ANNALES
POLONICI MATHEMATICI
LX.2 (1994)

Oscillation of a forced higher order equation

by Witold A. J. Kosmala (Boone, N.C.)

Abstract. We state and prove two oscillation results which deal with bounded solutions of a forced higher order differential equation. One proof involves the use of a nonlinear functional.

Introduction. The main objective of this paper is to present two oscillation results for bounded solutions of the differential equation

$$x^{(n)} + p(t)x^{(n-1)} + q(t)x^{(n-2)} + H(t,x) = Q(t) \tag{$*$}$$

where $n \geq 3$ is an integer and $H : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous, decreasing in its second variable and such that $uH(t,u) < 0$ for all $u \neq 0$. Here $\mathbb{R}$ denotes the real line and $\mathbb{R}^+$ the interval $[0,\infty)$. The differential equation $(*)$ has not been much studied under the assumptions on H as described above. The only oscillation result known to the author is given in [5]. In that paper $Q(t)$ is identically zero and conditions on H are stronger. There is no oscillation result known for $(*)$ with H as described above in the case of n even. As in [5], in this paper we also use a nonlinear functional to prove the result. This approach came in useful to Erbe [1], Heidel [2], Kartsatos [3], Kartsatos and Kosmala [4], and others in proving their theorems. In [6] the author also uses nonlinear functionals to prove a variety of asymptotic properties of the differential equation $(*)$. The reader might also wish to explore [7] where H is different but some other assumptions as well as methods are similar.

In what follows, we say that $x(t)$, $t \in [t_x,\infty) \subset \mathbb{R}^+$, is a solution of $(*)$ if it is n times continuously differentiable and satisfies $(*)$ on $[t_x,\infty)$. The number $t_x \geq 0$ depends on the particular solution $x(t)$ under consideration. We say that the function is *oscillatory* if it has an unbounded set of zeros. Moreover, we say that a property P holds *eventually* or *for all large t* if there

---

1991 *Mathematics Subject Classification*: Primary 34C10.
Key words and phrases: oscillation, nonlinear higher order equation, nonlinear functional.
Paper completed during author's sabbatical at the University of Alberta in Edmonton, Alberta, Canada.

exists $T \geq 0$ such that P holds for all $t \geq T$. We denote by $C^n(I)$ the space of all n times continuously differentiable functions $f : I \to \mathbb{R}$. We write $C(I)$ instead of $C^0(I)$. Throughout this paper we assume that $p \in C^1[t_0, \infty)$ and $q \in C[t_0, \infty)$ with

$$2q(t) \leq p'(t) \tag{1}$$

for $t \geq t_0$. Moreover, we assume that S is a solution of

$$S^{(n)} + p(t)S^{(n-1)} + q(t)S^{(n-2)} = Q(t)$$

which tends to zero.

Lemma 1 in [5] can be extended to the forced equation without too much difficulty. For the sake of completeness we state it formally and provide the proof.

LEMMA. *If x is an eventually positive solution of* $(*)$, *then either* $[x(t) - S(t)]^{(n-2)} \leq 0$ *or* $[x(t) - S(t)]^{(n-2)} > 0$ *for all large* t.

Proof. Suppose $x(t) > 0$ and $2q(t) \leq p'(t)$ for all $t \geq t_0 \geq 0$. Let $u = x - S$ with $t \geq t_0$. Then the equation $(*)$ becomes

$$u^{(n)}(t) + p(t)u^{(n-1)}(t) + q(t)u^{(n-2)}(t) + H(t, u(t) + S(t)) = 0. \tag{2}$$

Now, we suppose to the contrary that $u^{(n-2)}(t_1) = u^{(n-2)}(t_2) = 0$ with $u^{(n-2)}(t) > 0$ for $t_0 \leq t_1 < t < t_2$. This implies that $u^{(n-1)}(t) \not\equiv 0$ on (t_1, t_2). Now, multiply (2) by $u^{(n-2)}(t)$ and integrate from t_1 to t_2 to obtain

$$\int_{t_1}^{t_2} u^{(n-2)}(t)H(t, u(t) + S(t))\, dt$$
$$= \int_{t_1}^{t_2} (u^{(n-1)}(t))^2\, dt - \int_{t_1}^{t_2} \left(q(t) - \frac{p'(t)}{2}\right)(u^{(n-2)}(t))^2\, dt > 0.$$

Since the left hand side cannot be positive, we obtain a contradiction. Hence, the proof is complete.

