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Convolution of radius functions on $\mathbb{R}^3$

by Konstanty Holly (Kraków)

Abstract. We reduce the convolution of radius functions to that of 1-variable functions. Then we present formulas for computing convolutions of an abstract radius function on $\mathbb{R}^3$ with various integral kernels—given by elementary or discontinuous functions. We also prove a theorem on the asymptotic behaviour of a convolution at infinity. Lastly, we deduce some estimates which enable us to find the asymptotics of the velocity and pressure of a fluid (described by the Navier–Stokes equations) in the boundary layer.

1. Reduction of the convolution of radius functions on $\mathbb{R}^3$ to a convolution on $\mathbb{R}^1$. The *convolution* of Borel measurable functions $f, g : \mathbb{R}^n \to [0, \infty]$ is the function

$$f * g : \mathbb{R}^n \ni x \mapsto \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \in [0,\infty],$$

where dy denotes the Lebesgue measure on $\mathbb{R}^n$.

(1.1) Example. For $a, b > 0$,

$$\left(\frac{a}{\pi}\right)^{n/2} e^{-a|\cdot|^2} * \left(\frac{b}{\pi}\right)^{n/2} e^{-b|\cdot|^2} = \left(\frac{1}{\pi}\frac{ab}{a+b}\right)^{n/2} e^{-(ab/(a+b))|\cdot|^2}. \tag{1.2}$$

(1.3) Example. For $\lambda \in \,]0, n[$ consider the hyperboloid

$$K_\lambda : \mathbb{R}^n \ni x \mapsto |x|^{-\lambda} \in [0,\infty].$$

If $0 < \alpha, \beta < n$ and $\alpha + \beta > n$, then

$$C(n,\alpha)K_\alpha * C(n,\beta)K_\beta = C(n,\alpha+\beta-n)K_{\alpha+\beta-n}, \tag{1.4}$$

where $C(n,\lambda) := 2^\lambda(4\pi)^{-n/2}\Gamma(\lambda/2)\Gamma((n-\lambda)/2)^{-1}$. Identity (1.4) is known as the law of composition of the M. Riesz kernels (see [4] and [3]). ∎

The proofs of (1.2) and of Theorem (1.6) are included in Section 1′. Section 2′ contains the proofs of the theorems from Section 2, etc.

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(1.5) From now on it is assumed that all Borel measurable functions which are integrated have separable ranges.

Consider Banach spaces W_1, W_2, W over a number field K ($\in \{\mathbb{R}, \mathbb{C}\}$) and a continuous K-bilinear operator $W_1 \times W_2 \ni (v_1, v_2) \mapsto v_1 v_2 \in W$. The *convolution* of Borel measurable functions $f_1 : \mathbb{R}^n \to W_1$, $f_2 : \mathbb{R}^n \to W_2$ is the function

$$(f_1 * f_2)(x) := \int_{\mathbb{R}^n} f_1(x-y) f_2(y)\, dy$$

with values in W, defined on

$$\operatorname{dom}(f_1 * f_2) := \{x \in \mathbb{R}^n : (|f_1| * |f_2|)(x) < \infty\}.$$

A function of the form

$$\mathbb{R}^n \ni x \mapsto \kappa(|x|) \in W,$$

where $\kappa : \mathbb{R}_+ \to W$, is called a *radius function*. The functions mentioned in Examples (1.1), (1.3) are such functions.

(1.6) Theorem. *Consider Borel measurable functions $w_i : \mathbb{R} \to W_i$, $i = 1, 2$, such that w_1 is odd, while w_2 is even and absolutely continuous. Fix $x \in \mathbb{R}^3$. Then*

$$(w_1 * w_2)(|x|) = \frac{|x|}{2\pi} \left(-\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot{w}_2(|\cdot|)}{|\cdot|} \right)(x) \tag{1.7}$$

provided

$$(|w_1| * |w_2|)(|x|) < \infty \quad \textit{and} \quad \left(\left| \frac{w_1(|\cdot|)}{|\cdot|} \right| * \left| \frac{\dot{w}_2(|\cdot|)}{|\cdot|} \right| \right)(x) < \infty.$$

In practice the functions $f_i : \mathbb{R}^3 \to W_i$ ($i = 1, 2$) are usually given and we look for functions $w_i : \mathbb{R} \to W_i$ ($i = 1, 2$) such that

$$f_1 = -\frac{w_1(|\cdot|)}{|\cdot|}, \qquad f_2 = \frac{\dot{w}_2(|\cdot|)}{|\cdot|}.$$

We do not know any analogue of Theorem (1.6) for $\mathbb{R}^n$, $n \neq 3$. For instance, for $n = 2$ on the right-hand side of the analogue of $(1'.5)$ there appears the Bessel function J_0, which excludes any further analogies with our reasoning.

2. Various applications of Theorem (1.6)**.** First, we will show that the boundary value problem $\{\Delta u = \kappa(|\cdot|),\ u(\infty) = 0\}$ in $\mathbb{R}^3$ may be reduced to the study of a one-dimensional integral.

(2.1) Remark. *Let $\kappa :]0, \infty[\to W$ be a Borel measurable function such that $\int_0^1 r^2 |\kappa(r)|\, dr < \infty$ and $\int_1^\infty r|\kappa(r)|\, dr < \infty$. Then for every $x \in \mathbb{R}^3 \setminus \{0\}$*

we have

$$\left(\frac{1}{4\pi|\cdot|} * \kappa(|\cdot|)\right)(x) = \frac{1}{|x|}\int\limits_0^{|x|} r^2\kappa(r)\,dr + \int\limits_{|x|}^{\infty} r\kappa(r)\,dr.$$

This formula is "logically equivalent" to the formula from Corollary (1′.15)—see Lemma (1′.11) and Digression (1′.15)*.

(2.2) COROLLARY. *For all* $c > 0$ *and* $x \in \mathbb{R}^3$,

$$\left(\frac{1}{4\pi|\cdot|} * e^{-c|\cdot|^2}\right)(x) = \frac{1}{2c}\int\limits_0^1 e^{-c|x|^2\tau^2}\,d\tau. \ \blacksquare$$

The convolution of the integral kernel $(-4\pi|\cdot|)^{-1}$ of the stationary heat equation

$$0 = \Delta u + f$$

in $\mathbb{R}^3$, with the kernel $(4\pi\nu t)^{-3/2}\exp(-|\cdot|^2/(4\nu t))$ of the heat evolution equation

$$\frac{\partial u}{\partial t}(t,\cdot) = \nu\Delta u(t,\cdot) + f(t,\cdot)$$

appears in the integral kernel of the nonstationary Navier–Stokes equations in $\mathbb{R}^3$—see (0.5) in [1]. Corollary (2.2) may also be derived from the following:

(2.3) REMARK. *If* $\kappa :]0,\infty[\ \to [0,\infty]$ *is a Borel measurable function and* $c > 0$, *then for all* $x \in \mathbb{R}^3 \setminus \{0\}$,

$$\frac{1}{4\pi}\left(\kappa(|\cdot|) * e^{-(c/2)|\cdot|^2}\right)(x) = e^{-(c/2)|x|^2}\int\limits_0^{\infty} \kappa(r) r^2 e^{-(c/2)r^2}\frac{\operatorname{sh}(c|x|r)}{c|x|r}\,dr.$$

We now give a few formulas for the convolution of a radius function with the rational function $(a + |\cdot|^2)^{-m}$ $(a > 0,\ m \in \mathbb{N})$. Differentiation or integration with respect to the parameter a sometimes enables us to obtain some formulas from simpler ones. For instance, for the relations given below we have

$$\frac{d}{da}(2.5) = (2.7) \ \Rightarrow\ (2.10), \qquad \int (2.10)\,db = (2.9),$$
$$\frac{d}{da}(2.10) = (2.11), \qquad \frac{d^2}{da^2}(2.11) = (2.12).$$

(2.4) LEMMA. *Let* $\kappa :]0,\infty[\ \to W$ *be a Borel measurable function such that* $\int_0^1 r^2|\kappa(r)|\,dr < \infty$ *and* $\int_1^\infty |\kappa(r)|\,dr < \infty$. *Then for all* $x \in \mathbb{R}^3 \setminus \{0\}$,

$$(2.5)\qquad \frac{1}{\pi}\left(\frac{1}{a + |\cdot|^2} * \kappa(|\cdot|)\right)(x) = \frac{1}{|x|}\int\limits_0^{\infty} r\left(\ln\frac{a + (r + |x|)^2}{a + (r - |x|)^2}\right)\kappa(r)\,dr.$$

