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Markov inequality on sets with polynomial parametrization

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Abstract. The main result of this paper is the following: if a compact subset E of $\mathbb{R}^n$ is UPC in the direction of a vector $v \in S^{n-1}$ then E has the Markov property in the direction of v. We present a method which permits us to generalize as well as to improve an earlier result of Pawłucki and Pleśniak [PP1].

1. Introduction. Let E be a compact subset of $\mathbb{R}^n$ with nonempty interior. Consider the following two classical problems for polynomials:

• (*Bernstein's problem*) Estimate the derivatives of polynomials at interior points of E;

• (*Markov's problem*) Estimate the derivatives of polynomials at all points of E.

For Markov's problem, the most interesting situation is when E has the Markov property.

A set E is said to have the *Markov property* if there exist positive constants M and r such that the following Markov inequality holds:

$$|\operatorname{grad} p(x)| \le M(\deg p)^r \|p\|_E,$$

for every $x \in E$ and every polynomial $p : \mathbb{R}^n \to \mathbb{R}$. (Here $\|p\|_E$ stands for $\sup |p|(E)$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.)

Markov's inequality plays an important role in the constructive theory of functions. Pawłucki and Pleśniak have shown connections between the Markov property and the construction of a continuous linear extension operator $L : C^\infty(E) \to C^\infty(\mathbb{R}^n)$ (see [PP2]). Pleśniak [P] has proved that if E is a C^∞ determining compact set in $\mathbb{R}^n$ then the existence of such an operator is equivalent to the Markov property. Pawłucki and Pleśniak [PP1]

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showed that the closure of a fat subanalytic subset of $\mathbb{R}^n$ has the Markov property. They introduced a class of *uniformly polynomially cuspidal subsets* of $\mathbb{R}^n$ (briefly, UPC) and proved Markov's inequality for them. There are several classes of sets which are UPC. In particular, compact convex subsets of $\mathbb{R}^n$ with nonempty interior, fat subanalytic subsets of $\mathbb{R}^n$ and sets in Goetgheluck's paper [G] (where a first example of Markov's inequality on sets with cusps was proved) belong to this class.

The UPC sets are compact sets which have a polynomial parametrization satisfying some additional (geometrical) conditions. These conditions imply Markov's inequality.

In this paper we present a new approach to the notion of UPC sets. Observe that

$$|\mathrm{grad}\, p(x)| = \sup\{|D_v p(x)| : v \in S^{n-1}\},$$

where S^{n-1} is the unit Euclidean sphere in $\mathbb{R}^n$, and $D_v p$ denotes the derivative of p in the direction of the vector v. We shall say that a compact set E has *the Markov property in the direction of* $v \in S^{n-1}$ if there exist positive constants M and r such that

$$\|D_v p(x)\|_E \le M k^r \|p\|_E$$

for all polynomials of degree $\le k$. It is clear that having the Markov property is equivalent to the Markov property in n linearly independent directions. It can happen that a set E has the Markov property only in k, $1 \le k < n$, linearly independent directions (see Example 4.1). Hence the new notion is indeed more general.

In our investigations a crucial role is played by the following result which is strictly connected with Bernstein's problem.

1.1. Proposition ([B1], [B4], see also [B2]). *Let E be a compact subset of $\mathbb{R}^n$. Then for all $x \in E$, all $v \in S^{n-1}$ and all polynomials p of degree $\le k$,*

$$|D_v p(x)| \le k D_{v+} V_E(x) \begin{cases} (\|p\|_E^2 - p(x)^2)^{1/2} & \text{if } p \in \mathbb{R}[x_1, \ldots, x_n], \\ \|p\|_E & \text{if } p \in \mathbb{C}[x_1, \ldots, x_n]. \end{cases}$$

Here V_E is the extremal function defined by

$$V_E(z) = \sup\{u(z) : u \in \mathcal{L}, u_{|E} \le 0\} \quad \text{for } z \in \mathbb{C}^n,$$

where $\mathcal{L}$ is the Lelong class of all plurisubharmonic functions in $\mathbb{C}^n$ with logarithmic growth: $u(z) \le \text{const.} + \log(1 + |z|)$ (see [S]), and

$$D_{v+} V_E(x) = \liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} V_E(x + i\varepsilon v)$$

(see [B1], [B4]). The above Dini derivatives of the extremal function play an important role in applications to Markov's problem. In the classical situation of $E = [-1, 1]$, Proposition 1.1 reduces to the Bernstein (if p is a real polynomial) and Markov–Bernstein (if p is a complex polynomial) inequalities.

