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A 4_3 configuration of lines and conics in $\mathbb{P}^5$

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Abstract. Studying the connection between the title configuration and Kummer surfaces we write explicit quadratic equations for the latter. The main results are presented in Theorems 8 and 16.

Introduction. The aim of this work is to find canonical equations for a 4_3 configuration of lines and conics in projective space $\mathbb{P}^5$ connected in a natural manner with embeddings of certain Kummer surfaces in $\mathbb{P}^5$. These surfaces are projections of the abelian surfaces which were described first by Adler and van Moerbeke [A-M].

In the clasical case the (singular) Kummer surface is embedded in $\mathbb{P}^3$ by means of the second tensor power of an ample line bundle of type (1,1). There is the famous 16_6 configuration discovered by Kummer and exhibited in all details in the beautiful book of Hudson [Hu]. One of the keys to the study of the classical situation is its symmetry group, which is the Heisenberg group $H_{2,2}$, more precisely its quotient by the involution.

Here we use a linear subsystem of the fourth tensor power, invariant under a group of order four. This group is the representation of an order 4 subgroup of the Heisenberg group $H_{4,4}$ and also plays a very important role here. The image surface has only 12 nodes and we call it an intermediate Kummer surface. Our objective is to get equations defining such surfaces. This might prompt a computer-assisted investigation of the latter. Finally, we are also motivated by Mumford [M], since there is a close relation between abelian and Kummer surfaces.

Hyperelliptic curves. We begin with the Jacobian variety of an algebraic curve Θ of genus 2. This curve is hyperelliptic and $J(\Theta)$ is an abelian

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surface. We recall some important facts from the theory of algebraic curves.

Proposition 1. *Algebraic smooth curves of genus* 2 *are parametrized by triplets of points in* $\mathbb{P}^1$ *up to permutation.*

We only sketch the proof, which can be found in [Ha]. Since each genus 2 curve Θ is hyperelliptic it can be realized as the Riemann surface of the square root of some polynomial. This polynomial has degree 6, which follows directly from the Riemann–Hurwitz formula. In other words, there is a 2 : 1 covering $\Theta \to \mathbb{P}^1$ ramified over six points. They are all distinct if the curve is smooth. Applying automorphisms of the Riemann sphere $\mathbb{P}^1$ to these six points we can arrange them so that three of them are 0, 1, ∞. This normal form is also called the Rosenhain form and was extensively studied by Igusa in [I]. The other three points are fixed up to permutation. ∎

Let now $A = J(\Theta)$ be the Jacobian surface of a smooth curve Θ of genus 2. This curve can be identified on A via the Abel–Jacobi map as the theta divisor and defines the principal polarization of A. For construction and proof see [F] and [L-B]. There are exactly six halfperiods on Θ. Having in mind the above proposition it seems natural to distinguish three of them, say e_0, e_1, e_2. We fix e_0 as the neutral element for the group operation of A on itself. Defining $e_3 := e_1 + e_2$ we get the order 4 subgroup $e_0, \ldots, e_3$ of the order 16 group of halfperiods on A. We notice that $e_3 \notin \Theta$.

Let T_x denote the translation on A by x. Set

$$\Theta_3 = \Theta, \quad \Theta_2 = T_{e_1}(\Theta_3), \quad \Theta_1 = T_{e_2}(\Theta_3), \quad \Theta_0 = T_{e_3}(\Theta_3).$$

Thus $e_i \notin \Theta_i$ for $i = 0, \ldots, 3$. In what follows we write Θ if it is not important which of the above translates we mean.

The line bundles $\mathcal{O}_A(\Theta_0), \ldots, \mathcal{O}_A(\Theta_3)$ are not isomorphic. But already their second tensor powers are. We study this situation more closely in the next sections.

The group action on sections of a line bundle. We now recall the basic ideas of Mumford [M] and apply them in our situation of surfaces. We stick as far as possible to the notation of Mumford. We do not copy any proofs. They can all be found in [M].

As usual, let A be an abelian variety and $\widehat{A}$ its dual. For any ample invertible sheaf L on A we denote by $\Lambda(L)$ the associated homomorphism

$$\Lambda(L) : A \ni x \mapsto T_x^*(L) \otimes L^{-1} \in \widehat{A}.$$

Let $H(L)$ be the kernel of $\Lambda(L)$. It is a subgroup of A consisting of all points x such that $T_x^*(L) \simeq L$. We concentrate our study on the group $\mathcal{G}(L)$ which is defined as follows:

DEFINITION 2. $\mathcal{G}(L)$ is the set of pairs (x,φ), where $x \in A$ and φ is an isomorphism

$$\varphi : L \mapsto T_x^*(L).$$

The group structure is given by

$$(y,\psi) \bullet (x,\varphi) = (x+y, T_x^*(\psi) \circ \varphi).$$

The connection between $\mathcal{G}(L)$ and $H(L)$ is given by the exact sequence

$$0 \to \mathbb{C}^* \to \mathcal{G}(L) \to H(L) \to 0.$$

$\mathcal{G}(L)$ is in fact a central extension of $H(L)$. This extension defines the following skew-symmetric bilinear form on $H(L)$. Let $x, y \in H(L)$ and let $\widetilde{x}, \widetilde{y} \in \mathcal{G}(L)$ lie over x, y. Then set

$$e^L(x,y) = [\widetilde{x}, \widetilde{y}] = \widetilde{x} \bullet \widetilde{y} \bullet \widetilde{x}^{-1} \bullet \widetilde{y}^{-1}.$$

Notice that for any $x \in H(L)$ it is possible to choose $\widetilde{x} \in \mathcal{G}(L)$ of the same order.