(2.6) Corollary. *If* $\kappa :]0,\infty[\to W$ *is a Borel measurable function such that* $\int_0^1 r^2|\kappa(r)|\,dr < \infty$ *and* $\int_1^\infty r^{-2}|\kappa(r)|\,dr < \infty$, *then for all* $x \in \mathbb{R}^3$,

$$(2.7)\quad \frac{1}{4\pi}\left(\frac{1}{(a+|\cdot|^2)^2} * \kappa(|\cdot|)\right)(x) = \int_0^\infty \frac{r^2\kappa(r)\,dr}{(a+(r-|x|)^2)(a+(r+|x|)^2)}.$$

(2.8) Corollary. *Let* $a, b > 0$, *and let* $*$ *denote convolution in* $\mathbb{R}^3$. *Then*

$$(2.9)\qquad \frac{1}{a^2+|\cdot|^2} * \frac{1}{b^2+|\cdot|^2} = \frac{2\pi^2}{|\cdot|}\arctan\frac{|\cdot|}{a+b}$$

(*for* $x = 0$ *see* (3.11) *and* (3.12)),

$$(2.10)\qquad \frac{1}{a^2+|\cdot|^2} * \frac{1}{(b^2+|\cdot|^2)^2} = \frac{\pi^2}{b}\,\frac{1}{(a+b)^2+|\cdot|^2},$$

$$(2.11)\quad \left(\frac{\sqrt{a}}{\pi}\,\frac{1}{a^2+|\cdot|^2}\right)^2 * \left(\frac{\sqrt{b}}{\pi}\,\frac{1}{b^2+|\cdot|^2}\right)^2 = \left(\frac{\sqrt{a+b}}{\pi}\,\frac{1}{(a+b)^2+|\cdot|^2}\right)^2,$$

$$(2.12)\quad \left(\frac{1}{a^2+|\cdot|^2}\right)^4 * \left(\frac{1}{b^2+|\cdot|^2}\right)^2 = \frac{\pi^2}{a^3}\left(\frac{1}{(a+b)^2+|\cdot|^2}\right)^2 \times\left(\frac{1}{8a^2} + \frac{a+b}{2a}\,\frac{1}{(a+b)^2+|\cdot|^2} + \frac{(a+b)^3}{b}\left(\frac{1}{(a+b)^2+|\cdot|^2}\right)^2\right).$$

From the point of view of symmetry, (2.11) resembles the formulas (1.2), (1.4).

From now on we compute the convolution of a radius function with specific discontinuous functions.

(2.13) Remark. *Fix* $0 < \varrho < \infty$. *Let* χ_ϱ *denote the characteristic function of the interval* $[-\varrho, \varrho]$. *If* $\kappa \in L^1_{\mathrm{loc}}(]0,\infty[, W)$ *and* $\int_0^1 r|\kappa(r)|\,dr < \infty$, *then for all* $x \in \mathbb{R}^3 \setminus \{0\}$,

$$(\chi_\varrho(|\cdot|) * \kappa(|\cdot|))(x) = \frac{\pi}{|x|}\int_{|x|-\varrho}^{|x|+\varrho} (\varrho^2 - |r-|x|\,|^2)r\kappa(|r|)\,dr,$$

$$((|\cdot|\chi_\varrho(|\cdot|)) * \kappa(|\cdot|))(x) = \frac{2\pi}{3|x|}\int_{|x|-\varrho}^{|x|+\varrho} (\varrho^3 - |r-|x|\,|^3)r\kappa(|r|)\,dr.$$

(2.14) Remark. *Assume that* $0 < \varrho < \infty$, $3 < \lambda < \infty$ *and let* $\kappa :]0,\infty[\to W$ *be a Borel measurable function such that* $\int_0^1 r|\kappa(r)|\,dr < \infty$

and $\int_1^\infty r^{3-\lambda}|\kappa(r)|\,dr<\infty$. *Define*

$$f(r):=\begin{cases}0 & \text{if } r\le\varrho,\\ r^{-\lambda} & \text{if } r>\varrho.\end{cases}\tag{2.15}$$

Then for all $x\in\mathbb{R}^3\setminus\{0\}$,

$$\begin{aligned}&(f(|\cdot|)*\kappa(|\cdot|))(x)\\ &\quad=\frac{2\pi}{\lambda-2}\frac{1}{|x|}\Big(\varrho^{2-\lambda}\int\limits_{|r-|x||<\varrho} r\kappa(|r|)\,dr+\int\limits_{|r-|x||>\varrho}|r-|x||^{2-\lambda}r\kappa(|r|)\,dr\Big).\end{aligned}\tag{2.16}$$

3. Asymptotic behaviour of a convolution as $|x|\to\infty$**.** We start with two estimates of a convolution that involves the function $(1+|\cdot|)^{-\gamma}$.

(3.1) Remark. *Let* $f:\mathbb{R}^n\to[0,\infty]$ *be a Borel measurable function with compact support*, *i.e.* $\varrho:=\sup_{f(x)\ne0}|x|<\infty$. *Then for all* $\gamma\in\mathbb{R}$,

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^\gamma((1+|\cdot|)^{-\gamma}*f)(x)\le(1+\varrho)^{|\gamma|}\|f\|_{L^1}.\tag{3.2}$$

(3.3) Lemma. *If* $0\le\gamma\le\lambda$ *and* $\lambda>3$ *then*

$$\sup_{x\in\mathbb{R}^3}(1+|x|)^\gamma((1+|\cdot|)^{-\gamma}*(1+|\cdot|)^{-\lambda})(x)<\infty.\tag{3.4}$$

In spite of the resemblance between (3.2) and (3.4), Lemma (3.3) does not follow from Remark (3.1), because $\mathrm{supp}(1+|\cdot|)^{-\lambda}$ is unbounded.

The functions φ and w considered in (3.5) below are not necessarily radius functions.

(3.5) Theorem. *Assume that* $\lambda>3$, $0\le\gamma<\lambda$, $\varphi:\mathbb{R}^3\to W_1$, *and* $w:\mathbb{R}^3\to W_2$. *Assume that*

(3.6) *the functions* $(1+|\cdot|)^\lambda\varphi$, $(1+|\cdot|)^\gamma w$ *are bounded and Borel measurable.*

Moreover, *suppose that*

$$\lim_{x\to\infty}(|x|^\gamma w(x)-w_\infty(x/|x|))=0,\tag{3.7}$$

where $w_\infty:S_2\to W_2$ *is continuous. Then*

$$\lim_{x\to\infty}\Big(|x|^\gamma(\varphi*w)(x)-\int\limits_{\mathbb{R}^3}\varphi(y)\,dy\cdot w_\infty(x/|x|)\Big)=0.\tag{3.8}$$

(3.9)* The need for such a theorem appeared in a natural way in the theory of the Navier–Stokes equations. It is used in the proof of a theorem (announced in [2]) on the asymptotic behaviour of the velocity $v(t,x)$ and pressure $p(t,x)$ of the free fluid in $\mathbb{R}^3$ as $|x|\to\infty$.

When $\gamma = \lambda$, Theorem (3.5) is not true, as shown by the following example:

(3.10) Example. The functions

$$\varphi = w := \left(\frac{1}{\pi}\frac{1}{1+|\cdot|^2}\right)^2$$

satisfy assumptions (3.6), (3.7) for $\gamma = \lambda = 4$, $w_\infty \equiv \pi^{-2}$. Note that

(3.11) the function $\dfrac{1}{1+|\cdot|^2} * \dfrac{1}{1+|\cdot|^2} : \mathbb{R}^3 \to \mathbb{R}$ is continuous,

since $1/(1+|\cdot|^2) \in L^2(\mathbb{R}^3)$. Bearing that in mind and using (2.9) we compute:

$$(3.12)\quad \int\limits_{\mathbb{R}^3} \varphi(y)\,dy = \frac{1}{\pi^2}\int\limits_{\mathbb{R}^3}\frac{dy}{(1+|y|^2)^2} = \frac{1}{\pi^2}\left(\frac{1}{1+|\cdot|^2} * \frac{1}{1+|\cdot|^2}\right)(0)$$
$$= \frac{1}{\pi^2}\lim_{0\neq x\to 0}\left(\frac{1}{1+|\cdot|^2} * \frac{1}{1+|\cdot|^2}\right)(x)$$
$$= \frac{1}{\pi^2}\lim_{0\neq x\to 0}\frac{2\pi^2}{|x|}\arctan\frac{|x|}{2} = 1.$$

For $a = b = 1$ the formula (2.11) takes the form

$$\varphi * w = \left(\frac{\sqrt{2}}{\pi}\frac{1}{4+|\cdot|^2}\right)^2.$$

Finally,

$$\lim_{x\to\infty}|x|^4(\varphi * w)(x) = \lim_{x\to\infty}\frac{2}{\pi^2}\frac{|x|^4}{(4+|x|^2)^2}$$
$$= \frac{2}{\pi^2} \neq 1\cdot\frac{1}{\pi^2} \equiv \int\limits_{\mathbb{R}^3}\varphi(y)\,dy\cdot w_\infty. \ \blacksquare$$

4. Some special estimates. We present three inequalities needed in the proof of the theorem mentioned in Digression (3.9)*. From now on the parameters a, ν are fixed positive numbers.

