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Some inequalities involving multivalent functions

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Abstract. The object of the present paper is to derive some inequalities involving multivalent functions in the unit disk. One of our results is an improvement and a generalization of a result due to R. M. Robinson [4].

1. Introduction. Let $\mathbb{A}(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z : |z| < 1\}$.

In 1947, Robinson [4] proved the following

Theorem A. *Let $S(z)$ and $T(z)$ be analytic in $\mathbb{U}$, and let $\mathrm{Re}\{zS'(z)/S(z)\} > 0$ $(z \in \mathbb{U})$. If $|T'(z)/S'(z)| < 1$ $(z \in \mathbb{U})$ and $T(0) = 0$, then $|T(z)/S(z)| < 1$ $(z \in \mathbb{U})$.*

In the present paper, we derive an improvement and generalization of Theorem A for functions belonging to $\mathbb{A}(p)$.

To establish our results, we have to recall the following lemmas.

Lemma 1 ([1], [2]). *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value in the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then we can write*

$$z_0 w'(z_0) = k w(z_0), \tag{1.2}$$

where k is real and $k \geq 1$.

Lemma 2 ([3]). *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$. If there exists a*

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point $z_0 \in \mathbb{U}$ *such that*

$$\operatorname{Re}(p(z)) > 0 \quad (|z| < |z_0|), \quad \operatorname{Re}(p(z_0)) = 0, \quad \text{and} \quad p(z_0) \neq 0,$$

then $p(z_0) = ia$ $(a \neq 0)$ *and*

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\frac{k}{2}\left(a + \frac{1}{a}\right), \tag{1.3}$$

where k *is real and* $k \geq 1$.

2. Some counterparts of Theorem A. Our first result for functions in the class $\mathbb{A}(p)$ is contained in

THEOREM 1. *Let* $S(z) \in \mathbb{A}(m)$, $T(z) \in \mathbb{A}(n)$ *with* $p = n - m \geq 1$. *Let* $S(z)$ *satisfy* $\operatorname{Re}\{S(z)/zS'(z)\} > \alpha$ $(0 \leq \alpha < 1/m)$. *If*

$$\left|\frac{T'(z)}{S'(z)}\right| < (1 + p\alpha)|z|^{p-1} \qquad (z \in \mathbb{U}), \tag{2.1}$$

then

$$\left|\frac{T(z)}{S(z)}\right| < |z|^{p-1} \qquad (z \in \mathbb{U}). \tag{2.2}$$

Proof. Since $T(z)/S(z) = z^p + \ldots \in \mathbb{A}(p)$, we define the function $w(z)$ by $T(z) = z^{p-1}w(z)S(z)$. Then $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$. It follows from the definition of $w(z)$ that

$$\frac{T'(z)}{S'(z)} = z^{p-1}w(z)\left\{1 + \left(p - 1 + \frac{zw'(z)}{w(z)}\right)\frac{S(z)}{zS'(z)}\right\}. \tag{2.3}$$

If we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives $w(z_0) = e^{i\theta}$ and

$$z_0 w'(z_0) = kw(z_0) \qquad (k \geq 1).$$

Therefore,

$$\begin{aligned} \left|\frac{T'(z_0)}{z_0^{p-1}S'(z_0)}\right| &= \left|1 + \left(p - 1 + \frac{z_0 w'(z_0)}{w(z_0)}\right)\frac{S(z_0)}{z_0 S'(z_0)}\right| \\ &\geq 1 + (p - 1 + k)\operatorname{Re}\left(\frac{S(z_0)}{z_0 S'(z_0)}\right) > 1 + p\alpha. \end{aligned} \tag{2.4}$$

This contradicts our condition (2.1), so that $|w(z)| < 1$ for all $z \in \mathbb{U}$. This completes the proof of Theorem 1.

Remark. If we take $p = 1$ and $\alpha = 0$ in Theorem 1, then we recover Theorem A due to Robinson [4].

