Some families of pseudo-processes

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Abstract. We introduce several types of notions of dispersive, completely unstable, Poisson unstable and Lagrange unstable pseudo-processes. We try to answer the question of how many (in the sense of Baire category) pseudo-processes with each of these properties can be defined on the space $\mathbb{R}^m$. The connections are discussed between several types of pseudo-processes and their limit sets, prolongations and prolongational limit sets. We also present examples of applications of the above results to pseudo-processes generated by differential equations.

I. Introduction. The notion of the pseudo-process is a direct generalization of the notion of the process introduced by Dafermos in [2].

Let $X$ be a non-empty set, $(G, +)$ be an abelian semi-group with neutral element 0, and $H$ be a sub-semi-group of $G$ such that $0 \in H$.

Definition 1.1 (see [6]). The quadruple $(X, G, H, \mu)$ is said to be a pseudo-process iff $\mu$ is a mapping from $G \times X \times H$ into $X$ such that

(1.1) $\mu(t, x, 0) = x$,
(1.2) $\mu(t + s, \mu(t, x, s), r) = \mu(t, x, r + s)$

for all $t \in G, x \in X, s, r \in H$.

Definition 1.2 (see [8]). The triple $(X, H, \pi)$ is said to be a pseudo-dynamical semi-system iff $\pi$ is a mapping from $H \times X$ into $X$ such that

(1.3) $\pi(0, x) = x$,
(1.4) $\pi(s, \pi(r, x)) = \pi(s + r, x)$

for all $x \in X, s, r \in H$.

It is known that we can replace a pseudo-process by a pseudo-dynamical semi-system (we will write briefly “a pseudo-dynamical system”) analogously to the transition from non-autonomous to autonomous systems of

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ordinary differential equations. For a given pseudo-process \((X, G, H, \mu)\) we define the pseudo-dynamical system \((Y, H, \pi)\), where

\[
Y := G \times X,
\]

\[
\pi(s, (t, x)) := (t + s, \mu(t, x, s)) \quad \text{for} \quad (s, (t, x)) \in H \times Y.
\]

In particular, we can reduce problems concerning stability for pseudo-processes to corresponding problems for pseudo-dynamical systems. This idea is presented in the paper of A. Pelczar [6].

However, we will not use this method in the present paper. Limit sets and prolongational limit sets are empty for the pseudo-dynamical system \((Y, H, \pi)\) defined in (1.5), (1.6). Therefore, systems defined in this way are always dispersive, completely unstable, Poisson unstable and Lagrange unstable. So, if for a given pseudo-process \(\mu\) we investigate problems associated with limit sets and prolongational limit sets it is necessary to consider the pseudo-process \(\mu\) itself, and not the pseudo-dynamical system \((Y, H, \pi)\) defined above.

Therefore we try to transfer the methods used for investigation of dynamical systems (see [5]) to pseudo-processes. We show differences and resemblances between the results presented in [5] and in this paper.

II. Connections between pseudo-processes and their limit sets, prolongations and prolongational limit sets. Unless otherwise stated, we assume throughout the paper that the triple \((X, G, H)\) satisfies the following assumption:

(A) \((X, d)\) is a metric space, \((G, +, \prec)\) is a topological, ordered, abelian semi-group with neutral element 0 and with topology induced by an ordering relation which does not admit the last element, \((H, +, \prec)\) is a sub-semi-group of \(G\) (of the same type as \(G\)).

Let \(\{s_n\} \subset H\) be a sequence of elements of \(H\). We say that \(s_n \to \infty\) if for every \(s \in H\) there is \(n_0 \in \mathbb{N}\) such that \(s \prec s_n\) for every \(n \geq n_0\).

Let \((X, G, H, \mu)\) be a pseudo-process and \((t, x) \in G \times X\).

**Definition 2.1** (see [7]). The set

\[
\Lambda_\mu(t, x) := \{y \in X : \exists\{s_n\} \subset H, s_n \to \infty \quad \text{such that} \quad \mu(t, x, s_n) \to y \text{ as } n \to \infty\}
\]

is called the limit set for \((t, x)\).

**Definition 2.2** (see [7]). The set

\[
D_\mu(t, x) := \{y \in X : \exists\{t_n\} \subset G, \exists\{x_n\} \subset X, \exists\{s_n\} \subset H \quad \text{such that} \quad t_n \to t, x_n \to x \text{ and } \mu(t_n, x_n, s_n) \to y \text{ as } n \to \infty\}
\]

is called the prolongation of the point \((t, x)\).
Analogously to the different types of prolongations of the point \((t,x)\)
(see the definitions of \(D^1_\mu(t,x)\) and \(D^2_\mu(t,x)\) in [7]) we can introduce