The paper is organized as follows: in Section 2 we prove the Bernstein and Markov inequalities on a polynomial curve; in Section 3 we define UPC sets in the direction of a vector v and give a Markov type inequality in the direction of v—this is the main result of this paper. In the special case of a convex symmetric subset with nonempty interior we obtain another proof of a sharp result which was earlier obtained in [B4]. In Section 4 we give some examples where we apply the results of Sections 2 and 3.

2. Bernstein and Markov inequalities on a polynomial curve. Fix $v \in S^{n-1}$. For a given subset E of $\mathbb{R}^n$ and $x \in E$, we define the distance of x from $\mathbb{R}^n \setminus E$ in the direction of v by

$$\varrho_v(x) = \operatorname{dist}_v(x, \mathbb{R}^n \setminus E) := \sup\{t \geq 0 : [x - tv, x + tv] \subset E\}.$$

One can easily verify that if E is compact then ϱ_v is upper semicontinuous on E. Moreover,

$$\varrho_v(x) \geq \varrho(x) := \operatorname{dist}(x, \mathbb{R}^n \setminus E) \quad \text{and} \quad \varrho(x) = \inf\{\varrho_v(x) : v \in S^{n-1}\}.$$

The following result plays a crucial role in this section.

2.1. PROPOSITION. *Let E be a compact subset of $\mathbb{R}^n$ and let $\phi : \mathbb{R} \to \mathbb{R}^n$ be a polynomial mapping such that $\phi([0,1]) \subset E$. Put $d = \max(1, \deg \phi)$. Then*

$$D_{v+}V_E(\phi(t)) \leq 2d \sup_{0 \leq r \leq 1} \frac{\sqrt{r(1-r)}}{\varrho_v(\phi(rt))}$$

for $0 \leq t < 1$ and $v \in S^{n-1}$.

Proof. Fix $t \in [0,1)$, $\varepsilon > 0$ and $R > 1$. Assume that the right hand side of the inequality is finite. Denote by $\widetilde{\phi}$ the natural extension of ϕ to the whole plane $\mathbb{C}$. Define

$$f(\zeta) = \widetilde{\phi}\left(\frac{1}{2}at(g(\zeta) + 1)\right) + \frac{i}{2}(\zeta - \zeta^{-1})b\varepsilon v$$

for $|\zeta| \geq 1$, where $g(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1})$ is the Joukowski function and $a = 2/(g(R) + 1)$, $b = 2/(R - R^{-1})$.

Assume for the moment that

$$f(S^1) \subset E.$$

Then, by the maximum principle for subharmonic functions and by the definition of V_E, we obtain $V_E(f(\zeta)) \leq d \log |\zeta|$ for $|\zeta| \geq 1$. In particular,

$$V_E(\phi(t) + i\varepsilon v) \leq d \log R.$$

Now notice that

$$f(e^{i\theta}) = \phi\left(\frac{1}{2}at(\cos\theta + 1)\right) - \sin\theta b\varepsilon v$$

and the condition $f(S^1) \subset E$ is equivalent to

$$\phi(atr) \pm 2\sqrt{r(1-r)}\, b\varepsilon v \in E \quad \text{for each } 0 \le r \le 1.$$

This condition will be satisfied if

$$2\sqrt{r(1-r)}\, b\varepsilon \le \varrho_v(\phi(atr)),$$

or equivalently,

$$b \sup_{0\le r\le 1} \frac{2\sqrt{r(1-r)}}{\varrho_v(\phi(atr))} \le \frac{1}{\varepsilon}.$$

We have

$$\begin{aligned} b \sup_{0\le r\le 1} \frac{2\sqrt{r(1-r)}}{\varrho_v(\phi(atr))} &\le \frac{b}{\sqrt{a}} \sup_{0\le r\le 1} \frac{2\sqrt{ar(1-ar)}}{\varrho_v(\phi(atr))} \\ &\le \frac{b}{\sqrt{a}} \sup_{0\le r\le 1} \frac{2\sqrt{r(1-r)}}{\varrho_v(\phi(tr))}. \end{aligned}$$