(4.1) Lemma. *Let $E :]0,\infty[\times\mathbb{R}^3 \to \mathbb{R}$ be the heat kernel in $\mathbb{R}^3$, i.e.*

$$E(t,x) := (\sqrt{2\nu t})^{-3}\exp\left(-\frac{1}{2}\left|\frac{x}{\sqrt{2\nu t}}\right|^2\right).$$

Then for all $t > 0$,

$$(2\pi)^{-3/2}\frac{1}{(a+|\cdot|^2)^2} * E(t,\cdot) \le \left(1+\frac{3\nu t}{2a}\right)\frac{1}{(a+|\cdot|^2)^2}.$$

(4.2) Lemma. *Define*

$$F :]0,\infty[\times\mathbb{R}^3 \ni (\tau,x) \mapsto \begin{cases} (2\nu\tau)^{-5/2}|x| & \text{if } |x| < \sqrt{2\nu\tau}, \\ 0 & \text{if } |x| \ge \sqrt{2\nu\tau}. \end{cases}$$

Then there exists an increasing continuous function $M : \mathbb{R}_+ \to \mathbb{R}_+$ (see (4′.12)) *such that for all* $\tau > 0$,

$$(a+|\cdot|^2)^{-4} * F(\tau,\cdot) \le (\nu\tau)^{-1/2} M(\nu\tau/a)(a+|\cdot|^2)^{-4}.$$

(4.3) Lemma. *Define*

$$G :]0,\infty[\times\mathbb{R}^3 \ni (\tau,x) \mapsto \begin{cases} 0 & \text{if } |x| < \sqrt{2\nu\tau}, \\ |x|^{-4} & \text{if } |x| \ge \sqrt{2\nu\tau}. \end{cases}$$

Then there exist a constant $C \in \mathbb{R}_+$ (see (4′.18)) *and an increasing continuous function* $M : \mathbb{R}_+ \to \mathbb{R}_+$ (see (4′.19)) *such that for all* $\tau > 0$,

$$(a+|\cdot|^2)^{-4} * G(\tau,\cdot) \le Ca^{-5/2}(a+|\cdot|^2)^{-2} + (\nu\tau)^{-1/2} M(\nu\tau/a)(a+|\cdot|^2)^{-4}.$$

1′. Proofs. The Fourier transform of a summable function $u : \mathbb{R}^n \to W$ (in the case of $K = \mathbb{C}$) is the function $\widehat{u} = \mathcal{F}u : \mathbb{R}^n \to W$ given by

$$\widehat{u}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx,$$

where $x\xi = (x \mid \xi) := \sum_{k=1}^n x_k \xi_k$. Throughout Sections 1′, 2′, 3′ the following abbreviations are used:

(L↗) = the Lebesgue monotone convergence theorem,
($\overline{\text{L}}$) = the Lebesgue dominated convergence theorem.

(1′.1) Proof of (1.2). It is known that $\widehat{\varphi} = \varphi$ for $\varphi := e^{-(1/2)|\cdot|^2}$ ($\in S(\mathbb{R}^n,\mathbb{C})$). Let h_λ stand for the homothety with scale $\lambda \in \mathbb{R}\setminus\{0\}$. Then of course $(\varphi\circ h_\lambda)^\wedge = |\lambda|^{-n}\widehat{\varphi}\circ h_{1/\lambda}$. Therefore

$$e^{-a|\cdot|^2} = \varphi\circ h_{\sqrt{2a}} = (2a)^{-n/2}(\sqrt{2a})^n \widehat{\varphi}\circ h_{\sqrt{2a}} = (2a)^{-n/2}(\varphi\circ h_{\sqrt{1/(2a)}})^\wedge.$$

Hence

$$\begin{aligned} e^{-a|\cdot|^2} \,\widetilde{*}\, e^{-b|\cdot|^2} &= (4ab)^{-n/2}\Big(\varphi\circ h_{\sqrt{1/(2a)}}\Big)^\wedge \,\widetilde{*}\, \Big(\varphi\circ h_{\sqrt{1/(2b)}}\Big)^\wedge \\ &= (4ab)^{-n/2}\Big(\Big(\varphi\circ h_{\sqrt{1/(2a)}}\Big)\Big(\varphi\circ h_{\sqrt{1/(2b)}}\Big)\Big)^\wedge \\ &= (4ab)^{-n/2}\Big(\varphi\circ h_{\sqrt{(a+b)/(2ab)}}\Big)^\wedge \\ &= (4ab)^{-n/2}\bigg(\sqrt{\frac{a+b}{2ab}}\bigg)^{-n} \widehat{\varphi}\circ h_{\sqrt{(2ab)/(a+b)}} \\ &= (2(a+b))^{-n/2} e^{-(ab/(a+b))|\cdot|^2}, \end{aligned}$$

where

$$u \mathbin{\widetilde{*}} v := (2\pi)^{-n/2} u * v. \quad \blacksquare \tag{1'.2}$$

(1′.3) LEMMA. *Assume that $K = \mathbb{C}$ and an even Borel measurable function $f : \mathbb{R} \to W$ satisfies $\int_0^1 r|f(r)|\,dr < \infty$ and $\int_1^\infty r^2|f(r)|\,dr < \infty$. Then $f(|\cdot|) \in L^1(\mathbb{R}^3, W)$ and*

$$(f(|\cdot|))^\wedge(\xi) = \frac{i}{|\xi|} \cdot (\mathrm{id}_{\mathbb{R}} \cdot f)^\wedge(|\xi|) \tag{1'.4}$$

for all $\xi \in \mathbb{R}^3 \setminus \{0\}$.

P r o o f. The function $f(|\cdot|)$ is summable, since

$$\frac{1}{4\pi} \int_{\mathbb{R}^3} |f(|x|)|\,dx = \int_0^\infty r^2|f(r)|\,dr \le \int_0^1 r|f(r)|\,dr + \int_1^\infty r^2|f(r)|\,dr.$$

Applying the formula

$$\int_{S_2} e^{-i(v|\zeta)}\,d\zeta = \int_{S_2} \cos(v|\zeta)\,d\zeta = 4\pi \frac{\sin|v|}{|v|}, \tag{1'.5}$$

where $v \in \mathbb{R}^3 \setminus \{0\}$ and $d\zeta$ is the surface element on the sphere S_2, we obtain

$$\begin{aligned}(f(|\cdot|))^\wedge(\xi) &= (2\pi)^{-3/2} \int_0^\infty r^2 \int_{S_2} e^{-ir(\xi|\zeta)}\,d\zeta\, f(r)\,dr \\ &= \frac{1}{|\xi|}\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty (\sin r|\xi|) r f(r)\,dr.\end{aligned}$$

The function $r \to (\sin r|\xi|) r f(r)$ is even and $r \to (\cos r|\xi|) r f(r)$ is odd. So

$$\begin{aligned}\frac{1}{|\xi|}\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty (\sin r|\xi|) r f(r)\,dr &= \frac{1}{|\xi|}\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{2} \int_{\mathbb{R}} (\sin r|\xi|) r f(r)\,dr \\ &= -\frac{i}{|\xi|} \int_{\mathbb{R}} i(\sin r|\xi|) r f(r)\,\frac{dr}{\sqrt{2\pi}} = -\frac{i}{|\xi|} \int_{\mathbb{R}} e^{ir|\xi|} r f(r)\,\frac{dr}{\sqrt{2\pi}} \\ &= \frac{i}{|\xi|} \int_{\mathbb{R}} e^{-is|\xi|} s f(s)\,\frac{ds}{\sqrt{2\pi}} = \frac{i}{|\xi|} (\mathrm{id}_{\mathbb{R}}\, f)^\wedge(|\xi|),\end{aligned}$$

and this is precisely the assertion of the lemma. ■

(1′.6) P r o o f o f T h e o r e m (1.6). We can choose admissible norms in W_1, W_2 and W such that $|v_1 v_2| \le |v_1|\,|v_2|$ for any $(v_1, v_2) \in W_1 \times W_2$. Assuming some regularity conditions, we will easily prove (1.7). We will then reduce the assumptions gradually till we gain Theorem (1.6).