\textbf{Definition 2.3.} The sets

\[(2.3) \quad J_\mu(t,x) := \{y \in X : \exists \{t_n\} \subset G, \exists \{x_n\} \subset X, \exists \{s_n\} \subset H \]

such that \(t_n \to t, \, x_n \to x, \, s_n \to \infty\) and \(\mu(t_n, x_n, s_n) \to y\) as \(n \to \infty\),

\[(2.4) \quad J^1_\mu(t,x) := \{y \in X : \exists \{x_n\} \subset X, \exists \{s_n\} \subset H \text{ such that } x_n \to x, \, s_n \to \infty \text{ and } \mu(t_n, x_n, s_n) \to y\text{ as } n \to \infty\},

\[(2.5) \quad J^2_\mu(t,x) := \{y \in X : \exists \{t_n\} \subset G, \exists \{s_n\} \subset H \text{ such that } t_n \to t, \, s_n \to \infty \text{ and } \mu(t_n, x_n, s_n) \to y\text{ as } n \to \infty\}

are called the \textit{prolongational limit set}, the \((1)\)-\textit{prolongational limit set} and the \((2)\)-\textit{prolongational limit set} for \((t,x)\) respectively.

\textbf{Remark 2.1.} If a map \(\mu\) does not depend on the first variable then

\[J^1_\mu(t,x) = J_\mu(t,x) = J_\mu(0,x), \]

\[J^2_\mu(t,x) = \Lambda_\mu(t,x) = \Lambda_\mu(0,x)

for all \((t,x) \in G \times X\) (see also (3.1)).

\textbf{Definition 2.4.} The set

\[(2.6) \quad \mu[t,x] := \{\mu(t,x,s) : s \in H\}

is called the \textit{trajectory} of \(\mu\) which starts at \((t,x)\).

If we consider one fixed pseudo-process \(\mu\) we will write for short \(A(t,x), D(t,x), J(t,x), \ldots\) instead of \(A_\mu(t,x), D_\mu(t,x), J_\mu(t,x), \ldots\) respectively.

Let \((X,G,H,\mu)\) be a pseudo-process and \((t,x) \in G \times X\) be fixed.

\textbf{Theorem 2.1.} The sets \(A(t,x), D(t,x)\) and \(J(t,x)\) are closed.

\textbf{Proof.} We only prove the closedness of \(J(t,x)\). The proof of the closedness of the sets \(A(t,x), D(t,x)\) is presented in [7].

Let \(\{y_n\} \subset J(t,x)\) and \(y_n \to y\). From the definition of \(J(t,x)\) it follows that for every \(n \in \mathbb{N}\) there are sequences \(\{t^n_k\} \subset G, \{x^n_k\} \subset X, \{s^n_k\} \subset H\)

such that \(t^n_k \to t, \, x^n_k \to x, \, s^n_k \to \infty\) and \(\mu(t^n_k, x^n_k, s^n_k) \to y_n\) as \(k \to \infty\). Hence, for every \(n \in \mathbb{N}\) there is \(k_n \in \mathbb{N}\) such that

\[d(\mu(t^n_k, x^n_k, s^n_k), y_n) \leq 1/n \quad \text{for each } k \geq k_n \]

and

\[(2.7) \quad t_n := t^n_{k_n} \to t, \, x_n := x^n_{k_n} \to x, \, s_n := s^n_{k_n} \to \infty \quad \text{as } n \to \infty.

For every \(\varepsilon > 0\) there is \(n_0 \in \mathbb{N}\) such that for every \(n \geq n_0\) we have

\[d(\mu(t_n,x_n,s_n),y) \leq d(\mu(t_n,x_n,s_n),y_n) + d(y_n,y) \leq \varepsilon,\]
i.e. \( \mu(t_n, x_n, s_n) \to y \) as \( n \to \infty \). From (2.7) and (2.3) it follows that \( y \in J(t, x) \), which completes the proof.

**Remark 2.2.** The sets \( D^i(t, x) \) and \( J^i(t, x) \) \( (i = 1, 2) \) are also closed. The proof is analogous.

For any topological spaces \( Y \) and \( X \) we denote by \( \mathcal{F}(Y, X) (\mathcal{C}(Y, X)) \) the family of all maps (continuous maps) from \( Y \) into \( X \). Put

\[
\mathcal{F} := \{ \mu \in \mathcal{F}(G \times X \times H, X) : (X, G, H, \mu) \text{ is a pseudo-process} \},
\]

(2.8)

\[
\mathcal{F}_1 := \{ \mu \in \mathcal{F} : \mu \in \mathcal{C}([-t] \times X \times [-s], X) \text{ for each } (t, s) \in G \times H \},
\]

(2.9)

\[
\mathcal{F}_2 := \{ \mu \in \mathcal{F} : \text{ for every fixed } \tau \in H, \text{ the one-parameter family of maps } \mu(t, \cdot, \tau) : X \to X, \text{ with } t \in G, \text{ is equicontinuous} \},
\]

(2.10)

\[
\mathcal{F}_3 := \{ \mu \in \mathcal{F} : \mu \in \mathcal{C}(G \times X \times H, X) \}.
\]

(2.11)

**Remark 2.3.** The family \( \mathcal{F}_2 \) is the set of all maps for which the quadruple \((X, G, H, \mu)\) is a process in the sense of Dafermos (see [2]).