Now let L be a symmetric invertible sheaf, i.e. $L \simeq \iota^* L$, where $\iota : A \ni x \mapsto -x \in A$ is the usual involution on the abelian surface. Let $\alpha : L \to \iota^* L$ be the normalized isomorphism, i.e. α is the identity on the fibre L_{e_0} of L over e_0. Furthermore, let A_2 be the set of halfperiods on A. For any $x \in A_2$ we define $e_*^L(x)$ to be the integer k such that α is multiplication by k in the fibre L_x. Since $\alpha_x = \iota^* \alpha_x$ and $\iota^* \alpha \circ \alpha$ is the identity, it is clear that k can only be ± 1. Since α is normalized, $e_*^L(0) = 1$.

Unfortunately, there is no connection between the bilinear form e^L and the quadratic form e_*^L. However, we have the following

PROPOSITION 3 (Mumford). *Let L be a symmetric line bundle on the abelian surface A. Then $A_2 \subset H(L^2)$ and for every $x, y \in A_2$,*

$$e^{L^2}(x,y) = e_*^L(x+y) \cdot e_*^L(x) \cdot e_*^L(y). \tag{1}$$

For our calculations the following fact is of importance:

PROPOSITION 4 (Mumford). *Let D be a symmetric divisor on A and $L = \mathcal{O}_A(D)$. Then for any $x \in A_2$,*

$$e_*^L(x) = (-1)^{m(x)-m(e_0)}, \tag{2}$$

where $m(y)$ denotes the multiplicity of D in y.

The group $\mathcal{G}(L)$ operates on $H^0(L)$ in the following way.

DEFINITION 5. For $\widetilde{x} = (x,\varphi) \in \mathcal{G}(L)$ and $s \in H^0(L)$ set $U_{\tilde{x}}(s) = T_{-x}^*(\varphi(s))$.

This defines indeed a group action as the following calculation shows. For $\widetilde{y}=(y,\psi)\in\mathcal{G}(L)$ we have

$$\begin{aligned}U_{\tilde y}(U_{\tilde x}(s)) &= T^*_{-y}(\psi(T^*_{-x}(\varphi(s)))) = (T^*_{-y}\circ T^*_{-x})(T^*_x(\psi(T^*_{-x}(\varphi(s)))))\\ &= T^*_{-(x+y)}(T^*_x(\psi)(\varphi(s))) = U_{(x+y,T^*_x(\psi)\circ\varphi)}(s) = U_{\tilde y\bullet\tilde x}(s).\end{aligned}$$

In what follows we make no difference between $U_{\tilde x}$ and $\widetilde{x}$.

The linear system $H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$. For a symmetric line bundle L on an abelian surface A we denote by $H^0(L)^{\mathrm{ev}}$ the eigenspace of 1 of the mapping $H^0(L)\ni s\mapsto \iota_L s\iota\in H^0(L)$, where ι_L is the lifting of ι to an involution on the total space of L. The elements of $H^0(L)^{\mathrm{ev}}$ are called the *even sections* in the line bundle L.

We have seen that the translations T_{e_1} and T_{e_2} lift to involutions $\widetilde{e}_1,\widetilde{e}_2$ of the line bundle $\mathcal{O}_A(2\Theta)$ and canonically to involutions σ and τ of $\mathcal{O}_A(4\Theta)$. Let G denote the group generated by σ and τ, and let $\mathcal{I}$ be the ideal sheaf of the 0-dimensional variety consisting of $e_0,\dots,e_3$.

In the next proposition we describe explicitly a basis of the 6-dimensional vector space $H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$. Let t_3 be a generator of $H^0(\mathcal{O}_A(\Theta_3))$ and t_0,t_1,t_2 its translates vanishing on $\Theta_0,\Theta_1,\Theta_2$ respectively. Thus t_i is a section in the bundle $\mathcal{O}_A(\Theta_i)$ and we have the following

PROPOSITION 6. *The sections $u_1=u_{23}=t_3^2t_2^2$, $u_2=u_{01}=t_1^2t_0^2$, $u_3=u_{03}=t_3^2t_0^2$, $u_4=u_{12}=t_2^2t_1^2$, $u_5=u_{13}=t_3^2t_1^2$, $u_6=u_{02}=t_2^2t_0^2$ form a basis of $H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$ on which σ and τ operate as*

$$\sigma=\begin{bmatrix}1&0&&&&\\0&1&&&&\\&&0&1&&\\&&1&0&&\\&&&&0&1\\&&&&1&0\end{bmatrix}\quad\textit{and}\quad\tau=\begin{bmatrix}0&-1&&&&\\-1&0&&&&\\&&0&-1&&\\&&-1&0&&\\&&&&1&0\\&&&&0&1\end{bmatrix}.$$