This Lemma can be rephrased for an eventually negative solution as well.

THEOREM 1. *Consider the differential equation* $(*)$ *with the following additional assumptions*:

(i) $n \geq 3$ *is an odd integer*,
(ii) $p(t) \leq 0$, $q(t) \geq 0$ *and*

$$t[q(t) - p'(t)] \leq 2p(t) \tag{3}$$

eventually, *and*

(iii) *for any positive real constant* k,

$$-\int^{\infty} t^2 H(t, \pm k)\, dt = \pm\infty.$$

Then every bounded solution of $(*)$ *is oscillatory or tending to zero.*

Remarks. (a) If $Q(t) \equiv 0$, then every bounded solution of $(*)$ must oscillate.

(b) The function p cannot be a negative constant because if it is, by assumption (ii) and (1), $q(t) \equiv 0$. But this contradicts condition (3).

(c) Suppose $p(t) \le 0$ and $q(t) \ge 0$ eventually. Then assumption (1) does not imply assumption (3). Indeed, $p(t) = -1/t$ and $q(t) = 1/(5t^2)$ satisfy (1) but not (3). Moreover, condition (3) does not imply condition (1). For example, $p(t) = -1/t^5$ and $q(t) = 2.8/t^6$ satisfy (3) but not (1). It can be proven, however, that if $p(t)$ satisfies

$$p(t) \le \left(\frac{t^*}{t}\right)^4 p(t^*)$$

with $t \ge t^*$ for any fixed $t^* > 0$ for which $p(t^*) < 0$ then, together with assumption (3), the condition (1) must hold.

(d) A familiar differential equation

$$x''' - 8x = 0$$

fits all the assumptions of Theorem 1. It is easy to verify that since three linearly independent solutions are e^{2t}, $e^{-t}\sin\sqrt{3}t$, $e^{-t}\cos\sqrt{3}t$, all the solutions of this equation are either unbounded or bounded and oscillatory.

(e) Every homogeneous differential equation has a trivial bounded oscillatory solution. In particular, the differential equation

$$x''' - \frac{1}{t^4}x'' + \frac{1}{t^6}x' - \left(1 - \frac{1}{t^4} + \frac{1}{t^6}\right)x = 0$$

has a bounded oscillatory solution $x(t) = 0$ and an unbounded solution $x(t) = e^t$ for $t > 0$. Since the coefficient functions satisfy all the conditions in Theorem 1, every solution of this equation is unbounded and/or oscillatory.

(f) We observe that the differential equation

$$x''' - \frac{1}{t^4}x'' + \frac{1}{t^6}x' - \frac{1}{t}\arctan(tx) = -\frac{1}{t^8}(6t^4 + 2t + 1)$$

involves functions which satisfy all the required conditions in Theorem 1, and hence, every solution of this equation is either unbounded, oscillatory or tending to zero. In fact, $S(t) = 1/t$ in the above equation.

(g) The differential equation $(*)$ has some applications in stock market fluctuations, generalized mechanics, and astrophysics.

Proof of Theorem 1. We proceed by contradiction. Without loss of generality, we will assume that x is a bounded, positive solution of $(*)$ which does not tend to zero, and we will also assume that all the conditions on the functions p and q are satisfied for $t \geq t_0 \geq 0$. We let $u = x - S$ with $t \geq t_0$. Then equation $(*)$ can be written as equation (2). Also, by the above Lemma, we have either $u^{(n-2)}(t) \leq 0$ or $u^{(n-2)}(t) > 0$. In order to prove the theorem, we need to consider both cases and find a contradiction in each.

Case 1. We assume that $u^{(n-2)}(t) \leq 0$ for $t \geq t_1 \geq t_0$. Moreover, we suppose that there exists $t_2 \geq t_1$ such that $u^{(n-1)}(t_2) = 0$. Then we get

$$u^{(n)}(t_2) = -q(t_2)u^{(n-2)}(t_2) - H(t_2, u(t_2) + S(t_2)) > 0.$$

Thus, $u^{(n-1)}(t)$ is increasing at any t_2, $t_2 \geq t_1$, for which it is zero. Therefore, $u^{(n-1)}(t)$ cannot have any zeros larger than t_2. Moreover, $u^{(n-1)}(t)$ cannot be eventually negative, because together with the fact that $u^{(n-2)}(t) \leq 0$ we get $\lim_{t\to\infty} u(t) = -\infty$. Thus, $\lim_{t\to\infty}[x(t) - S(t)] = -\infty$. Since $\lim_{t\to\infty} S(t) = 0$, we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the positivity of x.