Let \((X, d)\) be a metric space. We define the function \( \varrho : \mathcal{F} \times \mathcal{F} \to [0, +\infty] \) by

\[
\varrho(\mu, \nu) := \sup\{d(\mu(t, x, s), \nu(t, x, s)) : (t, x, s) \in G \times X \times H\}
\]

for \( \mu, \nu \in \mathcal{F} \).

**Remark 2.4.** If \((X, d)\) is a metric space then \((\mathcal{F}, \varrho_1), (\mathcal{F}_i, \varrho_1) \) \( (i = 1, 2, 3) \) are metric spaces.

**Lemma 2.1.** If \((X, d)\) is a complete metric space then \((\mathcal{F}_i, \varrho_1) \) \( (i = 1, 2, 3) \) are complete metric spaces.

**Proof.** First we show that \((\mathcal{F}_1, \varrho_1)\) is complete if so is \((X, d)\). Let \( \{\mu_n\} \subset \mathcal{F}_1 \) be a Cauchy sequence. There is a function \( \mu \in \mathcal{F}(G \times X \times H, X) \) such that \( \{\mu_n\} \) is uniformly convergent to \( \mu \). Hence \( \mu \in \mathcal{C}([-t] \times X \times [-s], X) \) for each \( (t, s) \in G \times H \), because \( \mu_n \) has this property for every \( n \in \mathbb{N} \). We have

\[
|\mu_n(t + s, \mu_n(t, x, s), r) - \mu(t + s, \mu(t, x, s), r)|
\]

\[
\leq |\mu_n(t + s, \mu_n(t, x, s), r) - \mu(t + s, \mu_n(t, x, s), r)|
\]

\[
+ |\mu(t + s, \mu_n(t, x, s), r) - \mu(t + s, \mu(t, x, s), r)|
\]

for \( (t, x, s, r) \in G \times X \times H \times X \) and \((X, G, H, \mu_n)\) is a pseudo-process for \( n \in \mathbb{N} \), i.e. \( \mu_n \) satisfies (1.1), (1.2). Hence \((X, G, H, \mu)\) is a pseudo-process, so \( \mu \in \mathcal{F}_1 \).

Analogously we prove the completeness of \( \mathcal{F}_2 \) and \( \mathcal{F}_3 \).
For a non-empty metric space \((X,d)\) we denote—as usual—by \(2^X\) the family of all subsets of \(X\) and we put
\[
\text{Cl}(X) := \{ A \in 2^X : \overline{A} = A \}.
\]
We define a function \(\tilde{d} : 2^X \times 2^X \to \mathbb{R}\) by the formulae
\[
\tilde{d}(\emptyset, A) := \begin{cases} 0 & \text{for } A = \emptyset, \\ \infty & \text{for } A \in 2^X \setminus \{\emptyset\}, \end{cases}
\]
\[
\tilde{d}(A,B) := \max(\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)) \quad \text{for } A,B \in 2^X \setminus \{\emptyset\},
\]
where \(d(x,B) := \inf_{y \in B} d(x,y)\), i.e. \(\tilde{d}\) is the Hausdorff metric in \(\text{Cl}(X) \setminus \{\emptyset\}\) (see [3]).

**Lemma 2.2.** \((\text{Cl}(X),d_1)\) with \(d_1(A,B) := \min(1,\tilde{d}(A,B))\) for \(A,B \in \text{Cl}(X)\) is a metric space.

In the sequel we shall consider pseudo-processes in \(X = \mathbb{R}^m\).

**Theorem 2.2.** For all \(\mu, \nu \in \mathcal{F}, \delta \in \mathbb{R}\) and \(W := \Lambda, D, J\) or \(W \mu(t,x) := [\mu(t,x)]\) we have the implication
\[
\varrho(\mu,\nu) \leq \delta \Rightarrow \tilde{d}(W \mu(t,x), W \nu(t,x)) \leq \delta \quad \text{for each } (t,x) \in G \times X.
\]