Proof. We begin by computing the operation of σ and τ on the sections u_{ij}. It is enough to compute how $\widetilde{e}_1,\widetilde{e}_2$ operate on $t_0^2,\dots,t_3^2$, since σ and τ are their squares. Let $L=\mathcal{O}_A(\Theta)$; then $\widetilde{e}_1,\widetilde{e}_2$ are in fact elements of $\mathcal{G}(L^2)$. Using formulas (1) and (2) we compute their commutator

$$\begin{aligned}[\widetilde{e}_1,\widetilde{e}_2] &= e^{L^2}(e_1,e_2) = e^L_*(e_1+e_2)\cdot e^L_*(e_1)\cdot e^L_*(e_2)\\ &= (-1)^{m(e_3)-m(e_0)}\cdot(-1)^{m(e_1)-m(e_0)}\cdot(-1)^{m(e_2)-m(e_0)} = -1,\end{aligned}$$

since $m(e_0)=m(e_1)=m(e_2)=1$ and $m(e_3)=0$. This means that $\widetilde{e}_1\bullet\widetilde{e}_2=-\widetilde{e}_2\bullet\widetilde{e}_1$. If we now specify $\widetilde{e}_1=(e_1,\varphi_1)$, $\widetilde{e}_2=(e_2,\varphi_2)$, where $\varphi_i:L^2\to T^*_{e_i}L^2$ are fixed isomorphisms, then

$$\widetilde{e}_1\bullet\widetilde{e}_2=(e_1,\varphi_1)\bullet(e_2,\varphi_2)=(e_3,T^*_{e_2}(\varphi_1)\circ\varphi_2)$$

and $\widetilde{e}_2 \bullet \widetilde{e}_1 = (e_3, T^*_{e_1}(\varphi_2) \circ \varphi_1)$, hence

$$T^*_{e_2}(\varphi_1) \circ \varphi_2 = -T^*_{e_1}(\varphi_2) \circ \varphi_1.$$

For the sections $t_0, \dots, t_3$ we have

$$\begin{aligned} t_2^2 &= \varphi_1^{-1} T^*_{e_1} t_3^2 = T^*_{e_1} \varphi_1(t_3^2), \\ t_1^2 &= \varphi_2^{-1} T^*_{e_2} t_3^2 = T^*_{e_2} \varphi_2(t_3^2), \\ t_0^2 &= \varphi_3^{-1} T^*_{e_3} t_3^2 = T^*_{e_3} \varphi_3(t_3^2), \end{aligned}$$

where φ_3 is chosen to be $T^*_{e_2}(\varphi_1) \circ \varphi_2$ (this means $\widetilde{e}_3 = \widetilde{e}_1 \bullet \widetilde{e}_2$).

As already mentioned we compute the action of $\widetilde{e}_1$ and $\widetilde{e}_2$. It seems quite instructive to make some calculations explicit. For example, for $\widetilde{e}_2$ we find

$$\begin{aligned} \widetilde{e}_2(t_3^2) &= T^*_{e_2} \varphi_2(t_3^2) = t_1^2, \\ \widetilde{e}_2(t_2^2) &= T^*_{e_2} \varphi_2(T^*_{e_1} \varphi_1(t_3^2)) = T^*_{e_2} T^*_{e_1} (T^*_{e_1} \varphi_2 T^*_{e_1}) \varphi_1(t_3^2) \\ &= T^*_{e_3} T^*_{e_1}(\varphi_2) \circ \varphi_1(t_3^2) = -T^*_{e_3} T^*_{e_2}(\varphi_1) \varphi_2(t_3^2) = -T^*_{e_3} \varphi_3(t_3^2) = -t_0^2. \end{aligned}$$

Since $\widetilde{e}_2$ is an involution it is clear that $\widetilde{e}_2(t_0^2) = -t_2^2$ and $\widetilde{e}_2(t_1^2) = t_3^2$. Computing in the same way the action of $\widetilde{e}_1$ we get

	t_0^2	t_1^2	t_2^2	t_3^2
$\widetilde{e}_1$	t_1^2	t_0^2	t_3^2	t_2^2
$\widetilde{e}_2$	$-t_2^2$	t_3^2	$-t_0^2$	t_1^2

Since $\sigma(u_{ij}) = \widetilde{e}_1(t_i^2) \otimes \widetilde{e}_1(t_j^2)$ and $\tau(u_{ij}) = \widetilde{e}_2(t_i^2) \otimes \widetilde{e}_2(t_j^2)$ we have

	u_{01}	u_{02}	u_{03}	u_{12}	u_{13}	u_{23}
σ	u_{01}	u_{13}	u_{12}	u_{03}	u_{02}	u_{23}
τ	$-u_{23}$	u_{02}	$-u_{12}$	$-u_{03}$	u_{13}	$-u_{01}$

This table gives the matrices in the assertion.

Now, notice that $h^0(\mathcal{O}_A(4\Theta)) = \frac{1}{2}(4\Theta)^2 = 16$. It is clear that $\mathcal{I}$ imposes 4 independent conditions on the sections in $H^0(\mathcal{O}_A(4\Theta))$. Hence we have $h^0(\mathcal{I}.\mathcal{O}_A(4\Theta)) = 12$. Since Θ is symmetric on A we obtain

$$H^0(\mathcal{O}_A(4\Theta)) = H^0(\mathcal{O}_A(4\Theta))^{\mathrm{ev}} \oplus H^0(\mathcal{O}_A(4\Theta))^{\mathrm{odd}}.$$

By [B-L, formula 4.7.5] we get

$$h^0(\mathcal{O}_A(4\Theta))^{\mathrm{ev}} = 10 \quad \text{and} \quad h^0(\mathcal{O}_A(4\Theta))^{\mathrm{odd}} = 6.$$

But $H^0(\mathcal{O}_A(4\Theta))^{\mathrm{odd}} \subset H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))$ since odd sections vanish at all half-periods to order at least one. Thus what is left is $H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$ and its dimension is 6.