Since the right-hand side tends to 0 as $R \to \infty$, and to ∞ as $R \to 1+$, we may choose $R = R(\varepsilon) > 1$ such that

$$\sup_{0\le r\le 1} \frac{2\sqrt{r(1-r)}}{\varrho_v(\phi(tr))} = \frac{\sqrt{a}}{2\varepsilon}(R - R^{-1}).$$

It is clear that the condition $f(S^1) \subset E$ is satisfied, and $R \to 1$ as $\varepsilon \to 0+$. Now, observe that

$$\lim_{R\to 1+} 2(R - R^{-1})^{-1} \log R = 1.$$

By the definition of $D_{v+}V_E$ we have

$$\begin{aligned} D_{v+}V_E(\phi(t)) &\le d \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \log R(\varepsilon) = d \lim_{\varepsilon\to 0+} \frac{\sqrt{a}}{2\varepsilon}(R(\varepsilon) - R(\varepsilon)^{-1}) \\ &= d \sup_{0\le r\le 1} \frac{2\sqrt{r(1-r)}}{\varrho_v(\phi(rt))}. \end{aligned}$$

This completes the proof.

2.2. Corollary. *If* $x \in \operatorname{int}(E)$, *then*

$$D_{v+}V_E(x) \le 1/\varrho_v(x).$$

Using a similar argument to that of the proof of Proposition 2.1 one can also prove the following

2.3. PROPOSITION. *Let Ω be a bounded, star-shaped (with respect to the origin) and symmetric domain in $\mathbb{R}^n$ and let $E = \overline{\Omega}$. Then*

$$D_{v+}V_E(x) \le \sup_{0\le r\le 1} \frac{\sqrt{1-r^2}}{\varrho_v(rx)} \quad \text{for } x \in \operatorname{int}(E),$$

with equality in the case where E is convex.

Proof. A star-shaped symmetric set has a natural parametrization $t \to tx$, $t \in [-1,1]$, $x \in E$. The inequality in Proposition 2.3 is obtained by a similar argument to that of Proposition 2.1 applied to the mapping

$$f(\zeta) = ag(\zeta)x + \frac{i}{2}(\zeta - \zeta^{-1})b\varepsilon v,$$

where $g(\zeta)$ and b have been defined in the proof of Proposition 2.1 and $a = 1/g(R)$.

Now consider the case where E is convex. Then

$$E = \{x \in \mathbb{R}^n : x\cdot w \le 1, \forall w \in E^*\},$$

where E^* denotes the polar of E. It is easy to see that

$$\varrho_v(rx) = \inf\left\{\frac{1-|r||x\cdot w|}{|v\cdot w|} : w \in E^*\right\}.$$

Hence

$$\sup_{0\le r\le 1} \frac{\sqrt{1-r^2}}{\varrho_v(rx)} \le \sup\left\{\frac{|v\cdot w|}{(1-(x\cdot w)^2)^{1/2}} : w \in E^*\right\}.$$

It was proved by the author (see [B1], [B4]) that the right-hand side of this inequality is equal to $D_{v+}V_E(x)$. This completes the proof.

We need the following lemma, which is a generalization of the well-known lemma of Pólya and Szegö (see [C]).

2.4. LEMMA. *Let p be a polynomial in one variable of degree $\le k-1$. If*

$$|p(t)| \le (1-t^2)^{-\alpha} \quad \text{for } t \in (-1,1),$$

where $\alpha \ge 1/2$ is fixed, then

$$\|p\|_{[-1,1]} \le k^{2\alpha}.$$

Proof. For $\alpha = 1/2$ we obtain the Pólya–Szegö lemma. The general case reduces to the case $\alpha = 1/2$ in the following way. Let $X_k = \{p \in \mathbb{C}[t] : \deg p \le k-1\}$. For $\alpha \ge 0$ we define a norm $\|\cdot\|_\alpha$ in X_k by

$$\|p\|_\alpha := \sup\{(1-t^2)^\alpha |p(t)| : t \in [-1,1]\}.$$

For $\alpha > 1/2$, we have $\|p\|_\alpha \le \|p\|_{1/2} \le \|p\|_0 = \|p\|_{[-1,1]}$. Observe that the Pólya–Szegö lemma is equivalent to the inequality $\|p\|_0 \le k\|p\|_{1/2}$. Since $(X_k, \|\cdot\|_{1/2})$ is an interpolation space between $(X_k, \|\cdot\|_\alpha)$ and $(X_k, \|\cdot\|_0)$ of exact exponent $\theta = 1-1/(2\alpha)$, i.e. $\|p\|_{1/2} \le \|p\|_\alpha^{1-\theta}\|p\|_0^\theta$, by the Pólya–Szegö lemma we obtain $\|p\|_0^{1-\theta} \le k\|p\|_\alpha^{1-\theta}$, which completes the proof.