**Proof.** We prove this theorem for \(W = \Lambda\). The other cases are proved in the same way.

Let \(\mu, \nu \in \mathcal{F}\) and \(\varrho(\mu, \nu) \leq \delta\). First we suppose that \(y \in A_\mu(t,x) \neq \emptyset\). In view of (2.1) there is a sequence \(\{s_n\} \subset H\) such that \(s_n \to \infty\) and \(\mu(t,x,s_n) \to y\) as \(k \to \infty\). So there is \(r > 0\) such that \(\mu(t,x,s_n) \in B(y,r)\) for all \(n \in \mathbb{N}\). For every \(n \in \mathbb{N}\) we have
\[
d(\nu(t,x,s_n), y) \leq d(\nu(t,x,s_n), \mu(t,x,s_n)) + d(\mu(t,x,s_n), y) \leq \delta + r.
\]
Hence, because of the boundedness of the sequence \(\{\nu(t,x,s_n)\}\) there are \(z \in \mathbb{R}^m\) and a subsequence \(\{\nu(t,x,s_{n_k})\}\) such that
\[
\nu(t,x,s_{n_k}) \to z \in A_\nu(t,x) \neq \emptyset \quad \text{as } k \to \infty.
\]
For every \(\varepsilon > 0\) there is \(k_0 \in \mathbb{N}\) such that for each \(k \geq k_0\) we have
\[
d(y,z) \leq d(y,\mu(t,x,s_{n_k})) + d(\mu(t,x,s_{n_k}), \nu(t,x,s_{n_k})) + d(\nu(t,x,s_{n_k}), z) \leq \delta + \varepsilon,
\]
i.e. \(d(y,z) \leq \delta\). So
\[
d(y,A_\nu(t,x)) \leq d(y,z) \leq \delta \quad \text{for every } y \in A_\mu(t,x).
\]
Analogously we can prove that
\[
d(z,A_\mu(t,x)) \leq d(z,y) \leq \delta \quad \text{for every } z \in A_\nu(t,x).
\]
Hence we obtain
\[ \tilde{d}(A_\mu(t,x), A_\nu(t,x)) \leq \delta. \]
Let now \( A_\mu(t,x) = \emptyset \). The hypothesis that there is \( \nu \in \mathcal{F} \) such that \( \rho(\mu, \nu) = \delta < \infty \) and \( A_\nu(t,x) \neq \emptyset \) gives a contradiction in view of the first part of the proof. This proves the theorem.

As in the theory of dynamical systems (see [5]) we can prove the following

**Theorem 2.3.** For each \((t,x) \in G \times X\) and \( W := A, D, J \) or \( W_\mu(t,x) := \mu[t,x] \) the map
\[ W(t,x) : F \ni \mu \rightarrow W_\mu(t,x) \in \text{Cl}(X) \]
is uniformly continuous from \((F, \rho_1)\) to \((\text{Cl}(X), d_1)\).

**Remark 2.5.** The theorems analogous to Theorems 2.2 and 2.3 hold for \( W := D^i, J^i \) \((i = 1, 2)\) (see [7] and (2.4), (2.5) in this paper).


**Definition 3.1.** A pseudo-process \((X, G, H, \mu)\) is called
(i) **dispersive** iff for each \( x \in X \),
\[ J_\mu(t,x) = \emptyset \quad \text{for every } t \in G, \]
(ii) **completely unstable** iff each \( x \in X \) is wandering, i.e.
\[ x \not\in J_\mu(t,x) \quad \text{for every } t \in G, \]
(iii) **Poisson unstable** iff for each \( x \in X \),
\[ x \not\in A_\mu(t,x) \quad \text{for every } t \in G, \]
(iv) **Lagrange unstable** iff for each \( x \in X \),
\[ \mu[t,x] \quad \text{is not compact for every } t \in G, \]
(v) **Lagrange stable** iff for each \( x \in X \),
\[ \mu[t,x] \quad \text{is compact for every } t \in G. \]

We can define corresponding weak notions by replacing “for every \( t \in G \)” by “there is \( t \in G \)”.

**Definition 3.2.** A pseudo-process \((X, G, H, \mu)\) is called **weakly dispersive** iff for each \( x \in X \) there is \( t \in G \) such that \( J_\mu(t,x) = \emptyset \).

If we replace the set \( J_\mu(t,x) \) by \( J^i_\mu(t,x) \) we get the definition of \( (i) \)-dispersive or \( (i) \)-weakly dispersive pseudo-processes \((i = 1, 2)\).

These definitions agree with the analogous ones for dynamical systems (see [1], [8]).
Let \((X, G, H, \mu)\) be a pseudo-process and suppose \(\mu\) does not depend on the first variable. Put

\[
\pi(s, x) := \mu(t, x, s) \quad \text{for } (t, x, s) \in G \times X \times H.
\]

The pseudo-dynamical system \((X, H, \pi)\) defined in this way is dispersive, completely unstable, Poisson unstable, Lagrange unstable or Lagrange stable if and only if so is the pseudo-process \((X, G, H, \mu)\).

We introduce the families of all maps \(\mu\) for which the corresponding pseudo-processes have one of these properties:

\[
\begin{align*}
D &:= \{\mu \in F : (X, G, H, \mu) \text{ is dispersive}\}, \\
K &:= \{\mu \in F : (X, G, H, \mu) \text{ is completely unstable}\}, \\
\tilde{P} &:= \{\mu \in F : (X, G, H, \mu) \text{ is Poisson unstable}\}, \\
\tilde{L} &:= \{\mu \in F : (X, G, H, \mu) \text{ is Lagrange unstable}\}, \\
\mathcal{L} &:= \{\mu \in F : (X, G, H, \mu) \text{ is Lagrange stable}\}.
\end{align*}
\]

Remark 3.1. Directly from the definitions (3.2)–(3.4) it follows that \(D \subset K \subset \tilde{P}\).

Remark 3.2. \(K \setminus \tilde{L} \neq \emptyset\), so the inclusion \(\tilde{P} \subset \tilde{L}\) is not true in the theory of pseudo-processes, in contrast to the theory of dynamical systems (see [5]).