Step 1. Assume that

(i) $K = \mathbb{C}$,

(ii) $\{w_1 \neq 0\} \cup \{w_2 \neq 0\}$ is bounded,

(iii) w_1 is bounded,

(iv) $\int_{\mathbb{R}} |\dot{w}_2(r)|^2\, dr < \infty$.

It is clear that the functions $f_1(r) := -w_1(r)/r$ and $f_2(r) := \dot{w}_2(r)/r$ are even. By (1′.4), we have

$$(f_k(|\cdot|))^\wedge(\xi) = \frac{i}{|\xi|}(\mathrm{id}_{\mathbb{R}}\, f_k)^\wedge(|\xi|)$$

for all $\xi \in \mathbb{R}^3 \setminus \{0\}$ and $k = 1, 2$. We can rewrite this as

$$f_1(|\cdot|)^\wedge(\xi) = -\frac{i}{|\xi|} w_1^\wedge(|\xi|), \qquad f_2(|\cdot|)^\wedge(\xi) = \frac{i}{|\xi|}(\dot{w}_2)^\wedge(|\xi|) = -w_2^\wedge(|\xi|).$$

Hence, by (1′.2), we compute:

$$\text{(1}'\text{.7)}\quad \begin{aligned}(f_1(|\cdot|) \,\widetilde{*}\, f_2(|\cdot|))^\wedge(\xi) &= f_1(|\cdot|)^\wedge(\xi) \cdot f_2(|\cdot|)^\wedge(\xi) \\ &= \frac{i}{|\xi|}(w_1^\wedge w_2^\wedge)(|\xi|) = \frac{i}{|\xi|}(w_1 \,\widetilde{*}\, w_2)^\wedge(|\xi|) \\ &= \frac{i}{|\xi|}(\mathrm{id}_{\mathbb{R}}\, f)^\wedge(|\xi|),\end{aligned}$$

where $f(r) := (1/r)(w_1 \,\widetilde{*}\, w_2)(r)$. Note that f is even and $\int_0^1 r|f(r)|\, dr < \infty$, $\int_1^\infty r^2|f(r)|\, dr < \infty$ since $w_1 * w_2 \in C_{\mathrm{c}}(\mathbb{R}, W)$. Applying Lemma (1′.3) to (1′.7) we obtain, for all $\xi \in \mathbb{R}^3 \setminus \{0\}$,

$$(f_1(|\cdot|) \,\widetilde{*}\, f_2(|\cdot|))^\wedge(\xi) = (f(|\cdot|))^\wedge(\xi).$$

Equality of the transforms of summable functions implies equality of these functions almost everywhere, so

$$\text{(1}'\text{.8)}\qquad f_1(|\cdot|) \,\widetilde{*}\, f_2(|\cdot|) = f(|\cdot|) \quad \text{in } L^1(\mathbb{R}^3, W).$$

Certainly $f(|\cdot|)$ is continuous on $\mathbb{R}^3 \setminus \{0\}$. The left-hand side of (1′.8) has the same property, because $f_k(|\cdot|) \in L^2(\mathbb{R}^3, W_k)$, $k = 1, 2$ (the condition (iv) guarantees this for $k = 2$). Therefore $(f_1(|\cdot|) \,\widetilde{*}\, f_2(|\cdot|))(x) = f(|x|)$, for $x \in \mathbb{R}^3 \setminus \{0\}$, i.e.

$$\text{(1}'\text{.9)}\qquad (w_1 * w_2)(|x|) = \frac{|x|}{2\pi}(f_1(|\cdot|) * f_2(|\cdot|))(x).$$

The function $w_1 * w_2$ is odd, thus (1′.9) also holds for $x = 0$. Finally,

(1′.10) formula (1.7) holds for any $x \in \mathbb{R}^3$.

At this moment we interrupt Proof (1′.6). It is continued at (1′.16).

(1′.11) LEMMA (scalar version of Remark (2.1)). *If* $\kappa :]0,\infty[\to [0,\infty]$ *is a Borel measurable function, then for all* $x \in \mathbb{R}^3 \setminus \{0\}$,

$$\Big(\kappa(|\cdot|) * \frac{1}{4\pi|\cdot|}\Big)(x) = \frac{1}{|x|}\int_0^{|x|} r^2\kappa(r)\,dr + \int_{|x|}^{\infty} r\kappa(r)\,dr.$$

Proof. We apply Step 1 of Proof (1′.6). In view of (L↗) one can assume that κ is a bounded function with bounded support. We choose an even nonnegative function $\psi \in D(\mathbb{R})$ such that $\psi(0) = 1$ and ψ is decreasing on $\mathbb{R}_+$. Let $\nu \in \mathbb{N} \setminus \{0\}$. The functions

$$w_1(r) := -r\kappa(|r|), \qquad w_{2,\nu}(r) := \psi(r/\nu)\cdot|r|$$

satisfy the conditions (i)–(iv) of Step 1. As a consequence of (1′.10) we get, for all $x \in \mathbb{R}^3$,

$$(w_1 * w_{2,\nu})(|x|) = \frac{|x|}{2\pi}\Big(-\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot{w}_{2,\nu}(|\cdot|)}{|\cdot|}\Big)(x). \tag{1′.12}$$

We compute the limit as $\nu \to \infty$ of the left-hand side of (1′.12) using (L↗):

$$\begin{aligned}(w_1 * w_{2,\nu})(r) &= \int_{\mathbb{R}} (-s)\kappa(|s|)\psi\Big(\frac{|r-s|}{\nu}\Big)|r-s|\,ds \\ &= \int_{-\infty}^{0} + \int_0^{\infty} \\ &= \int_0^{\infty} \tau\kappa(\tau)\psi\Big(\frac{|r+\tau|}{\nu}\Big)(r+\tau)\,d\tau \\ &\quad - \int_0^{\infty} s\kappa(s)\psi\Big(\frac{|r-s|}{\nu}\Big)|r-s|\,ds \\ &\to \int_0^{\infty} \tau\kappa(\tau)(r+\tau)\,d\tau - \int_0^{\infty} s\kappa(s)|r-s|\,ds \\ &= 2\int_0^{r} \tau^2\kappa(\tau)\,d\tau + 2r\int_r^{\infty} \tau\kappa(\tau)\,d\tau.\end{aligned} \tag{1′.13}$$

Next, we compute the limit of the right-hand side of (1′.12):

$$\begin{aligned}\Big(\frac{-w_1(|\cdot|)}{|\cdot|} * \frac{\dot{w}_{2,\nu}(|\cdot|)}{|\cdot|}\Big)(x) &= \frac{1}{\nu}\Big(\kappa(|\cdot|) * \psi'\Big(\frac{|\cdot|}{\nu}\Big)\Big)(x) \\ &\quad + \Big(\kappa(|\cdot|) * \Big(\psi\Big(\frac{|\cdot|}{\nu}\Big)\frac{1}{|\cdot|}\Big)\Big)(x) \\ &=: \frac{1}{\nu}J_1(\nu) + J_2(\nu).\end{aligned}$$

The sequence $(J_1(\nu))_{\nu=1}^{\infty}$ is bounded; from (L$\nearrow$) it follows that $\lim_{\nu\to\infty} J_2(\nu) = (\kappa(|\cdot|) * K_1)(x)$. Hence for all $x \in \mathbb{R}^3$,

$$(1'.14) \qquad \lim_{\nu\to\infty}\Big(-\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot{w}_{2,\nu}(|\cdot|)}{|\cdot|}\Big)(x) = (\kappa(|\cdot|) * K_1)(x).$$

We compare the limits of both sides of (1′.12), taking into account (1′.13) and (1′.14). ■

(1′.15) COROLLARY. *For any* $v \in \mathbb{R}^3$,

$$\frac{1}{4\pi}\int\limits_{S_2} \frac{d\zeta}{|\zeta - v|} = \begin{cases} 1 & \textit{if } |v| \le 1, \\ 1/|v| & \textit{if } |v| \ge 1. \end{cases}$$

