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Some families of pseudo-processes

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Abstract. We introduce several types of notions of dispersive, completely unstable, Poisson unstable and Lagrange unstable pseudo-processes. We try to answer the question of how many (in the sense of Baire category) pseudo-processes with each of these properties can be defined on the space $\mathbb{R}^m$. The connections are discussed between several types of pseudo-processes and their limit sets, prolongations and prolongational limit sets. We also present examples of applications of the above results to pseudo-processes generated by differential equations.

I. Introduction. The notion of the pseudo-process is a direct generalization of the notion of the process introduced by Dafermos in [2].

Let X be a non-empty set, $(G,+)$ be an abelian semi-group with neutral element 0, and H be a sub-semi-group of G such that $0 \in H$.

Definition 1.1 (see [6]). The quadruple (X, G, H, μ) is said to be a *pseudo-process* iff μ is a mapping from $G \times X \times H$ into X such that

$$\mu(t, x, 0) = x, \tag{1.1}$$

$$\mu(t+s, \mu(t, x, s), r) = \mu(t, x, r+s) \tag{1.2}$$

for all $t \in G$, $x \in X$, $s, r \in H$.

Definition 1.2 (see [8]). The triple (X, H, π) is said to be a *pseudo-dynamical semi-system* iff π is a mapping from $H \times X$ into X such that

$$\pi(0, x) = x, \tag{1.3}$$

$$\pi(s, \pi(r, x)) = \pi(s+r, x) \tag{1.4}$$

for all $x \in X$, $s, r \in H$.

It is known that we can replace a pseudo-process by a pseudo-dynamical semi-system (we will write briefly "a pseudo-dynamical system") analogously to the transition from non-autonomous to autonomous systems of

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ordinary differential equations. For a given pseudo-process (X, G, H, μ) we define the pseudo-dynamical system (Y, H, π), where

$$Y := G \times X, \tag{1.5}$$

$$\pi(s,(t,x)) := (t+s, \mu(t,x,s)) \quad \text{for } (s,(t,x)) \in H \times Y. \tag{1.6}$$

In particular, we can reduce problems concerning stability for pseudo-processes to corresponding problems for pseudo-dynamical systems. This idea is presented in the paper of A. Pelczar [6].

However, we will not use this method in the present paper. Limit sets and prolongational limit sets are empty for the pseudo-dynamical system (Y, H, π) defined in (1.5), (1.6). Therefore, systems defined in this way are always dispersive, completely unstable, Poisson unstable and Lagrange unstable. So, if for a given pseudo-process μ we investigate problems associated with limit sets and prolongational limit sets it is necessary to consider the pseudo-process μ itself, and not the pseudo-dynamical system (Y, H, π) defined above.

Therefore we try to transfer the methods used for investigation of dynamical systems (see [5]) to pseudo-processes. We show differences and resemblances between the results presented in [5] and in this paper.

II. Connections between pseudo-processes and their limit sets, prolongations and prolongational limit sets. Unless otherwise stated, we assume throughout the paper that the triple (X, G, H) satisfies the following assumption:

(A) (X, d) is a metric space, $(G, +, \prec)$ is a topological, ordered, abelian semi-group with neutral element 0 and with topology induced by an ordering relation which does not admit the last element, $(H, +, \prec)$ is a sub-semi-group of G (of the same type as G).

Let $\{s_n\} \subset H$ be a sequence of elements of H. We say that $s_n \to \infty$ if for every $s \in H$ there is $n_0 \in \mathbb{N}$ such that $s \prec s_n$ for every $n \geq n_0$.

Let (X, G, H, μ) be a pseudo-process and $(t, x) \in G \times X$.

Definition 2.1 (see [7]). The set

$$\Lambda_\mu(t,x) := \{y \in X : \exists \{s_n\} \subset H, s_n \to \infty \text{ such that } \mu(t,x,s_n) \to y \text{ as } n \to \infty\} \tag{2.1}$$

is called the *limit set* for (t, x).

Definition 2.2 (see [7]). The set

$$D_\mu(t,x) := \{y \in X : \exists \{t_n\} \subset G,\ \exists \{x_n\} \subset X,\ \exists \{s_n\} \subset H \text{ such that } t_n \to t,\ x_n \to x \text{ and } \mu(t_n,x_n,s_n) \to y \text{ as } n \to \infty\} \tag{2.2}$$

is called the *prolongation* of the point (t, x).

Analogously to the different types of prolongations of the point (t,x) (see the definitions of $D^1_\mu(t,x)$ and $D^2_\mu(t,x)$ in [7]) we can introduce

Definition 2.3. The sets

$$J_\mu(t,x) := \{y \in X : \exists\{t_n\} \subset G,\ \exists\{x_n\} \subset X,\ \exists\{s_n\} \subset H \text{ such that } t_n \to t,\ x_n \to x,\ s_n \to \infty \text{ and } \mu(t_n,x_n,s_n) \to y \text{ as } n \to \infty\}, \tag{2.3}$$

$$J^1_\mu(t,x) := \{y \in X : \exists\{x_n\} \subset X,\ \exists\{s_n\} \subset H \text{ such that } x_n \to x,\ s_n \to \infty \text{ and } \mu(t,x_n,s_n) \to y \text{ as } n \to \infty\}, \tag{2.4}$$

$$J^2_\mu(t,x) := \{y \in X : \exists\{t_n\} \subset G,\ \exists\{s_n\} \subset H \text{ such that } t_n \to t,\ s_n \to \infty \text{ and } \mu(t_n,x,s_n) \to y \text{ as } n \to \infty\} \tag{2.5}$$

are called the *prolongational limit set*, the (1)-*prolongational limit set* and the (2)-*prolongational limit set* for (t,x) respectively.