Of course the u_{ij}'s are sections in $H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$ and we only have to prove that they are linearly independent. Suppose that

$$\sum \lambda_{ij} u_{ij} \equiv 0.$$

This equation must be invariant under G. Applying σ to the left side we get

$$\lambda_{23}u_{23} + \lambda_{01}u_{01} + \lambda_{03}u_{12} + \lambda_{12}u_{03} + \lambda_{13}u_{02} + \lambda_{02}u_{13} \equiv 0.$$

Subtracting both equations we get

$$(\lambda_{03} - \lambda_{12})u_{03} + (\lambda_{12} - \lambda_{03})u_{12} + (\lambda_{13} - \lambda_{02})u_{13} + (\lambda_{02} - \lambda_{13})u_{02} \equiv 0.$$

Let us now restrict this equation to Θ_0:

$$(\lambda_{12} - \lambda_{03})u_{12} + (\lambda_{13} - \lambda_{02})u_{13} = 0 \quad \text{on } \Theta_0.$$

Rewriting the above equation with t_i we have

$$t_1^2((\lambda_{12} - \lambda_{03})t_2^2 + (\lambda_{13} - \lambda_{02})t_3^2) = 0 \quad \text{on } \Theta_0.$$

Since the first factor has only two zeros on Θ_0, the second one must be zero on the whole divisor. Evaluating it at e_2 and e_3 we get

$$\lambda_{12} - \lambda_{03} = 0, \qquad \lambda_{13} - \lambda_{02} = 0.$$

Repeating the above procedure with τ and $\sigma \circ \tau$ we get

$$\begin{aligned} \lambda_{23} + \lambda_{01} &= 0, & \lambda_{12} + \lambda_{03} &= 0, \\ \lambda_{23} - \lambda_{01} &= 0, & \lambda_{13} + \lambda_{02} &= 0. \end{aligned}$$

Hence all $\lambda_{ij} = 0$. This ends the proof. ■

Notice that the linear systems $H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$ and $H^0(\mathcal{I}^2.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$ are the same since each even section vanishes to even order at all halfperiods. Let $Bl : \widetilde{A} \to A$ be the blow-up at $e_0, \ldots, e_3$. Then the above linear system can be pulled back to $\widetilde{A}$, more precisely:

Proposition 7.

$$(Bl)^*(H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}) = H^0(\mathcal{O}_{\widetilde{A}}(4(Bl)^*\Theta - 2(E_0 + E_1 + E_2 + E_3)))^{\mathrm{ev}}$$

and this is a base point free linear system on $\widetilde{A}$. *Here* $E_0, \ldots, E_3$ *are exceptional divisors and* "ev" *on the right side is taken with respect to* $(Bl)^*(\iota)$.

Proof. The equality in the proposition is self-evident. It is enough to prove that the linear system has no base points. Obviously $e_0, \ldots, e_3$ are the only fixed points of $H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$. Hence we have to show that there is a section vanishing at e_i to order ≤ 2 (actually $= 2$). Its pullback does not vanish along E_i. Such sections are u_{ij}, where $j \neq i$, since $e_i \notin \Theta_i$. ■

The linear system described above defines a G-equivariant mapping $\phi : \widetilde{A} \to \mathbb{P}^5$ which factors over $\widetilde{K} = \widetilde{A}/\langle (Bl)^*(\iota)\rangle$:

$$\begin{array}{ccc} \widetilde{A} & & \\ \downarrow & \searrow & \\ \widetilde{K} & \to & \mathbb{P}^5 \end{array}$$

We call $\widetilde{K}$ an *intermediate Kummer surface* since it lies "between" the classical Kummer surface with 16 singularities and the smooth $K3$ surface.

The linear system $|4Bl^*(\Theta)-2(E_0+E_1+E_2+E_3)|^{\mathrm{ev}}$ defines an associated line bundle $\mathcal{M}^+$ on $\widetilde{K}$ in such a way that

$$H^0(\mathcal{M}^+) = H^0(\mathcal{O}_{\tilde{A}}(4Bl^*(\Theta) - 2(E_0 + E_1 + E_2 + E_3)))^{\mathrm{ev}}.$$

The next theorem states that this bundle defines an embedding of $\widetilde{K}$.

PROPOSITION 8. *The line bundle $\mathcal{M}^+$ defined above is very ample.*

In the proof of this theorem the following result of Saint-Donat turns out to be essential. It can be found in [S] and in [Ba]. In what follows we use the notation of Bauer's paper.

PROPOSITION 9 (Saint-Donat). *Let K be a $K3$ surface and let L be a line bundle on K such that $L^2 \geq 4$, $|L| \neq \emptyset$ and such that $|L|$ has no fixed components. Then $|L|$ has no base points. Furthermore*, *the morphism $K \to \mathbb{P}(H^0(L))$ is birational except in the following cases*:

(i) *There is an irreducible curve E such that $p_a(E) = 1$ and $L.E = 2$.*

(ii) *There is an irreducible curve H such that $p_a(H) = 2$ and $L = \mathcal{O}_K(2H)$.*

If the morphism is birational, *then it is an isomorphism away of the contracted curves.*

Proof of Proposition 8. Let $Bl_s : A_s \to A$ be the blow-up at all 16 halfperiods. By slight abuse of language we denote again the exceptional divisors by $E_0, \ldots, E_{15}$. Let $L_s = Bl_s^*(\mathcal{O}_A(\mathcal{I}^2.4\Theta)^{\mathrm{ev}})$ and $\mathcal{M}_s^+$ be the corresponding line bundle on the smooth Kummer surface $K_s = A_s/\iota_s$. Let $D_0, \ldots, D_{15}$ denote the (-2)-curves on K_s, which we get as the push-downs of $E_0, \ldots, E_{15}$.