Now we can formulate the main result of this section.

2.5. Proposition. *Let E be a compact subset of $\mathbb{R}^n$ and let $\phi : \mathbb{R} \to \mathbb{R}^n$ be a polynomial mapping of degree $d \ge 1$ such that $\phi([0,1]) \subset E$. Fix $v \in S^{n-1}$ and assume that $\operatorname{dist}_v(\phi(t), \mathbb{R}^n \setminus E) \ge M(1-t)^m$ for $0 \le t \le 1$, where $M > 0$ and $m \ge 1$ are constants. If $p \in \mathbb{C}[x_1, \ldots, x_n]$ and $\deg p \le k$, then*

$$|D_v p(\phi(t))| \le \frac{1}{M}(2dk)^{2m}\|p\|_E \quad \text{for } 0 \le t \le 1.$$

Proof. By Proposition 2.1 we obtain

$$\begin{aligned} D_{v+}V_E(\phi(t)) &\le \frac{2d}{M}\sup_{0\le r\le 1}\sqrt{r(1-r)}\,(1-rt)^{-m} \\ &\le \frac{2d}{M}(1-t)^{-(m-1/2)} \quad \text{for } 0 \le t < 1. \end{aligned}$$

It follows from Proposition 1.1 that

$$|D_v p(\phi(t^2))| \le \frac{2dk}{M}(1-t^2)^{-(m-1/2)}\|p\|_E$$

for $|t| < 1$. Since $D_v p(\phi(t^2))$ is a polynomial of degree $\le 2d(k-1)$, combining the last inequality with Lemma 2.4 gives our assertion.

3. Markov inequality on UPC sets. Our considerations suggest a modification of the notion of a UPC set introduced in [PP1].

Let E be a compact subset of $\mathbb{R}^n$ and let $m \ge 1$. Given $v \in S^{n-1}$, we shall say that E is *m-UPC in the direction of v* if there exist $E_0 \subset E$, a positive constant M and a positive integer d such that for each $x \in E_0$ one can choose a polynomial map $\phi_x : \mathbb{R} \to \mathbb{R}^n$ of degree at most d satisfying

$$\begin{gathered} \phi_x([0,1]) \subset E \quad \text{and} \quad \phi_x(1) = x, \\ \varrho_v(\phi_x(t)) \ge M(1-t)^m \quad \text{for all } x \in E_0 \text{ and } t \in [0,1], \\ \bigcup_{x\in E_0} \phi_x([0,1]) = E. \end{gathered}$$

Applying Propositions 2.1, 2.5 and 1.1 we obtain the following

3.1. Theorem. *Let E be an m-UPC subset of $\mathbb{R}^n$ in the direction of v. Then for every $p \in \mathbb{C}[x_1, \ldots, x_n]$ with $\deg p \le k$ we have*

$$\|D_v p\|_E \le Ck^{2m}\|p\|_E,$$

where $C = \frac{1}{M}(2d)^{2m}$.

3.2. Remark. In the special case where $E = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le x^p\}$ with $p \ge 1$, Theorem 3.1 was proved by Goetgheluck [G].

3.3. Corollary. *Assume that there exist n linearly independent vectors $v_i \in S^{n-1}$ such that E is UPC in the direction of each v_i (with a constant m_i). Then there exists a constant $C = C(E)$ such that for each $p \in \mathbb{C}[x_1, \ldots, x_n]$ with $\deg p \le k$ the following Markov inequality holds*:

$$|\operatorname{grad} p(x)| \le Ck^{2m}\|p\|_E \quad \textit{for all } x \in E,$$

where $m = \max_{i=1,\ldots,n} m_i$.