Remark 2.1. If a map μ does not depend on the first variable then

$$J^1_\mu(t,x) = J_\mu(t,x) = J_\mu(0,x),$$
$$J^2_\mu(t,x) = \Lambda_\mu(t,x) = \Lambda_\mu(0,x)$$

for all $(t,x) \in G \times X$ (see also (3.1)).

Definition 2.4. The set

$$\mu[t,x] := \{\mu(t,x,s) : s \in H\} \tag{2.6}$$

is called the *trajectory* of μ which starts at (t,x).

If we consider one fixed pseudo-process μ we will write for short $\Lambda(t,x)$, $D(t,x), J(t,x), \ldots$ instead of $\Lambda_\mu(t,x), D_\mu(t,x), J_\mu(t,x), \ldots$ respectively.
Let (X,G,H,μ) be a pseudo-process and $(t,x) \in G \times X$ be fixed.

Theorem 2.1. *The sets $\Lambda(t,x), D(t,x)$ and $J(t,x)$ are closed.*

Proof. We only prove the closedness of $J(t,x)$. The proof of the closedness of the sets $\Lambda(t,x), D(t,x)$ is presented in [7].

Let $\{y_n\} \subset J(t,x)$ and $y_n \to y$. From the definition of $J(t,x)$ it follows that for every $n \in \mathbb{N}$ there are sequences $\{t^n_k\} \subset G$, $\{x^n_k\} \subset X$, $\{s^n_k\} \subset H$ such that $t^n_k \to t$, $x^n_k \to x$, $s^n_k \to \infty$ and $\mu(t^n_k, x^n_k, s^n_k) \to y_n$ as $k \to \infty$. Hence, for every $n \in \mathbb{N}$ there is $k_n \in \mathbb{N}$ such that

$$d(\mu(t^n_k, x^n_k, s^n_k), y_n) \le 1/n \qquad \text{for each } k \ge k_n$$

and

$$t_n := t^n_{k_n} \to t,\ x_n := x^n_{k_n} \to x,\ s_n := s^n_{k_n} \to \infty \qquad \text{as } n \to \infty. \tag{2.7}$$

For every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ we have

$$d(\mu(t_n,x_n,s_n), y) \le d(\mu(t_n,x_n,s_n), y_n) + d(y_n, y) \le \varepsilon,$$

i.e. $\mu(t_n, x_n, s_n) \to y$ as $n \to \infty$. From (2.7) and (2.3) it follows that $y \in J(t, x)$, which completes the proof.

Remark 2.2. The sets $D^i(t, x)$ and $J^i(t, x)$ $(i = 1, 2)$ are also closed. The proof is analogous.

For any topological spaces Y and X we denote by $\mathcal{F}(Y, X)$ ($\mathcal{C}(Y, X)$) the family of all maps (continuous maps) from Y into X. Put

$$\mathcal{F} := \{\mu \in \mathcal{F}(G \times X \times H, X) : (X, G, H, \mu) \text{ is a pseudo-process}\}, \tag{2.8}$$

$$\mathcal{F}_1 := \{\mu \in \mathcal{F} : \mu \in \mathcal{C}(\{t\} \times X \times \{s\}, X) \text{ for each } (t, s) \in G \times H\}, \tag{2.9}$$

$$\mathcal{F}_2 := \{\mu \in \mathcal{F} : \text{ for every fixed } \tau \in H, \text{ the one-parameter family of maps } \mu(t, \cdot, \tau) : X \to X, \text{ with } t \in G, \text{ is equicontinuous}\}, \tag{2.10}$$

$$\mathcal{F}_3 := \{\mu \in \mathcal{F} : \mu \in \mathcal{C}(G \times X \times H, X)\}. \tag{2.11}$$

Remark 2.3. The family $\mathcal{F}_2$ is the set of all maps for which the quadruple (X, G, H, μ) is a process in the sense of Dafermos (see [2]).

Let (X, d) be a metric space. We define the function $\varrho : \mathcal{F} \times \mathcal{F} \to \overline{\mathbb{R}}_+ := [0, +\infty]$ by

$$\varrho(\mu, \nu) := \sup\{d(\mu(t, x, s), \nu(t, x, s)) : (t, x, s) \in G \times X \times H\} \tag{2.12}$$

for $\mu, \nu \in \mathcal{F}$.

Remark 2.4. If (X, d) is a metric space then $(\mathcal{F}, \varrho_1)$, $(\mathcal{F}_i, \varrho_1)$ $(i = 1, 2, 3)$ with

$$\varrho_1(\mu, \nu) := \min(1, \varrho(\mu, \nu)) \quad \text{for } \mu, \nu \in \mathcal{F} \tag{2.13}$$

are metric spaces.