Example 3.1. Let \((\mathbb{R}, \mathbb{R}, \mathbb{R}^+, \mu)\) be the pseudo-process generated by the equation

\[
x' = \frac{bt}{(t^2 + a)(1 + \ln^2(t^2 + a))} \quad (a, b > 0).
\]

Then \(\mu[t, x]\) is compact for every \((t, x) \in \mathbb{R}^2\). However, \(x \not\in \Lambda(t, x) = J(t, x) \neq \emptyset\) for all \((t, x) \in \mathbb{R}^2\), so \(\mu \in K \cap \mathcal{L} \neq \emptyset\) (see (3.3), (3.6)). Such a situation is impossible in the theory of dynamical systems.

Remark 3.3. We also have

\[
D_w \subset K_w \subset \tilde{P}_w \quad \text{and} \quad K_w \cap \mathcal{L}_w \neq \emptyset,
\]

where \((w)\) denotes a weak condition. For example,

\[
K_w := \{\mu \in F : (X, G, H, \mu) \text{ is weakly completely unstable}\}.
\]

Remark 3.4. The inclusions \(D \subset \tilde{L}, D_w \subset \tilde{L}_w\) are evident because for \((t, x) \in G \times X\) such that \(\mu[t, x]\) is compact we get \(\Lambda_\mu(t, x) \neq \emptyset\).

We have the same results for the families corresponding to the \((i)\)-prolongational limit sets \((i = 1, 2)\):

\[
D^i := \{\mu \in F : (X, G, H, \mu) \text{ is } (i)\text{-dispersive}\}.
\]
IV. A classification of pseudo-processes. Let \( X = \mathbb{R}^m \). In the set \( \mathcal{F} \) (see (2.8)) we introduce an equivalence relation \( S \). If \( \mu, \nu \in \mathcal{F} \) then
\[
(\mu, \nu) \in S \iff \varrho(\mu, \nu) < \infty,
\]
where \( \varrho \) is defined by (2.12). We denote by \( F_\mu \) the \( S \)-equivalence class of \( \mu \in \mathcal{F} \), i.e.
\[
\mathcal{F}/S := \{ F_\mu : \mu \in \mathcal{F} \}.
\]

Remark 4.1. If \( F_* \subset \mathcal{F} \) and \( \varrho_* := \varrho|_{F_*} \) gives a metric in \( F_* \) then \( F_* \subset F_\mu \) for every \( \mu \in F_* \). That is, for every \( \mu \in \mathcal{F} \) the \( S \)-equivalence class \( F_\mu \) is the largest subset \( F_* \) of \( \mathcal{F} \) (in the sense of inclusion) for which the restriction \( \varrho_* \) is a metric and \( \mu \in F_* \).

Theorem 4.1. The spaces \( \mathcal{F} \) and \( \mathcal{F}_i \) \( (i = 1, 2, 3) \) endowed with the uniform convergence topology are not connected (see (2.8)–(2.11)).

Proof. Let \( \mu \in \mathcal{F} \) and \( B(\mu, r) := \{ \nu \in \mathcal{F} : \varrho(\mu, \nu) < r \} \). Then \( F_\mu = \bigcup \{ B(\mu, 1) : \mu \in F_\mu \} \) and \( \mathcal{F} \setminus F_\mu = \bigcup \{ F_\nu : \nu \notin F_\mu \} \). So \( F_\mu \) is open and closed in the space \( (\mathcal{F}, \varrho_1) \), where \( \varrho_1 \) is defined by (2.13). The set \( F_\mu \cap \mathcal{F}_i \) is open and closed in the space \( (\mathcal{F}_i, \varrho_1) \) \( (i = 1, 2, 3) \) (see Lemma 2.1). This finishes the proof.

Let \( \{ \chi_{tx} \subset \mathcal{F} : (t, x) \in G \times X \} \) be a family satisfying the condition
\[
(\mu \in \chi_{tx} \iff F_\mu \subset \chi_{tx}) \quad \text{for every} \quad (t, x) \in G \times X.
\]

Lemma 4.1. Let \( (t, x) \in G \times X, T \subset G, Y \subset X \). The sets \( \chi_{tx}, \bigcap \{ \chi_{tx} : t \in T \} : x \in Y \} \) and \( \bigcup \{ \chi_{tx} : t \in T \} : x \in Y \} \) are open and closed in \( (\mathcal{F}, \varrho_1) \).

This follows from condition (C) and the fact that the set \( F_\mu \) is open.

Analogously to the theory of dynamical systems we show that the families
\[
\mathcal{P}_{tx} := \{ \mu \in \mathcal{F} : x \in A_\mu(t, x) \}, \quad (t, x) \in G \times X,
\]
\[
\mathcal{C}_i \setminus \mathcal{K}_{tx} := \{ \mu \in \mathcal{F} : x \in J_\mu(t, x) \}, \quad (t, x) \in G \times X,
\]
do not satisfy condition (C).

For other examples we refer the reader to [5].