3.4. Remark. If E is a UPC set in the direction of each $v \in S^{n-1}$ with $E_0 = E$, with the same family of polynomial mappings ϕ_x and with the same constants M and m, for each v, then

$$\operatorname{dist}(\phi_x(t), \mathbb{R}^n \setminus E) \ge M(1-t)^m \quad \text{for all } t \in [0,1],\ x \in E.$$

This is equivalent to the fact that E is UPC. In this case, by Theorem 3.1 we obtain

3.5. Corollary. *If E is an m-UPC subset of $\mathbb{R}^n$, then*

$$|\operatorname{grad} p(x)| \le Ck^{2m}\|p\|_E$$

for all $p \in \mathbb{C}[x_1, \ldots, x_n]$ with $\deg p \le k$, where $C = \frac{\sqrt{2}}{M}(2d)^{2m}$.

This corollary improves Pawłucki and Pleśniak's result from [PP1] where the Markov inequality for UPC sets was proved with constant $2m+2$.

We finish this section by proving a version of the Markov inequality for star-shaped sets.

3.6. Theorem. *Let Ω be a bounded, star-shaped (with respect to the origin) and symmetric domain in $\mathbb{R}^n$ and let $E = \overline{\Omega}$. Assume that*

$$\varrho_v(tx) \ge M(1-|t|)^m \quad \textit{for } t \in [-1,1],\ x \in \partial E,$$

where $M > 0$ and $m \ge 1$ are constants. If $p \in \mathbb{C}[x_1, \ldots, x_n]$ and $\deg p \le k$, then

$$|D_v p(x)| \le \sqrt{2}\, M^{-1/(2m)} k \varrho_v(x)^{-(1-1/(2m))}\|p\|_E \quad \textit{for } x \in \operatorname{int}(E)$$

and

$$\|D_v p\|_E \le \left(2 - \frac{1}{m}\right)^{m-1/2} \frac{m^{-1/2}}{M} k^{2m}\|p\|_E.$$

Proof. If $x \in \operatorname{int}(E)$, then $x = t_0 x_0$, where $t_0 \in [0,1)$ and $x_0 \in \partial E$. Thus we get $\varrho_v(tx) \ge M(1-|t|t_0)^m \ge M2^{-m}(\sqrt{1-t^2})^{2m}$, which implies

$$\sup_{0 \le r \le 1} \sqrt{1-t^2}\, \varrho_v(rx)^{-1} \le \sqrt{2}\, M^{-1/m} \varrho_v(x)^{-(1-1/(2m))}.$$

Applying Propositions 1.1 and 2.3 we obtain the first assertion of the theorem. We also have

$$\sup_{0\le r\le 1}\sqrt{1-r^2}\,(1-r|t|)^m \le \left(2-\frac{1}{m}\right)^{m-1/2} m^{-1/2}(1-t^2)^{-(m-1/2)}$$

for $t\in(-1,1)$. Hence we obtain, for all polynomials p with $\deg p\le k$,

$$|D_v p(tx)| \le k\frac{m^{-1/2}}{M}\left(2-\frac{1}{m}\right)^{m-1/2}(1-t^2)^{-(m-1/2)}\|p\|_E.$$

Applying Lemma 2.4 completes the proof.

3.7. Corollary. *Let* $E=\{x\in\mathbb{R}^n : f(x)\le 1\}$, *where* f *is a norm in* $\mathbb{R}^n$. *If* $v\in S^{n-1}$ *and* p *is a polynomial of degree* $\le k$, *then*

$$\|D_v p\|_E \le f(v)k^2\|p\|_E.$$

Proof. Let $x\in\partial E$, $t\in[-1,1]$ and $\tau\in\mathbb{R}$. If $|t|+f(v)|\tau|\le 1$, i.e.

$$|\tau|\le\frac{1-|t|}{f(v)},$$

then $f(tx+\tau v)\le 1$. So we have

$$\varrho_v(tx)\ge\frac{1}{f(v)}(1-|t|)$$

and we can apply Theorem 3.6.

3.8. Remark. It follows from the proof of Theorem 3.6 that the following implication holds: if there exist constants $M>0$ and $m\ge 1$ such that $\varrho_v(tx)\ge M(1-|t|)^m$ for $t\in[-1,1]$ and $x\in\partial E$, then there exist constants $C>0$ and $1/2\le\alpha<1$ such that $\sup_{0\le r\le 1}\sqrt{1-t^2}\varrho_v(rx)^{-1}\le C\varrho_v(x)^{-\alpha}$ for $x\in\operatorname{int}(E)$.

The converse implication is also true.