LEMMA 2.1. *If (X, d) is a complete metric space then $(\mathcal{F}_i, \varrho_1)$ $(i = 1, 2, 3)$ are complete metric spaces.*

Proof. First we show that $(\mathcal{F}_1, \varrho_1)$ is complete if so is (X, d). Let $\{\mu_n\} \subset \mathcal{F}_1$ be a Cauchy sequence. There is a function $\mu \in \mathcal{F}(G \times X \times H, X)$ such that $\{\mu_n\}$ is uniformly convergent to μ. Hence $\mu \in \mathcal{C}(\{t\} \times X \times \{s\}, X)$ for each $(t, s) \in G \times H$, because μ_n has this property for every $n \in \mathbb{N}$. We have

$$\begin{aligned} |\mu_n(t+s, \mu_n(t, x, s), r) - \mu(t+s, \mu(t, x, s), r)| \\ \le |\mu_n(t+s, \mu_n(t, x, s), r) - \mu(t+s, \mu_n(t, x, s), r)| \\ + |\mu(t+s, \mu_n(t, x, s), r) - \mu(t+s, \mu(t, x, s), r)| \end{aligned}$$

for $(t, x, s, r) \in G \times X \times H \times H$ and (X, G, H, μ_n) is a pseudo-process for $n \in \mathbb{N}$, i.e. μ_n satisfies (1.1), (1.2). Hence (X, G, H, μ) is a pseudo-process, so $\mu \in \mathcal{F}_1$.

Analogously we prove the completeness of $\mathcal{F}_2$ and $\mathcal{F}_3$.

For a non-empty metric space (X,d) we denote—as usual—by 2^X the family of all subsets of X and we put

$$\mathrm{Cl}(X) := \{A \in 2^X : \overline{A} = A\}.$$

We define a function $\widetilde{d} : 2^X \times 2^X \to \overline{\mathbb{R}}$ by the formulae

$$\widetilde{d}(\emptyset, A) := \begin{cases} 0 & \text{for } A = \emptyset, \\ \infty & \text{for } A \in 2^X \setminus \{\emptyset\}, \end{cases}$$

$$\widetilde{d}(A,B) := \max(\sup_{x\in A} d(x,B), \sup_{y \in B} d(y,A)) \quad \text{for } A, B \in 2^X \setminus \{\emptyset\},$$

where $d(x,B) := \inf_{y\in B} d(x,y)$, i.e. $\widetilde{d}$ is the Hausdorff metric in $\mathrm{Cl}(X)\setminus\{\emptyset\}$ (see [3]).

LEMMA 2.2. $(\mathrm{Cl}(X), d_1)$ *with*

$$d_1(A,B) := \min(1, \widetilde{d}(A,B)) \quad \textit{for } A, B \in \mathrm{Cl}(X)$$

is a metric space.

In the sequel we shall consider pseudo-processes in $X = \mathbb{R}^m$.

THEOREM 2.2. *For all* $\mu,\nu \in \mathcal{F}, \delta \in \mathbb{R}$ *and* $W := \Lambda, D, J$ *or* $W_\mu(t,x) := \overline{\mu[t,x]}$ *we have the implication*

$$\varrho(\mu,\nu) \le \delta \Rightarrow \widetilde{d}(W_\mu(t,x), W_\nu(t,x)) \le \delta \textit{ for each } (t,x) \in G \times X.$$

Proof. We prove this theorem for $W = \Lambda$. The other cases are proved in the same way.

Let $\mu,\nu \in \mathcal{F}$ and $\varrho(\mu,\nu) \le \delta$. First we suppose that $y \in \Lambda_\mu(t,x) \ne \emptyset$. In view of (2.1) there is a sequence $\{s_n\} \subset H$ such that $s_n \to \infty$ and $\mu(t,x,s_n) \to y$ as $k \to \infty$. So there is $r > 0$ such that $\mu(t,x,s_n) \in B(y,r)$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we have

$$d(\nu(t,x,s_n),y) \le d(\nu(t,x,s_n), \mu(t,x,s_n)) + d(\mu(t,x,s_n),y) \le \delta + r.$$

Hence, because of the boundedness of the sequence $\{\nu(t,x,s_n)\}$ there are $z \in \mathbb{R}^m$ and a subsequence $\{\nu(t,x,s_{n_k})\}$ such that

$$\nu(t,x,s_{n_k}) \to z \in \Lambda_\nu(t,x) \ne \emptyset \quad \text{as } k \to \infty.$$

For every $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that for each $k \ge k_0$ we have

$$\begin{aligned} d(y,z) &\le d(y, \mu(t,x,s_{n_k})) + d(\mu(t,x,s_{n_k}), \nu(t,x,s_{n_k})) + d(\nu(t,x,s_{n_k}),z) \\ &\le \delta + \varepsilon, \end{aligned}$$

i.e. $d(y,z) \le \delta$. So

$$d(y, \Lambda_\nu(t,x)) \le d(y,z) \le \delta \quad \text{for every } y \in \Lambda_\mu(t,x).$$

Analogously we can prove that

$$d(z, \Lambda_\mu(t,x)) \le d(z,y) \le \delta \quad \text{for every } z \in \Lambda_\nu(t,x).$$

Hence we obtain

$$\widetilde{d}(\Lambda_\mu(t,x),\Lambda_\nu(t,x)) \le \delta.$$

Let now $\Lambda_\mu(t,x)=\emptyset$. The hypothesis that there is $\nu\in\mathcal{F}$ such that $\varrho(\mu,\nu)=\delta<\infty$ and $\Lambda_\nu(t,x)\neq\emptyset$ gives a contradiction in view of the first part of the proof. This proves the theorem.