It is enough to prove that the morphism defined by $\mathcal{M}_s^+ : K_s \to \mathbb{P}^5$ is birational and contracts the curves $D_4, \ldots, D_{15}$. The second part is easy since we have

$$\mathcal{M}_s^+.D_i = \tfrac{1}{2}(4Bl_s^*(\Theta) - 2(E_0 + E_1 + E_2 + E_3)).2E_i = \begin{cases} 2 & \text{if } i = 0, \ldots, 3, \\ 0 & \text{if } i \geq 4. \end{cases}$$

The assumptions of Saint-Donat's theorem are clearly satisfied, so we are done if we prove that the following two cases cannot occur:

Case 1. Let E be an irreducible elliptic curve with $\mathcal{M}_s^+.E = 2$. Let $F = \pi_s^*(E)$. Since π_s is a morphism of degree 2, we have $0 = E^2 = \frac{1}{2}F^2$. F is again an elliptic curve on A_s, by the adjunction formula. There is a symmetric divisor G on A such that $Bl_s^*G = F + \sum_{i=0}^{15} m_i E_i$. One easily computes that $G^2 = \sum_{i=0}^{15} m_i^2$ and

$$\begin{aligned} 2 = \mathcal{M}_s^+.E &= \tfrac{1}{2}(4Bl_s^*(\Theta) - 2(E_0+E_1+E_2+E_3)).F \\ &= \tfrac{1}{2}(4Bl_s^*(\Theta) - 2(E_0+E_1+E_2+E_3)).\Big(Bl_s^*G - \sum_{i=0}^{15} m_i E_i\Big) \\ &= \tfrac{1}{2}(4\Theta.G - 2(m_0+\ldots+m_3)). \end{aligned}$$

Thus we get

$$\Theta.G = 1 + \tfrac{1}{2}(m_0+\ldots+m_3).$$

Since $g(\Theta) = 2$, we have $\Theta.G \geq 2$ and thus $m_0+\ldots+m_3 \geq 2$. Using the Hodge inequality (see [Ba]) we have

$$2\sum_{i=0}^{15} m_i^2 = \Theta^2 \cdot G^2 \leq (\Theta.G)^2 = \big(1 + \tfrac{1}{2}(m_0+\ldots+m_3)\big)^2.$$

There are only two possibilities for m_i's to satisfy the above inequality:

(i) There exist $i, j \in \{0,1,2,3\}$ with $i \neq j$ such that

$$m_k = \begin{cases} 1 & \text{if } k = i \text{ or } k = j, \\ 0 & \text{for all other } k. \end{cases}$$

In this case we have $G^2 = 2$, so G defines a principal polarization. This is not possible because then G must pass through 6 halfperiods.

(ii) $m_0 = \ldots = m_3 = 1$ and $m_k = 0$ for $k = 4,\ldots,15$. Then $G^2 = 4$ and $G.\Theta = 3$. Thus G defines a (1,2)-polarization, hence $h^0(A,|G|) = 2$. Let e be a halfperiod on Θ different from $e_0,\ldots,e_3$. In the pencil $|G|$ we can find an element G_0 passing through e. That means that G_0 contains four halfperiods e_0, e_1, e_2, e lying on Θ. Since $G_0.\Theta = 3$ it follows that these divisors have common components because otherwise $G_0.\Theta \geq 4$. What is more, G_0 must be irreducible, since if it had two components, say G_1 and G_2, we would have $G_1.\Theta \geq 2$, $G_2.\Theta \geq 2$ and again a contradiction. So we can assume that G_0 is irreducible. Since Θ is reduced and irreducible we must have $G_0 = \Theta + R$. Again $\Theta.R \geq 2$ and $4 = G_0^2 = \Theta^2 + 2\Theta.R + R^2 \geq 6$, a contradiction.

Case 2. Let H be a genus 2 curve such that $\mathcal{M}_s^+ = \mathcal{O}_{K_s}(2H)$ and let G be the corresponding symmetric divisor on A. Let again F be the proper transform of G on A_s. We have $Bl_s^*G = F + \sum_{i=0}^{15} m_i E_i$. Computing intersection numbers we get $G^2 = 8$ and $m_0 = \ldots = m_3 = 1$, $m_k = 0$ for

$k = 4, \ldots, 15$. This is a contradiction: G is numerically, hence algebraically equivalent to 2Θ and it must be totally symmetric.

Thus we have shown that $\mathcal{M}^+$ defines an isomorphism. ∎

We proceed with describing the title 4_3 configuration. The following lemmas are simple consequences of the above proposition and of the geometry of the abelian surface A.