Next, applying Lemma 2, we prove

THEOREM 2. *Let* $S(z)\in\mathbb{A}(m)$, $T(z)\in\mathbb{A}(n)$ *with* $p=n-m\ge 1$. *Let* $S(z)$ *satisfy* $\mathrm{Re}\{S(z)/zS'(z)\}>\alpha$ $(0\le\alpha<1/m)$ *and* $-\alpha/p\le\mathrm{Im}\{S(z)/(zS'(z))\}\le\alpha/p$ $(0\le\alpha<1/m)$. *If*

$$\mathrm{Re}\left(\frac{T'(z)}{z^pS'(z)}\right)>0 \quad (z\in\mathbb{U}), \tag{2.5}$$

then

$$\mathrm{Re}\left(\frac{T(z)}{z^pS(z)}\right)>0 \quad (z\in\mathbb{U}). \tag{2.6}$$

Proof. Defining the function $q(z)$ by $T(z)=z^pq(z)S(z)$, we see that $q(z)$ is analytic in $\mathbb{U}$ with $q(0)=1$. Note that

$$\frac{T'(z)}{S'(z)}=z^pq(z)\left\{1+\left(p+\frac{zq'(z)}{q(z)}\right)\frac{S(z)}{zS'(z)}\right\}. \tag{2.7}$$

Suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$\mathrm{Re}(q(z))>0 \quad (|z|<|z_0|), \quad \mathrm{Re}(q(z_0))=0, \quad \text{and} \quad q(z_0)\ne 0.$$

Then, applying Lemma 2, we have $q(z_0)=ia$ $(a\ne 0)$ and

$$\frac{z_0q'(z_0)}{q(z_0)}=i\frac{k}{2}\left(a+\frac{1}{a}\right) \quad (k\ge 1).$$

Therefore, writing $S(z_0)/(z_0S'(z_0))=\alpha_0+i\beta_0$, we obtain

$$\begin{aligned}\mathrm{Re}\left(\frac{T'(z_0)}{z_0^pS'(z_0)}\right)&=-ap\beta_0-\frac{ak\alpha_0}{2}\left(a+\frac{1}{a}\right)\\&=-ap\beta_0-\frac{k\alpha_0}{2}(1+a^2)\\&\le -ap\beta_0-\frac{\alpha_0}{2}(1+a^2)\le -ap\beta_0-\frac{\alpha}{2}(1+a^2).\end{aligned} \tag{2.8}$$

Since $-\alpha/p\le\beta_0\le\alpha/p$, if $a>0$, then

$$\begin{aligned}-ap\beta_0-\frac{\alpha}{2}(1+a^2)&\le a\alpha-\frac{\alpha}{2}(1+a^2)\\&=-\frac{\alpha}{2}(1-a)^2\le 0,\end{aligned} \tag{2.9}$$

and if $a<0$, then

$$\begin{aligned}-ap\beta_0-\frac{\alpha}{2}(1+a^2)&\le -a\alpha-\frac{\alpha}{2}(1+a^2)\\&=-\frac{\alpha}{2}(1+a)^2\le 0.\end{aligned} \tag{2.10}$$

This contradicts our condition (2.5). Consequently, $\mathrm{Re}(q(z))>0$ for all $z\in\mathbb{U}$, so that $\mathrm{Re}\{T(z)/(z^pS(z))\}>0$ $(z_0\in\mathbb{U})$.

Further, using the same technique as in the proof of Theorem 2, we obtain

THEOREM 3. *Let* $S(z) \in \mathbb{A}(m)$, $T(z) \in \mathbb{A}(n)$ *with* $p = m - n \geq 0$. *Let* $S(z)$ *satisfy* $\mathrm{Re}\{S(z)/(zS'(z))\} > \alpha$ $(0 \leq \alpha < 1/m)$ *and* $-\alpha/p \leq \mathrm{Im}\{S(z)/(zS'(z))\} \leq \alpha/p$ $(0 \leq \alpha < 1/m)$. *If*

$$\mathrm{Re}\left(\frac{z^p T'(z)}{S'(z)}\right) > 0 \quad (z \in \mathbb{U}), \tag{2.11}$$

then

$$\mathrm{Re}\left(\frac{z^p T(z)}{S(z)}\right) > 0 \quad (z \in \mathbb{U}), \tag{2.12}$$

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