As in the theory of dynamical systems (see [5]) we can prove the following

THEOREM 2.3. *For each* $(t,x)\in G\times X$ *and* $W:=\Lambda, D, J$ *or* $W_\mu(t,x):=\overline{\mu[t,x]}$ *the map*

$$W(t,x): F\ni\mu\to W_\mu(t,x)\in \mathrm{Cl}(X)$$

is uniformly continuous from $(\mathcal{F},\varrho_1)$ *to* $(\mathrm{Cl}(X),d_1)$.

Remark 2.5. The theorems analogous to Theorems 2.2 and 2.3 hold for $W:=D^i, J^i$ $(i=1,2)$ (see [7] and (2.4), (2.5) in this paper).

III. Dispersive, completely unstable, Poisson unstable and Lagrange unstable pseudo-processes. Suppose (X,G,H) satisfies assumption (A).

DEFINITION 3.1. A pseudo-process (X,G,H,μ) is called

(i) *dispersive* iff for each $x\in X$,

$$J_\mu(t,x)=\emptyset \quad \text{for every } t\in G,$$

(ii) *completely unstable* iff each $x\in X$ is wandering, i.e.

$$x\notin J_\mu(t,x) \quad \text{for every } t\in G,$$

(iii) *Poisson unstable* iff for each $x\in X$,

$$x\notin \Lambda_\mu(t,x) \quad \text{for every } t\in G,$$

(iv) *Lagrange unstable* iff for each $x\in X$,

$$\overline{\mu[t,x]} \quad \text{is not compact for every } t\in G,$$

(v) *Lagrange stable* iff for each $x\in X$,

$$\overline{\mu[t,x]} \quad \text{is compact for every } t\in G.$$

We can define corresponding weak notions by replacing "for every $t\in G$" by "there is $t\in G$". For example:

DEFINITION 3.2. A pseudo-process (X,G,H,μ) is called *weakly dispersive* iff for each $x\in X$ there is $t\in G$ such that $J_\mu(t,x)=\emptyset$.

If we replace the set $J_\mu(t,x)$ by $J^i_\mu(t,x)$ we get the definition of (i)-dispersive or (i)-weakly dispersive pseudo-processes $(i=1,2)$.

These definitions agree with the analogous ones for dynamical systems (see [1], [8]).

Let (X, G, H, μ) be a pseudo-process and suppose μ does not depend on the first variable. Put

$$\pi(s, x) := \mu(t, x, s) \quad \text{for } (t, x, s) \in G \times X \times H. \tag{3.1}$$

The pseudo-dynamical system (X, H, π) defined in this way is dispersive, completely unstable, Poisson unstable, Lagrange unstable or Lagrange stable if and only if so is the pseudo-process (X, G, H, μ).

We introduce the families of all maps μ for which the corresponding pseudo-processes have one of these properties:

$$\mathcal{D} := \{\mu \in \mathcal{F} : (X, G, H, \mu) \text{ is dispersive}\}, \tag{3.2}$$

$$\mathcal{K} := \{\mu \in \mathcal{F} : (X, G, H, \mu) \text{ is completely unstable}\}, \tag{3.3}$$

$$\widetilde{\mathcal{P}} := \{\mu \in \mathcal{F} : (X, G, H, \mu) \text{ is Poisson unstable}\}, \tag{3.4}$$

$$\widetilde{\mathcal{L}} := \{\mu \in \mathcal{F} : (X, G, H, \mu) \text{ is Lagrange unstable}\}, \tag{3.5}$$

$$\mathcal{L} := \{\mu \in \mathcal{F} : (X, G, H, \mu) \text{ is Lagrange stable}\}. \tag{3.6}$$

R e m a r k 3.1. Directly from the definitions (3.2)–(3.4) it follows that $\mathcal{D} \subset \mathcal{K} \subset \widetilde{\mathcal{P}}$.

R e m a r k 3.2. $\mathcal{K} \backslash \widetilde{\mathcal{L}} \neq \emptyset$, so the inclusion $\widetilde{\mathcal{P}} \subset \widetilde{\mathcal{L}}$ is not true in the theory of pseudo-processes, in contrast to the theory of dynamical systems (see [5]).

EXAMPLE 3.1. Let $(\mathbb{R}, \mathbb{R}, \mathbb{R}_+, \mu)$ be the pseudo-process generated by the equation

$$x' = \frac{bt}{(t^2 + a)(1 + \ln^2(t^2 + a))} \qquad (a, b > 0).$$

Then $\overline{\mu[t, x]}$ is compact for every $(t, x) \in \mathbb{R}^2$. However, $x \notin \Lambda(t, x) = J(t, x) \neq \emptyset$ for all $(t, x) \in \mathbb{R}^2$, so $\mu \in \mathcal{K} \cap \mathcal{L} \neq \emptyset$ (see (3.3), (3.6)). Such a situation is impossible in the theory of dynamical systems.