By Theorem 2.2 we deduce that the families
\[
\mathcal{A}_{tx} := \{ \mu \in \mathcal{F} : \overline{[t, x]} \text{ compact} \}, \quad (t, x) \in G \times X,
\]
\[
\mathcal{B}_{tx} := \{ \mu \in \mathcal{F} : J_\mu(t, x) \neq \emptyset \}, \quad (t, x) \in G \times X,
\]
satisfy condition (C).

From the above we obtain some important results on the families of pseudo-processes defined in the third section.
Theorem 4.2. Let $W := \mathcal{D}, \mathcal{D}_w, \tilde{\mathcal{L}}_w, \mathcal{L}, \mathcal{L}_w, \mathcal{D}_i$ or $\mathcal{D}_w^i$ ($i = 1, 2$). The set $W$ is both open and closed in the space $\mathcal{F}$ endowed with the uniform convergence topology.

Proof. This follows directly from (4.2) and (4.3). We have, for example,\[ \mathcal{F} \setminus \mathcal{D} = \bigcup \{ B_{tx} : (t, x) \in G \times X \}, \]
\[ \mathcal{L}_w = \bigcap \{ \bigcup \{ A_{tx} : t \in G \} : x \in X \}. \]
In view of Lemma 4.1 this proves closedness and openness of the sets $\mathcal{D}$ and $\mathcal{L}_w$. The proof for the remaining sets is similar.

Corollary 4.1. The set $\mathcal{D}$ is not dense in $\mathcal{K}$ because $\mathcal{D} \neq \mathcal{K}$.

In virtue of theorems of Baire category theory (see [4]) and from Lemma 2.1 we get

Theorem 4.3. Let $W := \mathcal{D}, \mathcal{D}_w, \tilde{\mathcal{L}}_w, \mathcal{L}, \mathcal{L}_w, \mathcal{D}_i$ and $\mathcal{D}_w^i$ ($i = 1, 2$). The set $W \cap F_i$ is of the second Baire category in the space $(F_i, \varrho_i)$ but it is not residual in this space ($i = 1, 2, 3$).

Corollary 4.2. Let $W := \mathcal{K}, \mathcal{K}_w, \mathcal{K}_w^j, P$ or $\mathcal{P}_w$ ($j = 1, 2$). The set $W \cap F_i$ is of the second Baire category in $(F_i, \varrho_i)$ ($i = 1, 2, 3$).

We can prove that pseudo-processes are either dispersive (Lagrange unstable, Lagrange stable) for all functions belonging to $F_\mu$ or are not dispersive (Lagrange unstable, Lagrange stable) for all these functions. We have

Theorem 4.4. Let $F_\ast \subset \mathcal{F}$ and suppose that $\varrho_\ast := \varrho|_{F_\ast}$ gives a metric in $F_\ast$. Then $F_\ast \cap W \neq \emptyset \iff F_\ast \subset W$ for $W := \mathcal{D}, \mathcal{D}_w, \tilde{\mathcal{L}}_w, \mathcal{L}, \mathcal{L}_w, \mathcal{D}_i$ and $\mathcal{D}_w^i$ ($i = 1, 2$).

Proof. We prove this assertion for $W := \tilde{\mathcal{L}}_w$. The other cases are similar. Let $\mu \in F_\ast$ and $\mu \notin \tilde{\mathcal{L}}_w$. By (4.2) we have
\[ \mathcal{F} \setminus \tilde{\mathcal{L}}_w = \bigcup \{ \bigcap \{ A_{tx} : t \in G \} : x \in X \}. \]
Hence there is $x_0 \in X$ such that $\mu \in A_{tx_0}$ for every $t \in G$. In view of Remark 4.1, $\mu \in F_\ast \subset F_\mu$ and because $A_{tx}$ satisfies condition (C), $F_\mu \subset A_{tx_0}$ for every $t \in G$. So $F_\ast \subset \mathcal{F} \setminus \tilde{\mathcal{L}}_w$, which finishes the proof for $W := \tilde{\mathcal{L}}_w$.

Corollary 4.3. Let $W := \mathcal{D}, \tilde{\mathcal{L}}, \mathcal{L}$. In the quotient set $\mathcal{F}/S$ we can introduce the following equivalence relation:
\[ F_\mu(W) F_\nu \iff \mu, \nu \in W \text{ or } \mu, \nu \notin W. \]
Of course, we can also define in $\mathcal{F}/S$ other relations of this type. For example,
\[ F_\mu \sim F_\nu \leftrightarrow \mu, \nu \in \mathcal{L} \text{ or } \mu, \nu \in \tilde{\mathcal{L}} \text{ or } \mu, \nu \notin \mathcal{L} \cup \tilde{\mathcal{L}}. \]
From Theorem 4.4 it follows that these relations are well defined, i.e. their definitions are independent of the choice of representatives of the classes $F_\mu, F_\nu$.