We conclude that $u^{(n-1)}(t) > 0$ eventually. However, this is also impossible because from (2) we get $u^{(n)}(t) > 0$ for all large t. Together with $u^{(n-1)}(t) > 0$, this implies that $u^{(n-2)}(t) > 0$ eventually. This again gives a contradiction. This takes us to the next case.

Case 2. We assume that $u^{(n-2)}(t) > 0$ for $t \geq t_3 \geq t_0$. Since $x(t) > 0$ and $\lim_{t\to\infty} S(t) = 0$, we have $u(t) = x(t) - S(t) > 0$, which must be bounded (otherwise x will be unbounded), which in turn implies that $u^{(n-3)}(t) < 0$ for all $t \geq t_4 \geq t_3$. Therefore, there exists $\varepsilon > 0$ such that $u(t_4) = x(t_4) - S(t_4) > \varepsilon$ and $-\varepsilon < S(t) < \varepsilon$ for all $t \geq t_4$. Keeping in mind that n is odd, we have $u'(t) > 0$ for $t \geq t_5 \geq t_4$. This enables us to write

$$u(t) + S(t) > u(t) - \varepsilon > u(t_5) - \varepsilon \equiv k > 0 \quad \text{for all } t \geq t_5.$$

We define the functional G by

$$G(u(t)) = 2u^{(n-3)}(t)u^{(n-1)}(t) + 2p(t)u^{(n-3)}(t)u^{(n-2)}(t) - [u^{(n-2)}(t)]^2. \tag{4}$$

We will prove that $G(u(t)) > 0$ eventually by assuming to the contrary. So, let $t_6 \geq t_5$ be such that $G(u(t_6)) \leq 0$. Note that if t_6 like this does not exist, we are done. So now, we write

$$\begin{aligned}
G'(u(t)) &= 2u^{(n-3)}(t)u^{(n)}(t) + 2u^{(n-2)}(t)u^{(n-1)}(t) + 2p(t)u^{(n-3)}(t)u^{(n-1)}(t) \\
&\quad + 2p(t)[u^{(n-2)}(t)]^2 + 2p'(t)u^{(n-3)}(t)u^{(n-2)}(t) - 2u^{(n-2)}(t)u^{(n-1)}(t) \\
&= 2u^{(n-3)}(t)[-p(t)u^{(n-1)}(t) - q(t)u^{(n-2)}(t) - H(t, u(t) + S(t))] \\
&\quad + 2p(t)u^{(n-3)}(t)u^{(n-1)}(t) + 2p(t)[u^{(n-2)}]^2 + 2p'(t)u^{(n-3)}(t)u^{(n-2)}(t)
\end{aligned}$$

$$= 2u^{(n-3)}(t)u^{(n-2)}(t)[p'(t) - q(t)] + 2p(t)[u^{(n-2)}(t)]^2 - 2u^{(n-3)}(t)H(t, u(t) + S(t)) < 0 \quad \text{for } t \ge t_6,$$

because $0 \le 2q(t) \le p'(t)$ implies $q(t) \le p'(t)$. Hence $G(u(t)) < 0$ for $t > t_6$.

Now we distinguish three cases.

(i) Suppose $u^{(n-1)}(t) \ge 0$ eventually. This together with $u^{(n-2)}(t) > 0$ contradicts the boundedness of $u(t)$.

(ii) Suppose $u^{(n-1)}(t) \le 0$ for $t \ge t_7 > t_6$. Since G is nonincreasing, this gives us

$$-[u^{(n-2)}(t)]^2 \le G(u(t)) \le G(u(t_7)) < 0, \qquad t \ge t_7.$$

So, in view of this and the fact that $u^{(n-2)}(t)$ is nonincreasing and positive, there exists a number $m > 0$ such that $\lim_{t\to\infty} u^{(n-2)}(t) = m > 0$. This implies that $u^{(n-3)}(t) \to \infty$ as $t \to \infty$, which is a contradiction.