R e m a r k 3.3. We also have

$$\mathcal{D}_{\mathrm{w}} \subset \mathcal{K}_{\mathrm{w}} \subset \widetilde{\mathcal{P}}_{\mathrm{w}} \quad \text{and} \quad \mathcal{K}_{\mathrm{w}} \cap \mathcal{L}_{\mathrm{w}} \neq \emptyset,$$

where (w) denotes a weak condition. For example,

$$\mathcal{K}_{\mathrm{w}} := \{\mu \in \mathcal{F} : (X, G, H, \mu) \text{ is weakly completely unstable}\}.$$

R e m a r k 3.4. The inclusions $\mathcal{D} \subset \widetilde{\mathcal{L}}$, $\mathcal{D}_{\mathrm{w}} \subset \widetilde{\mathcal{L}}_{\mathrm{w}}$ are evident because for $(t, x) \in G \times X$ such that $\overline{\mu[t, x]}$ is compact we get $\Lambda_\mu(t, x) \neq \emptyset$.

We have the same results for the families corresponding to the (i)-prolongational limit sets $(i = 1, 2)$:

$$\mathcal{D}^i := \{\mu \in \mathcal{F} : (X, G, H, \mu) \text{ is } (i)\text{-dispersive}\}.$$

IV. A classification of pseudo-processes. Let $X = \mathbb{R}^m$. In the set $\mathcal{F}$ (see (2.8)) we introduce an equivalence relation S. If $\mu, \nu \in \mathcal{F}$ then

$$(4.1) \qquad (\mu, \nu) \in S \overset{\text{df}}{\Leftrightarrow} \varrho(\mu, \nu) < \infty,$$

where ϱ is defined by (2.12). We denote by F_μ the S-equivalence class of $\mu \in \mathcal{F}$, i.e.

$$\mathcal{F}/S := \{F_\mu : \mu \in \mathcal{F}\}.$$

Remark 4.1. If $F_* \subset \mathcal{F}$ and $\varrho_* := \varrho|_{F_*}$ gives a metric in F_* then $F_* \subset F_\mu$ for every $\mu \in F_*$. That is, for every $\mu \in \mathcal{F}$ the S-equivalence class F_μ is the largest subset F_* of $\mathcal{F}$ (in the sense of inclusion) for which the restriction ϱ_* is a metric and $\mu \in F_*$.

THEOREM 4.1. *The spaces $\mathcal{F}$ and $\mathcal{F}_i$ $(i = 1, 2, 3)$ endowed with the uniform convergence topology are not connected* (see (2.8)–(2.11)).

Proof. Let $\mu \in \mathcal{F}$ and $B(\mu, r) := \{\nu \in \mathcal{F} : \varrho(\mu, \nu) < r\}$. Then $F_\mu = \bigcup\{B(\nu, 1) : \nu \in F_\mu\}$ and $\mathcal{F} \setminus F_\mu = \bigcup\{F_\nu : \nu \notin F_\mu\}$. So F_μ is open and closed in the space $(\mathcal{F}, \varrho_1)$, where ϱ_1 is defined by (2.13). The set $F_\mu \cap \mathcal{F}_i$ is open and closed in the space $(\mathcal{F}_i, \varrho_1)$ $(i = 1, 2, 3)$ (see Lemma 2.1). This finishes the proof.

Let $\{\chi_{tx} \subset \mathcal{F} : (t, x) \in G \times X\}$ be a family satisfying the condition

$$\text{(C)} \qquad (\mu \in \chi_{tx} \Leftrightarrow F_\mu \subset \chi_{tx}) \quad \text{for every } (t, x) \in G \times X.$$

LEMMA 4.1. *Let $(t, x) \in G \times X$, $T \subset G$, $Y \subset X$. The sets χ_{tx}, $\bigcap\{\bigcup\{\chi_{tx} : t \in T\} : x \in Y\}$ and $\bigcup\{\bigcap\{\chi_{tx} : t \in T\} : x \in Y\}$ are open and closed in $(\mathcal{F}, \varrho_1)$.*

This follows from condition (C) and the fact that the set F_μ is open.

Analogously to the theory of dynamical systems we show that the families

$$\mathcal{P}_{tx} := \{\mu \in \mathcal{F} : x \in \Lambda_\mu(t, x)\}, \quad (t, x) \in G \times X,$$
$$\mathcal{C} \setminus \mathcal{K}_{tx} := \{\mu \in \mathcal{F} : x \in J_\mu(t, x)\}, \quad (t, x) \in G \times X,$$

do not satisfy condition (C).

For other examples we refer the reader to [5].

By Theorem 2.2 we deduce that the families

$$(4.2) \qquad \mathcal{A}_{tx} := \{\mu \in \mathcal{F} : \overline{\mu[t, x]} \text{ compact}\}, \quad (t, x) \in G \times X,$$
$$(4.3) \qquad \mathcal{B}_{tx} := \{\mu \in \mathcal{F} : J_\mu(t, x) \neq \emptyset\}, \quad (t, x) \in G \times X,$$

satisfy condition (C).