Proof. First, we give a proof based on methods of the classical theory of the Laplace equation. If $|v| > 1$, then the function $(x \mapsto |x - v|^{-1})$ is harmonic in the ball $\{|x| \le 1\}$; therefore, by the Gauss theorem,

$$\frac{1}{4\pi}\int\limits_{S_2} \frac{d\zeta}{|\zeta - v|} = \frac{1}{|x - v|}\bigg|_{x=0} = \frac{1}{|v|}.$$

With the aid of (L$\nearrow$) one can prove that this formula extends to the case of $|v| = 1$ (see the analogous reasoning below). It remains to consider the case of $|v| < 1$. Let $0 < \Theta < 1$. The function

$$c : \Theta S_2 \ni v \mapsto \frac{1}{4\pi}\int\limits_{S_2} \frac{d\zeta}{|\zeta - v|} \in \mathbb{R}$$

is constant, because the Euclidean measure on S_2 is invariant with respect to any linear isometry of $\mathbb{R}^3$. The function

$$u(v) := \frac{1}{4\pi}\int\limits_{S_2} \frac{d\zeta}{|\zeta - v|}$$

is a solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \{|x| \le \Theta\}, \qquad c \subset u.$$

From the maximum principle it follows that $u \equiv \text{const} = c$. In particular, $c = u(0) = 1$. ■

In order to show the usefulness of Theorem (1.6) we will present another proof, based just on Lemma (1′.11). One can assume that $v \ne 0$. Fix $\zeta_0 \in S_2$. Let $0 < R < \infty$ and put

$$\kappa_R :]0,\infty[\, \ni r \mapsto \begin{cases} r^{-2} & \text{for } r \le R, \\ 0 & \text{for } r > R. \end{cases}$$

From Lemma (1′.11) it follows that

$$\left(\kappa_R(|\cdot|) * \frac{1}{4\pi|\cdot|}\right)(\zeta_0) = \int_0^1 r^2\kappa_R(r)\,dr + \int_1^\infty r\kappa_R(r)\,dr = \begin{cases} R & \text{for } R \le 1, \\ 1+\ln R & \text{for } R > 1. \end{cases}$$

Thus the function $0 < R \mapsto (\kappa_R(|\cdot|) * (1/(4\pi|\cdot|)))(\zeta_0)$ is of class C^1, and moreover,

$$g(R) := \frac{d}{dR}\left(\kappa_R(|\cdot|) * \frac{1}{4\pi|\cdot|}\right)(\zeta_0) = \begin{cases} 1 & \text{for } R \le 1, \\ 1/R & \text{for } R > 1. \end{cases}$$

On the other hand,

$$\left(\kappa_R(|\cdot|) * \frac{1}{4\pi|\cdot|}\right)(\zeta_0) = \frac{1}{4\pi}\int_{|x|\le R} \frac{dx}{|x|^2|\zeta_0 - x|} = \int_0^R \frac{1}{4\pi}\int_{S_2} \frac{d\zeta}{|\zeta_0 - r\zeta|}\,dr =: \int_0^R f(r)\,dr.$$

If $K \subset \mathbb{R}_+ \setminus \{1\}$ is a compact set, then the function

$$K \times S_2 \ni (r,\zeta) \mapsto \frac{1}{|\zeta_0 - r\zeta|} \in \mathbb{R}$$

is bounded, as a continuous function with compact domain. Therefore with the aid of $(\overline{\mathrm{L}})$ it is easy to verify the continuity of f on $\mathbb{R}_+ \setminus \{1\}$. We know that $\int_0^R f(r)\,dr < \infty$ for all $R > 0$. If $R_0 \in \mathbb{R}_+ \setminus \{1\}$, then R_0 is a point of continuity of f, thus

$$f(R_0) = \frac{d}{dR}\int_0^R f(r)\,dr\bigg|_{R=R_0} = g(R_0).$$

We proved that $f = g$ on $\mathbb{R}_+\setminus\{1\}$. It remains to compute $f(1)$. The function $]1,\infty[\, \ni r \mapsto |\zeta_0 - r\zeta|$ is increasing, since $(d/dr)|\zeta_0 - r\zeta|^2 > 0$ whenever $r > 1$. We choose a decreasing sequence $1 < r_\nu \to 1$. The sequence $(1/|\zeta_0 - r_\nu\zeta|)$ is increasing, so using $(\mathrm{L}\nearrow)$, we compute

$$g(1) \leftarrow g(r_\nu) = f(r_\nu) = \frac{1}{4\pi}\int_{S_2} \frac{d\zeta}{|\zeta_0 - r_\nu\zeta|} \to \frac{1}{4\pi}\int_{S_2} \frac{d\zeta}{|\zeta_0 - \zeta|} = f(1).$$

So, finally, $f = g$ on $\mathbb{R}_+$. It remains to put $\zeta_0 := v/|v|$ and to calculate $f(1/|v|)$. Therefore, Remark (2.1) implies Corollary (1′.15). ∎

(1′.15)* Digression. Conversely, from Corollary (1′.15) one can easily derive Remark (2.1). Namely,

$$\Big(\frac{1}{4\pi|\cdot|} * \kappa(|\cdot|)\Big)(x) = \int_0^\infty \Big(\frac{1}{4\pi}\int_{S_2} \frac{d\zeta}{|\zeta - r^{-1}x|}\Big)\cdot r\kappa(r)\,dr.$$

(1′.16) Continuation of Proof (1′.6). Step 2. Let us take the additional assumptions (i), (ii), (iii) from Step 1. We choose an approximate identity $(h_\nu)_{\nu=1}^\infty$ on $\mathbb{R}$ composed of even test functions. The smooth function $w_{2,\nu} := h_\nu * w_2$ is even, $\dot w_{2,\nu}$ $(= h_\nu * \dot w_2)$ has bounded support. According to Step 1, for all $x \in \mathbb{R}^3$,

$$(w_1 * w_{2,\nu})(|x|) = \frac{|x|}{2\pi}\Big(-\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot w_{2,\nu}(|\cdot|)}{|\cdot|}\Big)(x). \tag{1′.17}$$

Then $w_{2,\nu}$ tends uniformly to w_2, since w_2 is uniformly continuous. From this and $(\overline{\mathrm{L}})$,

$$w_1 * w_{2,\nu} \to w_1 * w_2 \text{ pointwise} \quad \text{as } \nu \to \infty. \tag{1′.18}$$

Next, we compute the limit of the right-hand side of (1′.17):

$$\begin{aligned}\Big(\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot w_{2,\nu}(|\cdot|)}{|\cdot|}\Big)(x) &= \int_{\mathbb{R}^3} \frac{w_1(|x-y|)}{|x-y|\,|y|}\cdot \dot w_{2,\nu}(|y|)\,dy \\ &= \int_0^\infty \int_{S_2} \frac{w_1(|x - r\zeta|)}{|x/r - \zeta|}\,d\zeta \cdot \dot w_{2,\nu}(r)\,dr.\end{aligned}$$

With the aid of Corollary (1′.15) we estimate

$$\frac{1}{4\pi}\int_{S_2} \frac{|w_1(|x - r\zeta|)|}{|x/r - \zeta|}\,d\zeta \le \begin{cases} \|w_1\|_{L^\infty} & \text{if } |x| \le r, \\ \|w_1\|_{L^\infty}\cdot (r/|x|) & \text{if } |x| \ge r. \end{cases} \tag{1′.19}$$

In particular, the function

$$]0,\infty[\, \ni r \mapsto \int_{S_2} \frac{w_1(|x - r\zeta|)}{|x/r - \zeta|}\,d\zeta \in W_1$$

is bounded on the (bounded) subset $\bigcup_\nu \{\dot w_{2,\nu} \ne 0\}$. At the same time, $\dot w_{2,\nu} = h_\nu * \dot w_2 \to \dot w_2$ in $L^1(\mathbb{R}, W_2)$, since $\int_{\mathbb{R}} |\dot w_2(r)|\,dr < \infty$. Hence

$$\begin{aligned}\int_0^\infty \int_{S_2} \frac{w_1(|x - r\zeta|)}{|x/r - \zeta|}\,d\zeta \cdot \dot w_{2,\nu}(r)\,dr \to \int_0^\infty \int_{S_2} \frac{w_1(|x - r\zeta|)}{|x/r - \zeta|}\,d\zeta \cdot \dot w_2(r)\,dr \\ = \int_{\mathbb{R}^3} \frac{w_1(|x-y|)}{|x-y|\,|y|}\cdot \dot w_2(|y|)\,dy \\ = \Big(\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot w_2(|\cdot|)}{|\cdot|}\Big)(x).\end{aligned}$$

Taking this result and $(1'.18)$ into consideration, we compare the limits of both sides of $(1'.17)$ and we obtain $(1'.10)$ by the additional assumptions (i), (ii), (iii).