LEMMA 10. *$\phi|E_i$ is a $1:1$ mapping for $i = 0, \ldots, 3$, and its image is a smooth conic C_i lying in some plane F_i in $\mathbb{P}^5$.*

Proof. It is enough to notice that $\pi_s : E_i \to D_i$ is $1:1$ and to compute the intersection number of D_i with $\mathcal{M}^+$:

$$\mathcal{M}^+.D_i = \tfrac{1}{2}(4Bl^*(\Theta) - 2(E_0 + E_1 + E_2 + E_3)).2E_i = -2E_i^2 = 2. \ \blacksquare$$

LEMMA 11. *The restriction of ϕ to $\widetilde{\Theta}_i$ is for $i = 0, \ldots, 3$ a $2:1$ mapping branched at 3 not blown-up halfperiods on $\widetilde{\Theta}_i$, and its image is a line L_i in $\mathbb{P}^5$ (here $\widetilde{\Theta}_i$ denotes the strict transform of Θ_i under the blowing up).*

Proof. We are done if we show that $\phi(\widetilde{\Theta}_i)$ is a line, since then the claim follows from the Riemann–Hurwitz formula. We compute again the intersection number:

$$\begin{aligned}\mathcal{M}^+.(\pi_*\widetilde{\Theta}_i) &= \tfrac{1}{2}(4Bl^*(\Theta) - 2(E_0 + E_1 + E_2 + E_3)).\widetilde{\Theta}_i \\ &= 2\Theta.(Bl_*\widetilde{\Theta}_i) - 3 = 4 - 3 = 1. \ \blacksquare\end{aligned}$$

The lines L_i and the conics C_i are 4_3 configured. This means that each line intersects three of the conics and each conic intersects three of the lines. The next lemma describes which line intersects which conic and vice versa.

LEMMA 12. *The incidences in the 4_3 configuration are: $L_i \cap C_j = \emptyset$ if and only if $i = j$ for $i, j = 0, \ldots, 3$.*

LEMMA 13. *Any two planes F_i and F_j, $i \neq j$, meet in exactly one point p_{ij}. In each of the configuration planes F_i the points p_{ij}, $j \neq i$, build a nondegenerate (coordinate) triangle in which the conic C_i is inscribed.*

Proof. Suppose that $F_i \cap F_j$ is a line. This means that F_i and F_j span some $\mathbb{P}^3 \subset \mathbb{P}^5$. The sections u_{ij}, u_{ik}, u_{il} vanish at e_i to order 2, hence they do not vanish identically on C_i and thus also on F_i. By analogy the sections u_{ji}, u_{jk}, u_{jl} do not vanish identically on F_j. Thus we have found five independent (by Proposition 6) sections spanning $\mathbb{P}^3$. This is of course a contradiction. This also shows that $F_i \cap F_j$ is not empty, since five sections cannot span $\mathbb{P}^5$.

We now show that for fixed i the points p_{ij}, p_{ik}, p_{il} are not collinear. To this end it is enough to notice that since $p_{ij} \in F_i \cap F_j$, it is the only point where $u_{ij} \neq 0$ and all other sections vanish, and similarly for all other p_{mn}.

If the points mentioned above were collinear, the sections u_{ij}, u_{ik}, u_{il} would not be independent, but this is not possible.

Now we have to show that the line through p_{ij} and p_{ik} is tangent to the conic C_i. We show instead that the 4-space F defined by $u_{il} = 0$ is tangent to the conic C_i. F contains of course the line in question. Let D be the divisor defined by F on $\widetilde{K}$. Then we have

$$2 = D.C_i = \tfrac{1}{2}(2\widetilde{\Theta}_i + 2\widetilde{\Theta}_l).2E_i = 2\widetilde{\Theta}_l.E_i$$

and thus $\widetilde{\Theta}_l.E_i = 1$. Since both divisors are effective there can only be one intersection point on the Kummer surface. ∎

Quadratic equations of the 4_3 configuration. In the coordinates u_{ij} we can easily compute the equations of the planes $F_0, \ldots, F_3$. To this end we look for sections in $H^0(\mathcal{I}.\mathcal{O}_A(4\Theta))^{\mathrm{ev}}$ vanishing to order ≥ 3 at a fixed halfperiod e_i. These are $u_{jk} = t_j^2 t_k^2$, $u_{jl} = t_j^2 t_l^2$, $u_{kl} = t_k^2 t_l^2$, where i, j, k, l are all different. In this way one gets

$$F_i = \{(u_{23} : u_{01} : u_{03} : u_{12} : u_{13} : u_{02}) \in \mathbb{P}^5 \mid u_{jk} = u_{jl} = u_{kl} = 0\}.$$

We note that $F_1 = \sigma(F_0)$, $F_2 = \tau(F_0)$, $F_3 = \sigma(\tau(F_0))$.

Let $p_{ij} = F_i \cap F_j$ for $0 \leq i < j \leq 4$. We have

$$\begin{aligned} p_{01} &= (0:1:0:0:0:0), & p_{12} &= (0:0:0:1:0:0),\\ p_{02} &= (0:0:0:0:0:1), & p_{13} &= (0:0:0:0:1:0),\\ p_{03} &= (0:0:1:0:0:0), & p_{23} &= (1:0:0:0:0:0). \end{aligned}$$

Now, the conic C_i is tangent to the line through p_{ij} and p_{ik} at the point $q_{il} = C_i \cap L_l = F_i \cap L_l$; here again i, j, k, l are all different. The points of tangency are not the vertices of the coordinate triangle in F_i, which follows from Lemma 13.