From the above we obtain some important results on the families of pseudo-processes defined in the third section.

THEOREM 4.2. *Let* $W := \mathcal{D}, \mathcal{D}_{\mathrm{w}}, \widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}_{\mathrm{w}}, \mathcal{L}, \mathcal{L}_{\mathrm{w}}, \mathcal{D}^i$ *or* $\mathcal{D}^i_{\mathrm{w}}$ $(i = 1, 2)$. *The set* W *is both open and closed in the space* $\mathcal{F}$ *endowed with the uniform convergence topology.*

Proof. This follows directly from (4.2) and (4.3). We have, for example,

$$\mathcal{F}\backslash\mathcal{D} = \bigcup\{\mathcal{B}_{tx} : (t, x) \in G \times X\},$$
$$\mathcal{L}_{\mathrm{w}} = \bigcap\Big\{\bigcup\{\mathcal{A}_{tx} : t \in G\} : x \in X\Big\}.$$

In view of Lemma 4.1 this proves closedness and openness of the sets $\mathcal{D}$ and $\mathcal{L}_{\mathrm{w}}$. The proof for the remaining sets is similar.

COROLLARY 4.1. *The set* $\mathcal{D}$ *is not dense in* $\mathcal{K}$ *because* $\mathcal{D} \neq \mathcal{K}$.

In virtue of theorems of Baire category theory (see [4]) and from Lemma 2.1 we get

THEOREM 4.3. *Let* $W := \mathcal{D}, \mathcal{D}_{\mathrm{w}}, \widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}_{\mathrm{w}}, \mathcal{L}, \mathcal{L}_{\mathrm{w}}, \mathcal{D}^j$ *or* $\mathcal{D}^j_{\mathrm{w}}$ $(j = 1, 2)$. *The set* $W \cap F_i$ *is of the second Baire category in the space* $(\mathcal{F}_i, \varrho_1)$ *but it is not residual in this space* $(i = 1, 2, 3)$.

COROLLARY 4.2. *Let* $W := \mathcal{K}, \mathcal{K}_{\mathrm{w}}, \mathcal{K}^j, \mathcal{K}^j_{\mathrm{w}}, P$ *or* $\mathcal{P}_{\mathrm{w}}$ $(j = 1, 2)$. *The set* $W \cap \mathcal{F}_i$ *is of the second Baire category in* $(\mathcal{F}_i, \varrho_1)$ $(i = 1, 2, 3)$.

We can prove that pseudo-processes are either dispersive (Lagrange unstable, Lagrange stable) for all functions belonging to F_μ or are not dispersive (Lagrange unstable, Lagrange stable) for all these functions. We have

THEOREM 4.4. *Let* $F_* \subset \mathcal{F}$ *and suppose that* $\varrho_* := \varrho|_{F_*}$ *gives a metric in* F_*. *Then*

$$F_* \cap W \neq \emptyset \Leftrightarrow F_* \subset W$$

for $W := \mathcal{D}, \mathcal{D}_{\mathrm{w}}, \widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}_{\mathrm{w}}, \mathcal{L}, \mathcal{L}_{\mathrm{w}}, \mathcal{D}^i$ *and* $\mathcal{D}^i_{\mathrm{w}}$ $(i = 1, 2)$.

Proof. We prove this assertion for $W := \widetilde{\mathcal{L}}_{\mathrm{w}}$. The other cases are similar.

Let $\mu \in F_*$ and $\mu \notin \widetilde{\mathcal{L}}_{\mathrm{w}}$. By (4.2) we have

$$\mathcal{F}\backslash\widetilde{\mathcal{L}}_{\mathrm{w}} = \bigcup\Big\{\bigcap\{\mathcal{A}_{tx} : t \in G\} : x \in X\Big\}.$$

Hence there is $x_0 \in X$ such that $\mu \in \mathcal{A}_{tx_0}$ for every $t \in G$. In view of Remark 4.1, $\mu \in F_* \subset F_\mu$ and because $\mathcal{A}_{tx}$ satisfies condition (C), $F_\mu \subset \mathcal{A}_{tx_0}$ for every $t \in G$. So $F_* \subset \mathcal{F}\backslash\widetilde{\mathcal{L}}_{\mathrm{w}}$, which finishes the proof for $W := \widetilde{\mathcal{L}}_{\mathrm{w}}$.

COROLLARY 4.3. *Let* $W := \mathcal{D}, \widetilde{\mathcal{L}}, \mathcal{L}$. *In the quotient set* $\mathcal{F}/S$ *we can introduce the following equivalence relation*:

$$F_\mu\,(W)\,F_\nu \Leftrightarrow \mu, \nu \in W \ \text{or}\ \mu, \nu \notin W.$$

Of course, we can also define in $\mathcal{F}/S$ other relations of this type. For example,

$$F_\mu \sim F_\nu \Leftrightarrow \mu,\nu \in \mathcal{L} \text{ or } \mu,\nu \in \widetilde{\mathcal{L}} \text{ or } \mu,\nu \notin \mathcal{L} \cup \widetilde{\mathcal{L}}.$$

From Theorem 4.4 it follows that these relations are well defined, i.e. their definitions are independent of the choice of representatives of the classes F_μ, F_ν.

