finite, or a union of disjoint $\mathbb{C}$-curves.

Now we prove the converse: if S is a finite subset of the plane, or a finite union of disjoint plane $\mathbb{C}$-curves, then there is $f \in \operatorname{Aut}(\mathbb{C}^2)$ such that $S = \{x \in \mathbb{C}^2 : f(x) = x\}$. Of course we can assume that S is non-empty.

First assume that S is finite. The following is proved in [Jel4]:

Lemma 3.5. *Let $n \geq 2$ and $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots, b_r\} \subset \mathbb{C}^n$, where $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. Then there is a polynomial automorphism F of $\mathbb{C}^n$ such that $F(a_i) = b_i$, $i = 1, \ldots, r$.*

Let $S = \{a_1, \ldots, a_r\}$ and suppose $F \in \operatorname{Aut}(\mathbb{C}^2)$ has the property that $F(a_i) = (i, 0)$, $i = 1, \ldots, r$. Let $B = \{(1,0), (2,0), \ldots, (r,0)\}$. If we construct an automorphism G with $B = \operatorname{Fix} G$ then $f = F^{-1} \circ G \circ F$ has $\operatorname{Fix} f = S$. It is easy to check that we can take for G the automorphism

$$G(x,y) = \Big(x + y + \prod_{i=1}^{r}(x-i),\, y + \prod_{i=1}^{r}(x-i)\Big).$$

Now let S be a finite union of disjoint plane curves, i.e., $S = \bigcup_{i=1}^{r} \Gamma_i$, where $\Gamma_i \cong \mathbb{C}$ and $\Gamma_i \cap \Gamma_j = \emptyset$. By the Abhyankar–Moh–Suzuki theorem (see Remark 2.6) we can reduce the problem (as above) to the case when $\Gamma_1 = \{(x,y) : x = 0\}$. Then necessarily $\Gamma_i = \{(x,y) : x = a_i\}$ for some non-zero distinct complex numbers a_i, $i = 2, \ldots, r$. Indeed, the polynomial x restricted to Γ_i is a non-zero function on Γ_i, hence it is some constant a_i and if h_i is an irreducible equation of Γ_i then h_i divides $x - a_i$, i.e. $h_i = \operatorname{const}(x - a_i)$.

Thus $\Gamma_i = \{x = a_i\}, i = 1, \ldots, r$ (here $a_1 = 0$). Now it is easy to check that the automorphism

$$G(x,y) = \Big(x,\, y + \prod_{i=1}^{r}(x - a_i)\Big)$$

has $\operatorname{Fix} G = S$. ∎

Now we use the above theorem to determine the one-dimensional identity sets in $\mathbb{C}^2$. Let us recall the definition:

Definition 3.6 (see [Jel1], [Jel2]). Let Γ be an affine curve in $\mathbb{C}^2$. We say that Γ is an *identity set for polynomial automorphisms* of $\mathbb{C}^2$ if any two polynomial automorphisms that coincide on Γ must be equal.

The following corollary generalizes some results from [M–W], [Jel1], [Jel2]:

COROLLARY 3.7. *An affine curve $\Gamma \subset \mathbb{C}^2$ is an identity set for polynomial automorphisms of $\mathbb{C}^2$ if and only if it is not isomorphic to a union of disjoint $\mathbb{C}$-curves.*

Proof. The condition is necessary by the last part of Theorem 3.3.

It is also sufficient. Indeed, suppose Γ is not isomorphic to a union of disjoint $\mathbb{C}$-curves and let $f, g \in \mathrm{Aut}(\mathbb{C}^2)$ be two automorphisms that coincide on Γ. Then Γ is a one-dimensional subset of $\mathrm{Fix}\, F$ for the automorphism $F := f \circ g^{-1}$, and by Theorem 3.3, F must be trivial. Hence $f = g$ and we have proved that Γ is an identity set. ∎

Now we are in a position to describe irreducible affine curves with infinite group Stab_Γ.

THEOREM 3.8. *Let $\Gamma \subset \mathbb{C}^2$ be an irreducible affine curve with $\mathrm{Stab}_\Gamma = \{f \in \mathrm{Aut}(\mathbb{C}^2) : f(\Gamma) = \Gamma\}$ infinite. Then only two cases are possible*:

1) $\Gamma \cong \mathbb{C}^*$,

2) *Γ is homeomorphic to $\mathbb{C}$, i.e., in some coordinates $\Gamma = \{(x, y) \in \mathbb{C}^2 : x^k = y^l,\ (k, l) = 1\}$.*

Proof. We can assume that Γ is not isomorphic to $\mathbb{C}$. Hence Γ is an identity set and consequently the restriction to Γ gives the inclusion $\mathrm{Stab}_\Gamma \subset \mathrm{Aut}(\Gamma)$. Since Stab_Γ is infinite, so is $\mathrm{Aut}(\Gamma)$. By Lemma 3.1 and Proposition 2.5 the proof is finished. ∎

We conclude this paper with the following theorem:

THEOREM 3.9. *Let $\{\Gamma_1, \ldots, \Gamma_s\}$ be a finite family of non-rational curves in $\mathbb{C}^2$. Then the automorphism group of the variety $X := \mathbb{C}^2 \setminus \bigcup_{i=1}^s \Gamma_i$ is finite.*

Proof. Let $f \in \mathrm{Aut}(X)$. By Corollary 54 in [Jel2] we can extend f to the whole of $\mathbb{C}^2$ and consequently $\mathrm{Aut}(X) = \mathrm{Stab}_\Gamma$, where $\Gamma := \bigcup_{i=1}^s \Gamma_i$. Moreover, if $H := \mathrm{Stab}_\Gamma \cap \mathrm{Stab}_{\Gamma_1}$ then $(\mathrm{Aut}(X) : H) \le s$ and it is enough to show that H is finite. But since Γ_1 is non-rational, Stab_{Γ_1} is finite according to Theorem 3.8. ∎

Remark 3.10. Wakabayashi (see [Iit2], pp. 15–16) gave conditions for the complement of a finite family of straight lines in $\mathbb{C}^2$ to have a finite automorphism group.

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