3.9. Proposition. *Let* E *be a compact, fat* ($\overline{\operatorname{int}(E)}=E$), *star-shaped and symmetric* (*with respect to the origin*) *subset of* $\mathbb{R}^n$. *Assume that*

$$\sup_{0\le r\le 1}\sqrt{1-r^2}\varrho_v(rx)^{-1}\le C\varrho_v(x)^{-\alpha}\quad \textit{for } x\in\operatorname{int}(E),$$

where $C>0$ *and* $1/2\le\alpha<1$ *are constants. Then*

$$\varrho_v(tx)\ge C^{-2m}2^{-2m^2}(1-|t|)^m\quad \textit{for } t\in[-1,1],\ x\in\partial E,$$

with $m=1/(2(1-\alpha))$.

Proof. Fix $x\in\operatorname{int}(E)$. By the assumptions,

$$\varrho_v(t^2x)\ge\frac{1}{C}\sqrt{1-t^2}\varrho_v(tx)^{\alpha}\ge\frac{1}{C}\sqrt{1-t^2}\left[\frac{1}{C}\sqrt{1-t^2}\varrho_v(x)^{\alpha}\right]^{\alpha},$$

which implies

$$\varrho_v(tx) \geq C^{-(1+\alpha)} 2^{-(1+\alpha)/2} (\sqrt{1-t^2})^{1+\alpha} \varrho_v(x)^{\alpha^2},$$

and, by recurrence,

$$\varrho_v(tx) \geq 2^{-(1+2\alpha+3\alpha^2+\ldots+k\alpha^{k-1}+k\alpha^k)/2} \left(\frac{\sqrt{1-t^2}}{C}\right)^{1+\alpha+\ldots+\alpha^k} \varrho_v(x)^{\alpha^{k+1}}.$$

Letting $k \to \infty$ gives

$$\varrho_v(tx) \geq C^{-2m} 2^{-2m^2} (1-t^2)^m \geq C^{-2m} 2^{-2m^2} (1-|t|)^m$$

for $x \in \operatorname{int}(E)$ and $t \in [-1,1]$. Since ϱ_v is upper semicontinuous, this inequality also holds for $x \in \partial E$. The proof is complete.

4. Examples

4.1. Example. Let $E = \{(x,y) \in \mathbb{R}^2 : |x| < 1,\ |y| \leq e^{-(1-|x|)^{-1}}\} \cup \{(-1,0),(1,0)\}$. If $v = (1,0)$, $(x,y) \in \partial E$ and $\phi(t) = t(x,y)$, then easy calculations show that

$$1 - |t| \geq \varrho_v(\phi(t)) \geq \frac{1}{2}(1-|t|).$$

By Theorem 3.6 we obtain

$$\|D_1 p\|_E \leq 2k^2 \|p\|_E,$$

where p is a polynomial of degree $\leq k$. However, applying a similar argument to that for Zerner's example [Z] one can prove that Markov's inequality on E does not hold for any positive constant m.

4.2. Example. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \geq 1$, $i = 1, \ldots, n$. Define

$$E_\alpha = \{x \in \mathbb{R}^n : |x_1|^{1/\alpha_1} + \ldots + |x_n|^{1/\alpha_n} \leq 1\}.$$

Let $e_1, \ldots, e_n$ be the standard orthonormal basis in $\mathbb{R}^n$. Then

$$\varrho_{e_i}(x) = \Big(1 - \sum_{j=1, j\neq i}^{n} |x_j|^{1/\alpha_j}\Big)^{\alpha_i} - |x_i|.$$

Let $\beta_i = \max_{j \neq i} \alpha_j$, $i = 1, \ldots, n$. We have

$$\begin{aligned}\varrho_{e_i}(tx) &= \Big(1 - \sum_{j=1, j\neq i}^{n} |x_j|^{1/\alpha_j} |t|^{1/\alpha_j}\Big)^{\alpha_i} - |t||x_i| \\ &\geq \Big(1 - |t|^{1/\beta_i} \sum_{j=1, j\neq i}^{n} |x_j|^{1/\alpha_j}\Big)^{\alpha_i} - |t|^{1/\beta_i} \Big(1 - \sum_{j=1, j\neq i}^{n} |x_j|^{1/\alpha_j}\Big)^{\alpha_i} \\ &\geq (1 - |t|^{1/\beta_i})^{\alpha_i} \geq A_i (1-|t|)^{\alpha_i},\end{aligned}$$