V. Examples. The results of Section IV can be applied to processes generated by differential equations.

Definition 5.1. We say that a process $(\mathbb{R}^m, \mathbb{R}, \mathbb{R}_+, \mu)$ (we will write briefly μ) is *generated by a differential equation*

$$x' = f(t,x) \tag{5.1}$$

if for every $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^m$ there exists exactly one, saturated to the right, solution $\varphi(t_0, x_0, \cdot)$ of the Cauchy problem

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{5.2}$$

defined on the interval $[t_0, \infty)$ and

$$\mu(t,x,\tau) = \varphi(t,x,t+\tau) \tag{5.3}$$

for every $(t,x) \in \mathbb{R} \times \mathbb{R}^m, \tau \in \mathbb{R}_+$.

Example 5.1. We consider the differential equation

$$x' = f_\varepsilon(t,x), \tag{ε}$$

where $f_\varepsilon(t,x) = \varepsilon$ for every $(t,x) \in \mathbb{R}^2$ ($\varepsilon \in \mathbb{R}_+$). We have $\sup\{|f_\varepsilon(t,x) - f_0(t,x)| : (t,x) \in \mathbb{R}^2\} = \varepsilon$, but for the process μ_ε generated by the equation (ε) we get $\varrho(\mu_\varepsilon, \mu_0) = \infty$ for $\varepsilon \neq 0$. It is easily seen that $\mu_\varepsilon \in \mathcal{D} \cap \widetilde{\mathcal{L}}$ for $\varepsilon \neq 0$ but $\Lambda_{\mu_0}(t,x) = J_{\mu_0}(t,x) = \overline{\mu_0[t,x]} = \{x\}$ for every $(t,x) \in \mathbb{R}^2$.

The above example shows that a small change of the right hand side of a differential equation can change the type of the process generated by this equation. This difficulty exists even for dynamical systems.

However, we can change the right hand side of a differential equation in a special way.

Let $x, \widetilde{x} \in \mathbb{R}^m$ and $x = (x_1, \ldots, x_m)$, $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_m)$. We will write

$$x \le \widetilde{x} \quad \text{if} \quad x_k \le \widetilde{x}_k \text{ for } k = 1, \ldots, m,$$

and for every fixed $i \in \{1, \ldots, m\}$,

$$x \overset{i}{\le} \widetilde{x} \quad \text{if} \quad x \le \widetilde{x} \text{ and } x_i = \widetilde{x}_i.$$

DEFINITION 5.2 (see [9]). A function $f = (f_1, \ldots, f_m)$ from $\mathbb{R}\times\mathbb{R}^m$ to $\mathbb{R}^m$ is said to satisfy *condition* (W_+) if for every $i \in \{1, \ldots, m\}$ and $x, \widetilde{x} \in \mathbb{R}^m$,

$$(\mathrm{W}_+) \qquad x \overset{i}{\leq} \widetilde{x} \Rightarrow f_i(t,x) \leq f_i(t,\widetilde{x}) \text{ for } t \in \mathbb{R}.$$

LEMMA 5.1 (see [9]). *Assume that* $f : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$ *is continuous and satisfies condition* (W_+) *and* μ *is the process generated by the differential equation* (5.1). *Let* $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^m$, *set*

$$\varphi(t) := \mu(t_0, x_0, t - t_0) \quad \textit{for every } t \geq t_0$$

and suppose a function ψ *from* $\mathbb{R}$ *into* $\mathbb{R}^m$ *is differentiable and satisfies the initial condition* $\psi(t_0) = x_0$. *Then*

(i) $\psi'(t) \leq f(t, \psi(t))$ *for* $t \geq t_0 \Rightarrow \psi(t) \leq \varphi(t)$ *for* $t \geq t_0$,
(ii) $\psi'(t) \geq f(t, \psi(t))$ *for* $t \geq t_0 \Rightarrow \psi(t) \geq \varphi(t)$ *for* $t \geq t_0$.

From the above we get

THEOREM 5.1. *Assume that* f_i $(i = 1, 2, 3)$ *are continuous functions from* $\mathbb{R}\times\mathbb{R}^m$ *into* $\mathbb{R}^m$, f_i $(i = 1, 2)$ *satisfy condition* (W_+) *and*

$$f_1(t,x) \leq f_3(t,x) \leq f_2(t,x), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^m.$$

Denote by μ_i *the process generated by the differential equation* $x' = f_i(t,x)$ $(i = 1, 2, 3)$. *Then*

$$\mu_1 \in F_{\mu_2} \Rightarrow F_{\mu_1} = F_{\mu_2} = F_{\mu_3}.$$

Proof. This will be proved by showing that

$$\mu_1(t,x,\tau) \leq \mu_3(t,x,\tau) \leq \mu_2(t,x,\tau)$$

for every $(t,x) \in \mathbb{R}\times\mathbb{R}^m, \tau \in \mathbb{R}_+$.