Our next aim is to find equations of the conics $C_0, \ldots, C_3$. It is in fact enough to find one of the equations since the other three are its transforms under the action of G.

For simplicity, let us study the situation in $\mathbb{P}^2 = \{(x_1 : x_2 : x_3)\}$. We choose two general points on the coordinate lines $P_1 = \{x_3 = 0\}$ and $P_2 = \{x_2 = 0\}$, say $X_1 = (\beta : \alpha : 0)$ and $X_2 = (\gamma : 0 : \alpha)$, where $\alpha, \beta, \gamma \in \mathbb{C}^*$. Then there are precisely two conics intersecting the coordinate lines at the fixed points X_1 and X_2 with multiplicity 2 and intersecting the third coordinate line at some point X_3 with the same multiplicity. One of these conics is the double line through X_1 and X_2 and the second is a smooth conic touching the third coordinate line at $X_3 = (0 : \gamma : \beta)$. The matrix of the quadratic form belonging to this smooth conic is

$$\begin{bmatrix} \alpha^2 & -\alpha\beta & -\alpha\gamma \\ -\alpha\beta & \beta^2 & -\beta\gamma \\ -\alpha\gamma & -\beta\gamma & \gamma^2 \end{bmatrix}.$$

Applying the above considerations to C_0 we get its equation in F_0:

$$\alpha^2 u_{01}^2 + \beta^2 u_{03}^2 + \gamma^2 u_{02}^2 - 2\alpha\beta u_{01}u_{03} - 2\alpha\gamma u_{01}u_{02} - 2\beta\gamma u_{03}u_{02} = 0.$$

We also have the coordinates of the tangency points q_{ij}, $i \neq j$. In particular, we note that

$$\begin{aligned} q_{03} &= (0:\gamma:0:0:0:\alpha), \\ q_{13} &= \sigma(q_{02}) = (0:\beta:0:\alpha:0:0), \\ q_{23} &= \tau(q_{01}) = (0:0:0:-\gamma:0:\beta). \end{aligned}$$

These points are all on the line L_3 and they determine its equation. It is an easy exercise to find nine other points of tangency and equations of the other three configuration lines. It is easy because we have already identified the symmetry group of the configuration.

Now we look for quadrics Q in $\mathbb{P}^5$ containing our configuration of conics and lines. We note that the conics alone contain all information required, more precisely:

LEMMA 14. *If Q is a quadric in $\mathbb{P}^5$ containing the conics $C_0, \ldots, C_3$ then it also contains the lines $L_0, \ldots, L_3$.*

Proof. Each line is tangent to three conics, so its intersection with Q contains at least three points. Hence the whole line lies on the quadric by a degree argument. ∎

Let Q be given as a symmetric matrix $[a_{ij}]$, $i,j=1,\ldots,6$. Its intersection with the plane F_i is, again by a degree argument, the conic C_i. Comparing the equations of C_i and $Q|F_i$ one gets conditions for the coefficients a_{ij}. We omit these boring calculations here.

PROPOSITION 15. *All quadrics containing the subject configuration form a 4-dimensional linear system spanned by*

$$Q_1 = \begin{bmatrix} \alpha^2 & 0 & -\alpha\beta & -\alpha\beta & \alpha\gamma & \alpha\gamma \\ 0 & \alpha^2 & -\alpha\beta & -\alpha\beta & -\alpha\gamma & -\alpha\gamma \\ -\alpha\beta & -\alpha\beta & \beta^2 & 0 & \beta\gamma & -\beta\gamma \\ -\alpha\beta & -\alpha\beta & 0 & \beta^2 & -\beta\gamma & \beta\gamma \\ \alpha\gamma & -\alpha\gamma & \beta\gamma & -\beta\gamma & \gamma^2 & 0 \\ \alpha\gamma & -\alpha\gamma & -\beta\gamma & \beta\gamma & 0 & \gamma^2 \end{bmatrix}, \quad Q_2 = \left[\begin{array}{cc|cc|cc} & 1 & & & & \\ 1 & & & & & \\ \hline & & & & & \\ & & & & & \\ \hline & & & & & \\ & & & & & \end{array}\right],$$

$$Q_3 = \left[\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline & & & 1 & & \\ & & 1 & & & \\ \hline & & & & & \\ & & & & & \end{array}\right], \quad Q_4 = \left[\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline & & & & & \\ & & & & & \\ \hline & & & & & 1 \\ & & & & 1 & \end{array}\right].$$

P r o o f. One easily verifies that these quadrics contain our configuration. The completeness of the list follows from the considerations above. ■

We now want to inspect more closely the intersections of the quadric hypersurfaces just found. It is clear that the last three are very singular, in fact they can each be decomposed into sums of two hyperplanes. The intersection $Q_2 \cap Q_3 \cap Q_4$ is a singular $K3$ surface S, which is the union of eight planes: $F_0, \ldots, F_3$ and $F_0', \ldots, F_3'$. If F_i is defined by $u_{jk} = u_{jl} = u_{kl} = 0$, then F_i' is defined by $u_{ij} = u_{ik} = u_{il} = 0$, where i, j, k, l are all different.