Step 3. We additionally assume that

$$\{w_1 \neq 0\} \text{ is bounded, } w_1 \text{ is bounded and } K = \mathbb{C}. \tag{1'.20}$$

We choose $\psi \in D(\mathbb{R})$ such that $\psi(0) = 1$ and for $\nu \in \mathbb{N} \setminus \{0\}$ we set $w_{2,\nu}(r) := \psi(|r|/\nu) \cdot w_2(r)$. According to Step 2, for all $x \in \mathbb{R}^3$,

$$(w_1 * w_{2,\nu})(|x|) = \frac{|x|}{2\pi} \cdot \left(- \frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot{w}_{2,\nu}(|\cdot|)}{|\cdot|} \right)(x). \tag{1'.21}$$

From $(\overline{\mathrm{L}})$ it follows that

$$w_1 * w_{2,\nu} \to w_1 * w_2 \text{ pointwise} \quad \text{as } \nu \to \infty. \tag{1'.22}$$

Let $x \in \mathbb{R}^3$. Then

$$\begin{aligned} \left(\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot{w}_{2,\nu}(|\cdot|)}{|\cdot|} \right)(x) &= \frac{1}{\nu} \int_{\mathbb{R}^3} \frac{w_1(|x-y|)}{|x-y|} \cdot \psi'\left(\frac{|y|}{\nu}\right) \cdot \frac{w_2(|y|)}{|y|}\, dy \\ &\quad + \int_{\mathbb{R}^3} \frac{w_1(|x-y|)}{|x-y|} \psi\left(\frac{|y|}{\nu}\right) \cdot \frac{\dot{w}_2(|y|)}{|y|}\, dy \\ &=: \frac{1}{\nu} J_1(\nu) + J_2(\nu). \end{aligned}$$

We estimate

$$\begin{aligned} |J_1(\nu)| &\le \int_{\mathbb{R}^3} \frac{|w_1(|x-y|)|}{|x-y|} \cdot \|\psi'\|_{L^\infty} \cdot \frac{|w_2(|y|)|}{|y|}\, dy \\ &= \|\psi'\|_{L^\infty} \cdot \int_0^{M+|x|} \int_{S_2} \frac{|w_1(|x-r\zeta|)|}{|x/r-\zeta|}\, d\zeta \cdot |w_2(r)|\, dr, \end{aligned}$$

where $M := \sup_{w_1(t)\neq 0} |t|$. From this and the inequality $(1'.19)$ it follows that the sequence $(J_1(\nu))$ is bounded. Also by $(1'.19)$ one can apply $(\overline{\mathrm{L}})$ to the calculation of $\lim_{\nu\to\infty} J_2(\nu)$, namely

$$\begin{aligned} J_2(\nu) &= \int_0^{M+|x|} \psi\left(\frac{r}{\nu}\right) \int_{S_2} \frac{w_1(|x-r\zeta|)}{|x/r-\zeta|}\, d\zeta \cdot \dot{w}_2(r)\, dr \\ &\to \int_0^{M+|x|} \int_{S_2} \frac{w_1(|x-r\zeta|)}{|x/r-\zeta|}\, d\zeta \cdot \dot{w}_2(r)\, dr \\ &= \left(\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot{w}_2(|\cdot|)}{|\cdot|} \right)(x). \end{aligned}$$

Putting all together gives

$$\lim_{\nu\to\infty}\left(\frac{w_1(|\cdot|)}{|\cdot|}*\frac{\dot w_{2,\nu}(|\cdot|)}{|\cdot|}\right)(x)=\left(\frac{w_1(|\cdot|)}{|\cdot|}*\frac{\dot w_2(|\cdot|)}{|\cdot|}\right)(x).$$

Taking this fact and (1′.22) into account, we compare the limits of the both sides of (1′.21) and we get (1′.10) by the additional assumptions (1′.20).

Step 4. Now we only assume that $K=\mathbb{C}$. Fix $x_0\in\mathbb{R}^3$ such that

$$(|w_1|*|w_2|)(|x_0|)<\infty\quad\text{and}\quad\left(\frac{|w_1(|\cdot|)|}{|\cdot|}*\frac{|\dot w_2(|\cdot|)|}{|\cdot|}\right)(x_0)<\infty.$$

By (1.5) there exists a sequence of simple Borel measurable functions $\widetilde w_{1,\nu}:\mathbb{R}_+\to W_1$, $\nu=1,2,\ldots$, with bounded supports such that $\{\widetilde w_{1,\nu}\to w_1|_{\mathbb{R}_+}$ pointwise as $\nu\to\infty\}$ and $|\widetilde w_{1,\nu}(r)|\le|w_1(r)|$ for all $r\in\mathbb{R}_+$ and all ν. For every ν the function

$$w_{1,\nu}:\mathbb{R}\ni r\mapsto\begin{cases}\widetilde w_{1,\nu}(r) & \text{if } r>0,\\ -\widetilde w_{1,\nu}(-r) & \text{if } r<0\end{cases}$$

is Borel measurable, simple, odd and has bounded support. Moreover, $w_{1,\nu}\to w_1$ pointwise as $\nu\to\infty$ and $|w_{1,\nu}(r)|\le|w_1(r)|$ for all $r\in\mathbb{R}$ and all ν. According to Step 3, for all $\nu\in\mathbb{N}$,

$$(1'.23)\qquad (w_{1,\nu}*w_2)(|x_0|)=\frac{|x_0|}{2\pi}\left(-\frac{w_{1,\nu}(|\cdot|)}{|\cdot|}*\frac{\dot w_2(|\cdot|)}{|\cdot|}\right)(x_0).$$

Applying ($\overline{\mathrm{L}}$) we pass to the limit on both sides of (1′.23) and we get

$$(1'.24)\qquad (w_1*w_2)(|x_0|)=\frac{|x_0|}{2\pi}\left(-\frac{w_1(|\cdot|)}{|\cdot|}*\frac{\dot w_2(|\cdot|)}{|\cdot|}\right)(x_0).$$

Step 5. It remains to deal with the case of $K=\mathbb{R}$. We recall the complexification method. W^2 is a complex linear space with the following multiplication of a scalar and a vector:

$$\alpha\cdot(v',v''):=(\alpha_1v'-\alpha_2v'',\alpha_1v''+\alpha_2v'),$$

where $\alpha\in\mathbb{C}$, $\alpha_1:=\operatorname{Re}\alpha$ and $\alpha_2:=\operatorname{Im}\alpha$. The locally convex space W^2 ($=W\times W$ with the product topology) has a bound7ed neighbourhood of zero, therefore it is a $\mathbb{C}$-normed space. One can choose an admissible $\mathbb{C}$-norm in W^2 such that

$$\forall(v',v'')\in W^2:\ |(v',v'')|\le|v'|+|v''|.$$

Likewise we fit $\mathbb{C}$-norms in W_1^2, W_2^2 (corresponding to the topologies). The map

$$(1'.25)\quad W_1^2\times W_2^2\ni((v_1',v_1''),(v_2',v_2''))\mapsto(v_1'v_2'-v_1''v_2'',v_1'v_2''+v_1''v_2')\in W^2$$

is continuous and $\mathbb{C}$-bilinear. Consider the canonical injection $l:W\ni v\mapsto(v,0)\in W^2$ and the analogously defined injections $l_1:W_1\to W_1^2$, $l_2:W_2\to$

W_2^2. It is clear that $l_1 \circ w_1$ is odd and Borel measurable, $l_2 \circ w_2$ is even and absolutely continuous, $(l_2 \circ w_2)' = l_2 \circ \dot{w}_2$. Fix $x_0 \in \mathbb{R}^3$ such that

$$(|w_1| * |w_2|)(|x_0|) < \infty \quad \text{and} \quad \left(\frac{|w_1(|\cdot|)|}{|\cdot|} * \frac{|\dot{w}_2(|\cdot|)|}{|\cdot|}\right)(x_0) < \infty.$$