Fix $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^m$ and $\tau \in \mathbb{R}_+$. Denote by $\varphi_i(t_0, x_0, \cdot)$ the solution of the Cauchy problem $x' = f_i(t,x)$, $x(t_0) = x_0$. By Lemma 5.1,

$$\varphi_1(t_0,x_0,t) \leq \varphi_3(t_0,x_0,t) \leq \varphi_2(t_0,x_0,t) \quad \text{for every } t \geq t_0.$$

In view of the definition of the process μ_i (see (5.3)) we have

$$\mu_i(t_0,x_0,\tau) = \varphi_i(t_0,x_0,\tau + t_0) \quad (i = 1,2,3),$$

which finishes the proof.

COROLLARY 5.1. *Assume that* $f : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$ *is continuous and there exist continuous functions* g, f_1, f_2 *from* $\mathbb{R}$ *into* $\mathbb{R}^m$ *for which*

$$f_1(t) \leq f(t,x) - g(t) \leq f_2(t) \quad \textit{for every } (t,x) \in \mathbb{R}\times\mathbb{R}^m$$

and the function $\alpha \to \int_0^\alpha f_i(s)\,ds$ *is bounded* $(i = 1, 2)$. *Denote by* μ, ν *the processes generated by the differential equations* $x' = f(t,x), x' = g(t)$ *respectively. Then* $\mu \in F_\nu$.

EXAMPLE 5.2. Let $a, b, c_i \in \mathbb{R}^m$ $(i = 1, \ldots, m)$. Denote by μ, ν the processes generated by the differential equations

$$x' = f(t,x) := \Big(\sum_{i=1}^{m} c_i \cos x_i\Big)(1+t^2)^{-1} + a + b\sin t,$$
$$x' = g(t) := a + b\sin t$$

respectively. There is $k \in \mathbb{R}^m$ such that for every $(t,x) \in \mathbb{R}\times\mathbb{R}^m$ we have

$$\frac{-k}{1+t^2} \le f(t,x) - g(t) \le \frac{k}{1+t^2}.$$

According to Corollary 5.1 we get $\mu \in F_\nu$. So, if $a \neq 0$ then $\mu \in \mathcal{D} \cap \widetilde{\mathcal{L}}$ and if $a = 0$ then $\mu \in \mathcal{L}$.

Define

$$\text{(5.4)}\quad \mathcal{P} := \{\mu \in \mathcal{F} : (X, G, H, \mu) \text{ is Poisson stable, i.e. } x \in \Lambda_\mu(t,x) \text{ for every } (t,x) \in G\times X\}.$$

Remark 5.1. Let the assumptions of Theorem 5.1 be satisfied and suppose that for every $(t,x) \in \mathbb{R}\times\mathbb{R}^m$ there exists a sequence $\{\tau_n(t,x)\} \subset \mathbb{R}_+$ such that $\tau_n(t,x) \to \infty$ and $\mu_i(t,x,\tau_n(t,x)) \to x$ $(i = 1,2)$ as $n \to \infty$. Then $\mu_3 \in \mathcal{P}$.

EXAMPLE 5.3. Let $a > 1$, $b \in \mathbb{R}$, v be a continuous bounded function from $\mathbb{R}$ into $\mathbb{R}$,

$$w(t) := c_s t^s + \ldots + c_1 t + c_0 \qquad (c_i \in \mathbb{R},\ i = 1, \ldots, s,\ s \in \mathbb{N}),$$
$$g(t) := w'(t)e^{-w(t)},$$
$$f(t,x) := \frac{bv(x)\cos t}{(1+\ln^2(\sin t + a))(\sin t + a)} + g(t)$$

for $t, x \in \mathbb{R}$. There exist $k_i \in \mathbb{R}$ $(i = 1,2)$ such that for

$$f_i(t) := \frac{k_i \cos t}{(1+\ln^2(\sin t + a))(\sin t + a)}$$

we have

$$f_1(t) \le f(t,x) - g(t) \le f_2(t) \qquad \text{for } t, x \in \mathbb{R}.$$

Denote by μ, ν, μ_i $(i = 1,2)$ the processes generated by the differential equations

$$x' = f(t,x), \quad x' = g(t), \quad x' = f_i(t) \quad (i = 1,2)$$

respectively. Because the assumptions of Corollary 5.1 are satisfied we get $\mu \in F_\nu$. So, for $s \neq 0$, $c_s > 0$ we have

$$x \notin J_\nu(t,x) = \Lambda_\nu(t,x) \neq \emptyset,$$

hence $\nu \in \mathcal{K} \cap \mathcal{L}$ and $\mu \in F_\nu \subset \mathcal{L}$. If $s \neq 0$, $c_s < 0$ then $\mu \in F_\nu \subset \mathcal{D} \cap \widetilde{\mathcal{L}}$. For $s = 0$ we get $g \equiv 0$. Now we see that $\mu_i \in \mathcal{L} \cap \mathcal{P}$ (see (5.4)) and

$$\mu_i(t, x, 2n\pi) \to x \quad \text{as } n \to \infty,$$

for every $(t, x) \in \mathbb{R}^2$, $i = 1, 2$. In view of Theorem 5.1 and Remark 5.1 we have $\mu \in \mathcal{L} \cap \mathcal{P}$ for $s = 0$.

Remark 5.2. If a process μ does not depend on the first variable we have the dynamical system (X, H, π) defined by (3.1). In this case for other examples we refer the reader to [5].

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