V. Examples. The results of Section IV can be applied to processes generated by differential equations.

**Definition 5.1.** We say that a process $(\mathbb{R}^m, \mathbb{R}, \mathbb{R}_+, \mu)$ (we will write briefly $\mu$) is generated by a differential equation
(5.1) 
\[ x' = f(t, x) \]
if for every $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^m$ there exists exactly one, saturated to the right, solution $\varphi(t_0, x_0, \cdot)$ of the Cauchy problem
(5.2) 
\[ x' = f(t, x), \quad x(t_0) = x_0, \]
defined on the interval $[t_0, \infty)$ and
(5.3) 
\[ \mu(t, x, \tau) = \varphi(t, x, t + \tau) \]
for every $(t, x) \in \mathbb{R} \times \mathbb{R}^m, \tau \in \mathbb{R}_+$.

**Example 5.1.** We consider the differential equation
(\varepsilon) 
\[ x' = f_\varepsilon(t, x), \]
where $f_\varepsilon(t, x) = \varepsilon$ for every $(t, x) \in \mathbb{R}^2 \ (\varepsilon \in \mathbb{R}_+)$. We have $\sup\{|f_\varepsilon(t, x) - f_0(t, x)| : (t, x) \in \mathbb{R}^2\} = \varepsilon$, but for the process $\mu_\varepsilon$ generated by the equation (\varepsilon) we get $\rho(\mu_\varepsilon, \mu_0) = \infty$ for $\varepsilon \neq 0$. It is easily seen that $\mu_\varepsilon \in \mathcal{D} \cap \tilde{\mathcal{L}}$ for $\varepsilon \neq 0$ but $\Lambda_{\mu_\varepsilon}(t, x) = J_{\mu_\varepsilon}(t, x) = \mu_0[t, x] = \{x\}$ for every $(t, x) \in \mathbb{R}^2$.

The above example shows that a small change of the right hand side of a differential equation can change the type of the process generated by this equation. This difficulty exists even for dynamical systems.

However, we can change the right hand side of a differential equation in a special way.

Let $x, \tilde{x} \in \mathbb{R}^m$ and $x = (x_1, \ldots, x_m), \ \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m)$. We will write
\[ x \leq \tilde{x} \text{ if } x_k \leq \tilde{x}_k \text{ for } k = 1, \ldots, m, \]
and for every fixed $i \in \{1, \ldots, m\}$,
\[ x \leq \tilde{x} \text{ if } x \leq \tilde{x} \text{ and } x_i = \tilde{x}_i. \]
**Definition 5.2 (see [9]).** A function \( f = (f_1, \ldots, f_m) \) from \( \mathbb{R} \times \mathbb{R}^m \) to \( \mathbb{R}^m \) is said to satisfy condition \((W_+)\) if for every \( i \in \{1, \ldots, m\} \) and \( x, \bar{x} \in \mathbb{R}^m \),

\[
(W_+) \quad x \leq \bar{x} \Rightarrow f_i(t, x) \leq f_i(t, \bar{x}) \text{ for } t \in \mathbb{R}.
\]

**Lemma 5.1 (see [9]).** Assume that \( f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \) is continuous and satisfies condition \((W_+)\) and \( \mu \) is the process generated by the differential equation \((5.1)\). Let \( (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^m \), set

\[
\varphi(t) := \mu(t_0, x_0, t - t_0) \quad \text{for every } t \geq t_0
\]

and suppose a function \( \psi \) from \( \mathbb{R} \) into \( \mathbb{R}^m \) is differentiable and satisfies the initial condition \( \psi(t_0) = x_0 \). Then

(i) \( \psi'(t) \leq f(t, \psi(t)) \) for \( t \geq t_0 \Rightarrow \psi(t) \leq \varphi(t) \) for \( t \geq t_0 \),

(ii) \( \psi'(t) \geq f(t, \psi(t)) \) for \( t \geq t_0 \Rightarrow \psi(t) \geq \varphi(t) \) for \( t \geq t_0 \).

From the above we get

**Theorem 5.1.** Assume that \( f_i \) \((i = 1, 2, 3)\) are continuous functions from \( \mathbb{R} \times \mathbb{R}^m \) into \( \mathbb{R}^m \), \( f_i \) \((i = 1, 2)\) satisfy condition \((W_+)\) and

\[
f_1(t, x) \leq f_3(t, x) \leq f_2(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^m.
\]

Denote by \( \mu_i \) the process generated by the differential equation \( x' = f_i(t, x) \) \((i = 1, 2, 3)\). Then

\[
\mu_1 \in F_{\mu_2} \Rightarrow F_{\mu_1} = F_{\mu_2} = F_{\mu_3}.
\]

**Proof.** This will be proved by showing that

\[
\mu_1(t, x, \tau) \leq \mu_3(t, x, \tau) \leq \mu_2(t, x, \tau)
\]

for every \((t, x, \tau) \in \mathbb{R} \times \mathbb{R}^m, \tau \in \mathbb{R}_+\).

Fix \((t_0, x_0) \in \mathbb{R} \times \mathbb{R}^m\) and \( \tau \in \mathbb{R}_+ \). Denote by \( \varphi_i(t_0, x_0, \cdot) \) the solution of the Cauchy problem \( x' = f_i(t, x), x(t_0) = x_0 \). By Lemma 5.1,

\[
\varphi_1(t_0, x_0, t) \leq \varphi_3(t_0, x_0, t) \leq \varphi_2(t_0, x_0, t) \quad \text{for every } t \geq t_0.
\]

In view of the definition of the process \( \mu_i \) \((see (5.3))\) we have

\[
\mu_i(t_0, x_0, \tau) = \varphi_i(t_0, x_0, \tau + t_0) \quad (i = 1, 2, 3),
\]

which finishes the proof.