Then

$$(|l_1 \circ w_1| * |l_2 \circ w_2|)(|x_0|) < \infty \quad \text{and}$$
$$\left(\frac{|(l_1 \circ w_1)(|\cdot|)|}{|\cdot|} * \frac{|(l_2 \circ w_2)'(|\cdot|)|}{|\cdot|}\right)(x_0) < \infty$$

and by Step 4,

$$(1'.26) \quad ((l_1 \circ w_1) * (l_2 \circ w_2))(|x_0|) = \frac{|x_0|}{2\pi}\left(-\frac{(l_1 \circ w_1)(|\cdot|)}{|\cdot|} * \frac{(l_2 \circ w_2)'(|\cdot|)}{|\cdot|}\right)(x_0),$$

where the convolutions on both sides of (1′.26) are defined with the aid of the "multiplication" (1′.25). We rearrange the equality (1′.26), using the identity $l_1(v_1)l_2(v_2) = l(v_1v_2)$ and linearity of the integral to obtain

$$l((w_1 * w_2)(|x_0|)) = l\left(\frac{|x_0|}{2\pi} \cdot \left(-\frac{w_1(|\cdot|)}{|\cdot|} * \frac{\dot{w}_2(|\cdot|)}{|\cdot|}\right)(x_0)\right),$$

which proves (1′.24). The proof of Theorem (1.6) is complete. ■

2′. Proofs

(2′.1) Proof of Remark (2.1). See Lemma (1′.11) (and reduction to the scalar version in (2′.5)) or Digression (1′.15)*. ■

(2′.2) Proof of Remark (2.3). By (L↗) one can assume that κ is bounded and has bounded support. In Theorem (1.6) we substitute

$$w_1(r) = -r\kappa(|r|), \qquad w_2(r) = -\frac{1}{c}\, e^{-(c/2)r^2}. \ ■$$

(2′.3) Lemma (scalar version of Lemma (2.4)). *If $\kappa :]0,\infty[\to [0,\infty]$ is a Borel measurable function, then for all $x \in \mathbb{R}^3 \setminus \{0\}$,*

$$(2'.4) \quad \frac{1}{\pi}\left(\frac{1}{a+|\cdot|^2} * \kappa(|\cdot|)\right)(x) = \frac{1}{|x|}\int_0^\infty r\left(\ln\frac{a+(r+|x|)^2}{a+(r-|x|)^2}\right)\kappa(r)\,dr.$$

Proof. In view of (L↗) one can assume that κ is a bounded function with bounded support. We put $w_1(r) = -r\kappa(|r|)$, $w_2(r) = \frac{1}{2}\ln(a+r^2)$ and apply Theorem (1.6). ■

(2′.5) Proof of Lemma (2.4). By (1.5) one can assume the separability of W. The left-hand side of (2.5) exists, since

$$(2'.6)\qquad \Big(\frac{1}{a+|\cdot|^2} * |\kappa(|\cdot|)|\Big)(x) = \int\limits_0^\infty \int\limits_{S_2} \frac{r^2 d\zeta}{a+|x-r\zeta|^2}\cdot|\kappa(r)|\,dr < \infty.$$

The right-hand side of (2.5) also exists, because by (2′.6) and Lemma (2′.3),

$$\int\limits_0^\infty r\Big(\ln\frac{a+(r+|x|)^2}{a+(r-|x|)^2}\Big)|\kappa(r)|\,dr = \frac{|x|}{\pi}\Big(\frac{1}{a+|\cdot|^2} * |\kappa(|\cdot|)|\Big)(x) < \infty.$$

Suppose that Lemma (2.4) is true for $W=\mathbb{R}$. Then in the general case, for all $l\in L(W,\mathbb{R})$,

$$\begin{aligned} l\Big(\frac{1}{\pi}\Big(\frac{1}{a+|\cdot|^2} * \kappa(|\cdot|)\Big)(x)\Big) &= \frac{1}{\pi}\Big(\frac{1}{a+|\cdot|^2} * (l\circ\kappa)(|\cdot|)\Big)(x)\\ &= \frac{1}{|x|}\int\limits_0^\infty r\Big(\ln\frac{a+(r+|x|)^2}{a+(r-|x|)^2}\Big)(l\circ\kappa)(r)\,dr\\ &= l\Big(\frac{1}{|x|}\int\limits_0^\infty r\Big(\ln\frac{a+(r+|x|)^2}{a+(r-|x|)^2}\Big)\kappa(r)\,dr\Big) \end{aligned}$$

and hence

$$(2'.7)\qquad \frac{1}{\pi}\Big(\frac{1}{a+|\cdot|^2} * \kappa(|\cdot|)\Big)(x) = \frac{1}{|x|}\int\limits_0^\infty r\Big(\ln\frac{a+(r+|x|)^2}{a+(r-|x|)^2}\Big)\kappa(r)\,dr,$$

because, by the separability of W and by the Hahn–Banach theorem, the linear continuous real functions separate points in W. Therefore one can take $W=\mathbb{R}$, and apply Lemma (2′.3) to the functions $\kappa^+ := \max\{0,\kappa\}$, $\kappa^- := \max\{0,-\kappa\}$. ■

(2′.8) Lemma (scalar version of Corollary (2.6)). *If* $\kappa :]0,\infty[\ \to [0,\infty]$ *is a Borel measurable function*, *then for all* $x\in\mathbb{R}^3$,

$$(2'.9)\qquad \frac{1}{4\pi}\Big(\frac{1}{(a+|\cdot|^2)^2} * \kappa(|\cdot|)\Big)(x) = \int\limits_0^\infty \frac{r^2\kappa(r)\,dr}{(a+(r-|x|)^2)(a+(r+|x|)^2)}.$$

Proof. In view of (L↗) one can assume that κ is a bounded function with bounded support. It is easy to verify (2′.9) for $x=0$, and for $x\neq 0$ it may be immediately obtained by differentiation of (2′.4) with respect to a. It is worth while, however, to investigate the proof of (2′.9) based directly on Theorem (1.6). Namely, for $w_1(r) = -r\kappa(|r|)$, $w_2(r) = -1/(2(a+r^2))$ the formula (1.7) takes the form

$$(2'.10)\qquad \Big(\kappa(|\cdot|) * \frac{1}{(a+|\cdot|^2)^2}\Big)(x) = \frac{\pi}{|x|}\int\limits_{\mathbb{R}} \frac{r\kappa(|r|)\,dr}{a+(r-|x|)^2},$$

which, after a change of variables, is equivalent to (2′.9). ■

(2′.11) Proof of Corollary (2.6). In view of Lemma (2′.8), it is a slight modification of the argument (2′.5). ■

(2′.12) Proof of (2.10). Both sides of (2.10) are continuous on $\mathbb{R}^3$, thus it suffices to show their equality on $\mathbb{R}^3 \setminus \{0\}$. From (2′.10) it follows that for all $x \in \mathbb{R}^3 \setminus \{0\}$,

$$\left(\frac{1}{a^2+|\cdot|^2} * \frac{1}{(b^2+|\cdot|^2)^2}\right)(x) = \frac{\pi}{|x|} \int\limits_{\mathbb{R}} \frac{r\,dr}{(a^2+r^2)(b^2+(r-|x|)^2)}$$
$$= \frac{\pi}{|x|} \lim_{R\to\infty} \Big(\int\limits_{-R}^{R} f(z)\,dz + \int\limits_{\Gamma_R} f(z)\,dz - \int\limits_{\Gamma_R} f(z)\,dz \Big), \tag{2′.13}$$

where $\Gamma_R : [0,\pi] \ni \varphi \mapsto Re^{i\varphi} \in \mathbb{C}$ and

$$f(z) := \frac{z}{(a^2+z^2)(b^2+(z-|x|)^2)} = \frac{z}{(z-ia)(z+ia)(z-|x|-ib)(z-|x|+ib)}.$$

For sufficiently large R we have

$$\int\limits_{-R}^{R} f(z)\,dz + \int\limits_{\Gamma_R} f(z)\,dz = 2\pi i(\operatorname{res}_{ia} f + \operatorname{res}_{|x|+ib} f),$$

where

$$\operatorname{res}_{ia} f = \frac{z}{(z+ia)(z-|x|-ib)(z-|x|+ib)}\bigg|_{z=ia} = \frac{1}{2(i(a-b)-|x|)(i(a+b)-|x|)},$$
$$\operatorname{res}_{|x|+ib} f = \frac{z}{(z-ia)(z+ia)(z-|x|+ib)}\bigg|_{z=|x|+ib} = -\,\frac{|x|+ib}{2ib(i(a-b)-|x|)(i(a+b)+|x|)}.$$