with $A_i = (\max_{j\neq i}\alpha_j)^{-\alpha_i}, i = 1,\ldots,n$, for $t\in[-1,1]$ and $x\in E_\alpha$. By Theorem 3.6 we obtain

$$\|D_i p\|_{E_\alpha} \le \left(2-\frac{1}{\alpha_i}\right)^{\alpha_i-1/2}\alpha_i^{-1/2}(\max_{j\neq i}\alpha_j)^{\alpha_i}k^{2\alpha_i}\|p\|_{E_\alpha}, \quad i=1,\ldots,n,$$

for all polynomials p of degree $\le k$.

This inequality is sharp in the case where $\alpha_1 = \ldots = \alpha_n = 1$ and generalizes the classical Markov inequality (see [B4]).

An easy calculation shows that we also have

$$\sup_{0\le r\le 1}\sqrt{1-r^2}\varrho_{e_i}(rx)^{-1} \le \max\left(1, \left(\frac{\beta_i}{\alpha_i}\right)^{1/2}\right)\varrho_{e_i}(x)^{-(1-1/(2\alpha_i))}$$

for $x\in \operatorname{int}(E_\alpha)$, $i=1,\ldots,n$. Thus, we obtain the following Bernstein–Markov inequality:

$$|D_i p(x)| \le \max\left(1, \left(\frac{1}{\alpha_i}\max_{j\neq i}\alpha_j\right)^{1/2}\right)k\varrho_{e_i}(x)^{-(1-1/(2\alpha_i))}\|p\|_{E_\alpha}$$

for $i=1,\ldots,n$, $x\in\operatorname{int}(E_\alpha)$, and $p\in\mathbb{C}[x_1,\ldots,x_n]$ with $\deg p\le k$.

4.3. Example. Let

$$E = \left\{(x,y)\in\mathbb{R}^2 : |x|\le 1, |y|\le (1-|x|)\left[1+\log\frac{1}{1-|x|}\right]^{-1}\right\}.$$

Let $e_1=(1,0), e_2=(0,1)$. One can check the following estimates:

$$\varrho_{e_1}(t(x,y)) \ge \frac{1}{2}(1-|t|)$$

and

$$\varrho_{e_2}(t(x,y)) \ge (1-|t|)\left[1+\log\frac{1}{1-|t|}\right]^{-1},$$

for $t\in[-1,1]$ and $(x,y)\in\partial E$. The first inequality implies

$$\|D_1 p\|_E \le 2k^2\|p\|_E$$

for any polynomial p of degree $\le k$. By the second inequality, we obtain

$$\begin{aligned} D_{e_2+}V_E(t(x,y)) &\le \sup_{0\le r\le 1}\sqrt{1-r^2}(1-r|t|)^{-1}\left[1+\log\frac{1}{1-r|t|}\right] \\ &\le (1-t^2)^{-1/2}\left[1+\log 2+\log\frac{1}{1-t^2}\right] \\ &\le (1-t^2)^{-1/2}\left[1+\sqrt{5}+\log\frac{1}{1-t^2}\right] \\ &\le (1-t^2)^{-1/2}(1+\sqrt{5})(1-t^2)^{-1/(1+\sqrt{5})}, \end{aligned}$$

for $t \in (-1, 1)$ and $(x, y) \in \partial E$. We now have, for every polynomial p with $\deg p \le k$,

$$|D_2 p(t(x, y))| \le (1 + \sqrt{5}) k^{2+2/(1+\sqrt{5})} \|p\|_E$$

for $t \in (-1, 1)$ and $(x, y) \in \partial E$, and

$$\begin{aligned} &|D_2 p(t(x, y))| \\ &\quad \le k(1 - t^2)^{-1/2} \\ &\qquad \times \min\left(1 + \sqrt{5} + \log \frac{1}{1 - t^2}, (1 + \sqrt{5}) k^{1+2/(1+\sqrt{5})} (1 - t^2)^{1/2}\right) \|p\|_E \\ &\quad \le k(1 - t^2)^{-1/2} (1 + \sqrt{5})(1 + \log k) \|p\|_E. \end{aligned}$$

Thus, we obtain $\|D_2 p\|_E \le (1 + \sqrt{5}) k^2 (1 + \log k) \|p\|_E$.

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