**Corollary 5.1.** Assume that \( f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \) is continuous and there exist continuous functions \( g, f_1, f_2 \) from \( \mathbb{R} \) into \( \mathbb{R}^m \) for which

\[
f_1(t) \leq f(t, x) - g(t) \leq f_2(t) \quad \text{for every } (t, x) \in \mathbb{R} \times \mathbb{R}^m
\]

and the function \( \alpha \to \int_0^\alpha f_i(s) \, ds \) is bounded \((i = 1, 2)\). Denote by \( \mu, \nu \) the processes generated by the differential equations \( x' = f(t, x), x' = g(t) \) respectively. Then \( \mu \in F_{\nu} \).
Example 5.2. Let $a, b, c_i \in \mathbb{R}^m \ (i = 1, \ldots, m)$. Denote by $\mu, \nu$ the processes generated by the differential equations

$$x' = f(t, x) := \left( \sum_{i=1}^{m} c_i \cos x_i \right) \left( 1 + t^2 \right)^{-1} + a + b \sin t,$$

$$x' = g(t) := a + b \sin t$$

respectively. There is $k \in \mathbb{R}^m$ such that for every $(t, x) \in \mathbb{R} \times \mathbb{R}^m$ we have

$$-\frac{k}{1 + t^2} \leq f(t, x) - g(t) \leq \frac{k}{1 + t^2}.$$ 

According to Corollary 5.1 we get $\mu \in F_\nu$. So, if $a \neq 0$ then $\mu \in \mathcal{D} \cap \tilde{\mathcal{L}}$ and if $a = 0$ then $\mu \in \mathcal{L}$.

Define

$$(5.4) \quad \mathcal{P} := \{ \mu \in \mathcal{F} : (X, G, H, \mu) \text{ is Poisson stable, i.e.} \}
\quad x \in A_\mu(t, x) \text{ for every } (t, x) \in \mathbb{R} \times X \}. $$

Remark 5.1. Let the assumptions of Theorem 5.1 be satisfied and suppose that for every $(t, x) \in \mathbb{R} \times \mathbb{R}^m$ there exists a sequence $\{\tau_n(t, x)\} \subset \mathbb{R}_+$ such that $\tau_n(t, x) \to \infty$ and $\mu_i(t, x, \tau_n(t, x)) \to x \ (i = 1, 2)$ as $n \to \infty$. Then $\mu_3 \in \mathcal{P}$.

Example 5.3. Let $a > 1, b \in \mathbb{R}, v$ be a continuous bounded function from $\mathbb{R}$ into $\mathbb{R}$,

$$w(t) := c_s t^s + \ldots + c_1 t + c_0 \quad (c_i \in \mathbb{R}, \ i = 1, \ldots, s, \ s \in \mathbb{N}),$$

$$g(t) := w'(t)e^{-w(t)},$$

$$f(t, x) := \frac{bv(x) \cos t}{(1 + \ln^2 (\sin t + a))(\sin t + a)} + g(t)$$

for $t, x \in \mathbb{R}$. There exist $k_i \in \mathbb{R} \ (i = 1, 2)$ such that for

$$f_i(t) := \frac{k_i \cos t}{(1 + \ln^2 (\sin t + a))(\sin t + a)}$$

we have

$$f_1(t) \leq f(t, x) - g(t) \leq f_2(t) \quad \text{for } t, x \in \mathbb{R}. $$

Denote by $\mu, \nu, \mu_i \ (i = 1, 2)$ the processes generated by the differential equations

$$x' = f(t, x), \quad x' = g(t), \quad x' = f_i(t) \ (i = 1, 2)$$

respectively. Because the assumptions of Corollary 5.1 are satisfied we get $\mu \in F_\nu$. So, for $s \neq 0, c_s > 0$ we have

$$x \notin J_\nu(t, x) = A_\nu(t, x) \neq \emptyset,$$
Some families of pseudo-processes

hence $\nu \in K \cap L$ and $\mu \in F_\nu \subset L$. If $s \neq 0$, $c_s < 0$ then $\mu \in F_\nu \subset D \cap \tilde{L}$. For $s = 0$ we get $g \equiv 0$. Now we see that $\mu_i \in L \cap P$ (see (5.4)) and

$$\mu_i(t, x, 2n\pi) \to x \quad \text{as } n \to \infty,$$

for every $(t, x) \in \mathbb{R}^2$, $i = 1, 2$. In view of Theorem 5.1 and Remark 5.1 we have $\mu \in L \cap P$ for $s = 0$.

Remark 5.2. If a process $\mu$ does not depend on the first variable we have the dynamical system $(X, H, \pi)$ defined by (3.1). In this case for other examples we refer the reader to [5].

References