(iii) Suppose that $u^{(n-1)}(t)$ changes sign for arbitrarily large t. Recall that $u^{(n-2)}(t) > 0$ for $t \ge t_6$. Thus $\liminf_{t\to\infty} u^{(n-2)}(t) \ge 0$. If this limit is greater than zero, then $u^{(n-2)}(t) \ge r$ for some $r > 0$. This contradicts the fact that $u^{(n-3)}(t)$ is negative. Hence

$$\liminf_{t\to\infty} u^{(n-2)}(t) = 0.$$

Since $u^{(n-1)}(t)$ oscillates, $u^{(n-2)}(t)$ has local extrema. Thus, there exists a sequence of local minima a_n such that $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} u^{(n-2)}(a_n) = 0$ and $u^{(n-1)}(a_n) = 0$. Consequently, if $a_m \ge t_8 > t_6$, we obtain

$$-[u^{(n-2)}(a_m)]^2 \le G(u(a_m)) \le G(u(t_8)) < 0,$$

contrary to $\lim_{n\to\infty} u^{(n-2)}(a_n) = 0$.

Hence, since $G(u(t)) \le 0$ prevents $u^{(n-1)}(t)$ from existing, we conclude that $G(u(t)) > 0$ for $t \ge t_9 \ge t_5$. Also, since $u^{(n-3)}(t) < 0$, we can drop the the last term in (4) and obtain

$$u^{(n-1)}(t) + p(t)u^{(n-2)}(t) < 0 \qquad \text{for } t \ge t_9. \tag{5}$$

Next, we multiply equation (2) by t^2 and integrate (the first two terms by parts) from t_9 to t, $t \ge t_9$, to obtain

$$\begin{aligned} t^2u^{(n-1)}(t) - (t_9)^2u^{(n-1)}(t_9) - 2\int_{t_9}^{t} su^{(n-1)}(s)\,ds \\ + p(t)t^2u^{(n-2)}(t) - p(t_9)(t_9)^2u^{(n-2)}(t_9) \\ + \int_{t_9}^{t} [s^2q(s) - (s^2p(s))']u^{(n-2)}(s)\,ds \\ = -\int_{t_9}^{t} s^2H(s, u(s) + S(s))\,ds. \end{aligned} \tag{6}$$

Since condition (3) implies that $t^2q(t) - (t^2p(t))' \le 0$, in view of (5) we can rewrite (6) as

$$M - 2\int_{t_9}^{t} su^{(n-1)}(s)\,ds > -\int_{t_9}^{t} s^2H(s,u(s)+S(s))\,ds > -\int_{t_9}^{t} s^2H(s,k)\,ds,$$

with M constant. From the hypotheses, since the right hand side tends to ∞, so must the left hand side. Therefore

$$\int_{t_9}^{\infty} tu^{(n-1)}(t)\,dt = -\infty. \tag{7}$$

Now, we rewrite (6) again, but this time we drop the fourth and sixth terms to obtain

$$t^2u^{(n-1)}(t) - 2\int_{t_9}^{t} su^{(n-1)}(s)\,ds + N > -\int_{t_9}^{t} s^2H(s,k)\,ds.$$

Since the right hand side tends to ∞, we can write

$$\lim_{t\to\infty}\Big[t^2u^{(n-1)}(t) - 2\int_{t_9}^{t} su^{(n-1)}(s)\,ds\Big] = \infty.$$

Next, we define

$$z(t) = \int_{t_9}^{t} su^{(n-1)}(s)\,ds.$$

Then $z'(t) = tu^{(n-1)}(t)$ and $\lim_{t\to\infty}[tz'(t) - 2z(t)] = \infty$. By Lemma 1 of [8], we know that $z(t)$ must tend to either ∞ or $-\infty$. Since we can write

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty}\int_{t_9}^{t} su^{(n-1)}(s)\,ds = \lim_{t\to\infty}[tu^{(n-2)}(t) - u^{(n-3)}(t)],$$

where the last term is positive, we must have $\lim_{t\to\infty} z(t) = \infty$. This contradicts (7). Therefore, we have a contradiction in this case as well. Hence, the proof of the theorem is complete.