Intersecting S with Q_1 we get our configuration: the lines L_i are, however, counted with multiplicity 2. Thus they are double lines (singular conics) in the planes F_i'. There is an amusing "external" symmetry in this situation. Let us recall that looking for the equation of a conic tangent to all sides of the coordinate triangle and passing through two fixed points X_1 and X_2 we found in fact two conics. One of them was smooth and the other one, which we left aside, was a double line. If we took this line instead of the smooth conic we would now get smooth conics in the "dual" planes $F_0', \ldots, F_3'$. Their equations would be exactly the equations of $C_0, \ldots, C_3$ after a change of coordinates with the matrix $Q_2 + Q_3 + Q_4$.

Equations of the intermediate Kummer surface. The 4_3 configuration studied in the previous section lies on the surface $\widetilde{K}$ defined after Proposition 7. We use the equations of this configuration to find equations defining our Kummer surface. Let us notice that two of them are self-evident. These are

$$S_1 = Q_2 - Q_3 = u_{01}u_{23} - u_{03}u_{12} = 0,$$
$$S_2 = Q_2 - Q_4 = u_{01}u_{23} - u_{02}u_{13} = 0.$$

If we hope to get the surface $\widetilde{K}$ as a complete intersection, as in the case of a smooth $K3$ surface in $\mathbb{P}^5$, we need another equation. It must be a linear combination of $Q_1, \ldots, Q_4$. Because of the form of S_1 and S_2 we can assume that $S_3 = \mu Q_1 + \lambda Q_2$. Actually, we have already studied in the previous section the case $\mu = 0$, so we can assume $S_3 = Q_1 + \lambda Q_2$.

To find λ we need additional information about the surface. This data is coded in its singular points. There are exactly 12 of them and they are in $1:1$ correspondence with the halfperiods on the abelian surface A which we have not blown up. Since there are precisely 3 of them on each divisor $\Theta_0, \ldots, \Theta_3$, there are three singular points of the surface on each line $L_0, \ldots, L_3$. They do not coincide with the points q_{ij}. By symmetry it is clear that the singularities on one of the lines, say L_3, determine all the others. In fact, already the choice of one singular point on L_3 determines the other two and thus all singularities of the surface. This is shown in the following

PROPOSITION 16. *Fix* $(-\alpha : \beta : \gamma) \in (F_3')^*$. *Let* $x = sq_{03} + tq_{23}$ *be a point on* L_3, *where* $(s : t) \in \mathbb{P}^1 \setminus \{(0:1), (1:0), (\beta : -\alpha)\}$. *Then for*

$$\lambda = -\frac{2\alpha\beta^2 t^3 + (3\alpha^2\beta + \beta\gamma^2 + \beta^3)st^2 + (3\alpha\beta^2 - \alpha\gamma^2 + \alpha^3)s^2 t + 2\alpha^2\beta s^3}{\beta st^2 + \alpha s^2 t}$$

the point x *is singular and* $\widetilde{K}$ *is the complete intersection of* S_1, S_2 *and* S_3.

Proof. For x to be singular in $\widetilde{K} = S_1 \cap S_2 \cap S_3$ it suffices that

$$\operatorname{rk}\begin{pmatrix} dS_1/d(u_{01}\ldots u_{23}) \\ dS_2/d(u_{01}\ldots u_{23}) \\ dS_3/d(u_{01}\ldots u_{23}) \end{pmatrix} < 3 \quad \text{at } x.$$

Instead of simply inserting λ we show how to calculate it knowing x. It is not only a more exciting and natural procedure but also, proceeding along this line, we get some additional information. Calculating the matrix above we find

$$\begin{pmatrix} \lambda s\gamma + 2\alpha\beta\gamma + s\alpha^2\gamma & 0 & -2s\alpha\beta\gamma - t\beta^2\gamma & 0 & -s\alpha\gamma^2 + t\beta\gamma^2 & 0 \\ s\gamma & 0 & t\gamma & 0 & 0 & 0 \\ s\gamma & 0 & 0 & 0 & -s\alpha - t\beta & 0 \end{pmatrix}.$$

It is clear that its rank is < 3 if the determinant

$$\begin{vmatrix} \lambda s\gamma + 2\alpha\beta\gamma + s\alpha^2\gamma & -2s\alpha\beta\gamma - t\beta^2\gamma & -s\alpha\gamma^2 + t\beta\gamma^2 \\ s\gamma & t\gamma & 0 \\ s\gamma & 0 & -s\alpha - t\beta \end{vmatrix}$$

vanishes. Since the determinant is linear in λ, we can simply calculate λ and thus verify the assertion of our proposition. ∎

Let us write down the above condition on the determinant as a homogeneous equation in t and s:

$$\gamma^2(2\alpha\beta^2 t^3 + (\lambda\beta + 3\alpha^2\beta + \beta\gamma^2 + \beta^3)st^2 + (\lambda\alpha + 3\alpha\beta^2 - \alpha\gamma^2 + \alpha^3)s^2 t + 2\alpha^2\beta s^3) = 0.$$

If we now set $t = \alpha p$ and $s = \beta q$, the above equation becomes a bit simpler:

$$2\alpha^2 p^3 + (\lambda + 3\alpha^2 + \gamma^2 + \beta^2)qp^2 + (\lambda + 3\beta^2 - \gamma^2 + \alpha^2)q^2 p + 2\beta^2 q^3 = 0.$$

It is not very surprising that we get an equation of degree 3. The three zeroes correspond to singular points on L_3 for fixed λ. This equation is also the starting point to study the moduli space of intermediate Kummer surfaces. But that is another story.

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