THEOREM 2. *Consider the differential equation* $(*)$ *with the following additional conditions*:

(i) $n \ge 3$ *is an odd integer*,

(ii) $p(t) \le 0$ *and* $q(t) \ge 0$ *eventually*, *and*

(iii) *for any positive real constant* k,

$$-\int^{\infty} H(t,\pm k)\,dt = \pm\infty.$$

Then every bounded solution of $(*)$ *must oscillate or tend to zero.*

Note that, as in Theorem 1, here also if $Q(t) \equiv 0$, then every bounded solution of $(*)$ must oscillate.

Proof of Theorem 2. We also argue by contradiction. Without loss of generality, we will assume that x is a bounded, positive solution of $(*)$ which does not tend to zero, and we will assume that all the conditions on functions p and q are satisfied for $t \geq t_0 \geq 0$. Let $u(t) = x(t) - S(t)$, $t \geq t_0$. Then the Lemma above guarantees that $u^{(n-2)}(t) \leq 0$ or $u^{(n-2)}(t) > 0$ eventually. In order to prove the theorem, we need to consider both cases and find a contradiction.

Case 1. We assume that $u^{(n-2)}(t) \leq 0$ for all large t. To obtain a contradiction we follow case 1 in the proof of Theorem 1 above.

Case 2. We assume that $u^{(n-2)}(t) > 0$ for $t \geq t_1 \geq t_0$. As in the proof of case 2 in Theorem 1, we know that $u(t) > 0$, $u'(t) > 0$, $u^{(n-3)}(t) < 0$ and $u(t) + S(t) \geq k$ for $k > 0$ constant, whenever $t \geq t_2 \geq t_1$. So, now we integrate equation (2) from t_2 to t, $t \geq t_2$, to get

$$\begin{aligned} u^{(n-1)}(t) &+ p(t)u^{(n-2)}(t) \\ &= u^{(n-1)}(t_2) + p(t_2)u^{(n-2)}(t_2) \\ &\quad + \int_{t_2}^{t} [p'(s) - q(s)]u^{(n-2)}(s)\, ds - \int_{t_2}^{t} H(s, u(s) + S(s))\, ds \\ &= M + f(t) - \int_{t_2}^{t} H(s, u(s) + S(s))\, ds, \end{aligned}$$

where M is a constant and $f(t)$ is the first integral above. Since $f(t) \geq 0$ (note $0 \leq 2q(t) \leq p'(t)$ implies $q(t) \leq p'(t)$), we can rewrite the above as

$$u^{(n-1)}(t) + p(t)u^{(n-2)}(t) > M + f(t) - \int_{t_2}^{t} H(s, k)\, ds.$$

Since $p(t) \leq 0$, $u^{(n-2)}(t) \geq 0$ and the right hand side tends to ∞, we conclude that $u^{(n-1)}(t)$ must also tend to ∞. Therefore, $\lim_{t\to\infty} u(t) = \infty$ implies $\lim_{t\to\infty} x(t) = \infty$, which means that x is unbounded. Contradiction. Hence, the result follows.

References

[1] L. Erbe, *Oscillation, nonoscillation and asymptotic behaviour for third order nonlinear differential equations*, Ann. Mat. Pura Appl. 110 (1976), 373–391.

[2] J. W. Heidel, *Qualitative behaviour of solutions of a third order nonlinear differential equation*, Pacific J. Math. 27 (1968), 507–526.

[3] A. G. Kartsatos, *The oscillation of a forced equation implies the oscillation of the unforced equation—small forcings*, J. Math. Anal. Appl. 76 (1980), 98–106.

[4] A. G. Kartsatos and W. A. Kosmala, *The behaviour of an nth-order equation with two middle terms*, ibid. 88 (1982), 642–664.

[5] W. A. Kosmala, *Properties of solutions of the higher order differential equations*, Differential Equations Appl. 2 (1989), 29–34.

[6] —, *Behavior of bounded positive solutions of higher order differential equations*, Hiroshima Math. J., to appear.

[7] W. A. Kosmala and W. C. Bauldry, *On positive solutions of equations with two middle terms*, Ann. Polon. Math. 50 (1990), 241–250.

[8] V. A. Staikos and Y. G. Sficas, *Forced oscillations for differential equations of arbitrary order*, J. Differential Equations 17 (1975), 1–11.

DEPARTMENT OF MATHEMATICAL SCIENCES
APPALACHIAN STATE UNIVERSITY
BOONE, NORTH CAROLINA 28608
U.S.A.

Reçu par la Rédaction le 20.4.1993
Révisé le 3.11.1993