After a rearrangement we obtain

$$\int\limits_{-R}^{R} f(z)\,dz + \int\limits_{\Gamma_R} f(z)\,dz = \frac{\pi|x|}{b} \cdot \frac{1}{(a+b)^2+|x|^2}. \tag{2′.14}$$

The formulas (2′.13), (2′.14) imply (2.10), since

$$\lim_{R\to\infty} \int\limits_{\Gamma_R} f(z)\,dz = 0. \ ■$$

(2′.15) Proof of (2.9). Let $x \in \mathbb{R}^3 \setminus \{0\}$. Applying (2.10) we compute:

$$\begin{aligned}\frac{d}{db}\left(\frac{1}{a^2+|\cdot|^2} * \frac{1}{b^2+|\cdot|^2}\right)(x) &= -2b\left(\frac{1}{a^2+|\cdot|^2} * \frac{1}{(b^2+|\cdot|^2)^2}\right)(x)\\ &= -\frac{2\pi^2}{(a+b)^2+|x|^2}\\ &= \frac{d}{db}\left(-\frac{2\pi^2}{|x|}\arctan\frac{a+b}{|x|}\right).\end{aligned}$$

Consequently, there exists $C \in \mathbb{R}$ such that for all $b > 0$,

$$\left(\frac{1}{a^2+|\cdot|^2} * \frac{1}{b^2+|\cdot|^2}\right)(x) = C - \frac{2\pi^2}{|x|}\arctan\frac{a+b}{|x|}. \tag{2′.16}$$

Passing to the limit on both sides of (2′.16) as $b \to \infty$, we get

$$C = \frac{2\pi^2}{|x|}\cdot\frac{\pi}{2}.$$

It remains to apply the formula $\forall r > 0:\ \arctan r + \arctan(1/r) = \pi/2$. ■

(2′.17) Proof of (2.11). We differentiate both sides of (2.10) with respect to a. ■

(2′.18) Proof of (2.12). We write (2.11) in the form

$$\frac{1}{(a^2+|\cdot|^2)^2} * \frac{1}{(b^2+|\cdot|^2)^2} = \frac{\pi^2}{b}\cdot\frac{a+b}{a}\cdot\frac{1}{((a+b)^2+|\cdot|^2)^2}$$

and differentiate it with respect to a. After a rearrangement we obtain

$$\begin{aligned}&\frac{1}{(a^2+|\cdot|^2)^3} * \frac{1}{(b^2+|\cdot|^2)^2}\\ &\quad= \frac{\pi^2}{a^2}\frac{1}{((a+b)^2+|\cdot|^2)^2}\left(\frac{1}{4a} + \frac{(a+b)^2}{b}\frac{1}{(a+b)^2+|\cdot|^2}\right).\end{aligned} \tag{2′.19}$$

We differentiate both sides of (2′.19) again with respect to a. ■

(2′.20) Proof of (2.13). In Theorem (1.6) we substitute

$\{w_1(r) = -r\kappa(|r|),\ w_2(r) = \frac{1}{2}(r^2-\varrho^2)\cdot\chi_\varrho(r)\}$ or, in the other case,
$\{w_1(r) = -r\kappa(|r|),\ w_2(r) = \frac{1}{3}(|r|^3-\varrho^3)\cdot\chi_\varrho(r)\}$. ■

(2′.21) Proof of Remark (2.14). In Theorem (1.6) we substitute

$$w_1(r) = -r\kappa(|r|), \qquad w_2(r) = \frac{1}{2-\lambda}(\varrho^{2-\lambda}\chi_\varrho(r) + |r|^{2-\lambda}(1-\chi_\varrho(r))).\ ■$$

3′. Proofs

(3′.1) Proof of Remark (3.1). First, consider the case of $\gamma \le 0$. Using the triangle inequality $|y| \le |x| + |y-x|$, we estimate

$$\begin{aligned}((1+|\cdot|)^{-\gamma} * f)(x) &= \int_{|x-y|\le\varrho} (1+|y|)^{|\gamma|} f(x-y)\,dy \\ &\le (1+|x|+\varrho)^{|\gamma|} \int_{|x-y|\le\varrho} f(x-y)\,dy \\ &= (1+|x|+\varrho)^{|\gamma|} \cdot \|f\|_{L^1},\end{aligned}$$

i.e.

$$(1+|x|)^{\gamma}((1+|\cdot|)^{-\gamma} * f)(x) \le \left(\frac{1+|x|+\varrho}{1+|x|}\right)^{|\gamma|} \cdot \|f\|_{L^1}. \tag{3′.2}$$

The right-hand side of (3′.2) is majorized by the right-hand side of (3.2), since

$$\frac{d}{dr}\frac{1+r+\varrho}{1+r} \le 0 \quad \text{on } \mathbb{R}_+.$$

It remains to consider the case of $\gamma > 0$. If $|x| \le \varrho$, then at once $(1+|x|)^{\gamma}((1+|\cdot|)^{-\gamma} * f)(x) \le (1+\varrho)^{\gamma}\|f\|_{L^1}$, because $(1+|\cdot|)^{-\gamma} * f \le 1 * f \equiv \|f\|_{L^1}$. Assume that $|x| > \varrho$. Again with the aid of the triangle inequality we estimate

$$\begin{aligned}((1+|\cdot|)^{-\gamma} * f)(x) &= \int_{|x-y|\le\varrho} (1+|y|)^{-\gamma} f(x-y)\,dy \\ &\le (1+|x|-\varrho)^{-\gamma} \int_{|x-y|\le\varrho} f(x-y)\,dy \\ &= (1+|x|-\varrho)^{-\gamma} \|f\|_{L^1},\end{aligned}$$

that is,

$$(1+|x|)^{\gamma}((1+|\cdot|)^{-\gamma} * f)(x) \le \left(\frac{1+|x|}{1+|x|-\varrho}\right)^{\gamma} \cdot \|f\|_{L^1}. \tag{3′.3}$$

The right-hand side of (3′.3) is majorized by the right-hand side of (3.2), since

$$\frac{d}{dr}\frac{1+r}{1+r-\varrho} \le 0 \quad \text{in } [\varrho, \infty[. \ \blacksquare$$

(3′.4) Proof of Lemma (3.3). Define

$$M := \sup_{x\in\mathbb{R}^3} (1+|x|)^{\lambda}((1+|\cdot|)^{-\lambda} * (1+|\cdot|)^{-\lambda})(x)$$

and let $x_0 \in \mathbb{R}^3$. Then

$$
\begin{aligned}
&(1+|x_0|)^\gamma((1+|\cdot|)^{-\gamma} * (1+|\cdot|)^{-\lambda})(x_0)\\
&\qquad = \int\limits_{|x|\le|x_0|} \left(\frac{1+|x_0|}{1+|x|}\right)^\gamma (1+|x_0-x|)^{-\lambda}\,dx\\
&\qquad\quad + \int\limits_{|x|>|x_0|} \left(\frac{1+|x_0|}{1+|x|}\right)^\gamma (1+|x_0-x|)^{-\lambda}\,dx\\
&\qquad \le \int\limits_{|x|\le|x_0|} \left(\frac{1+|x_0|}{1+|x|}\right)^\lambda (1+|x_0-x|)^{-\lambda}\,dx\\
&\qquad\quad + \int\limits_{|x|>|x_0|} (1+|x_0-x|)^{-\lambda}\,dx \le M + \int\limits_{\mathbb{R}^3} (1+|y|)^{-\lambda}\,dy.
\end{aligned}
$$

Therefore for every $\gamma \in [0,\lambda]$ the left-hand side of (3.4) is majorized by $M + \|H\|_{L^1}$, where $H := (1+|\cdot|)^{-\lambda}$. Thus it suffices to prove that $M < \infty$. Let $H_0 := \chi_1(|\cdot|)\cdot H$, $H_\infty := H - H_0$. Then

$$
(3'.5)\quad H*H = (H_0*H)+(H_\infty*H_0)+(H_\infty*H_\infty) \le 2(H*H_0)+f(|\cdot|)*f(|\cdot|),
$$

where f stands for the function (2.15) for $\varrho = 1$. By Remark (3.1),

$$
\sup_{x\in\mathbb{R}^3} (1+|x|)^\gamma (H*H_0)(x) \le 2^\lambda \|H\|_{L^1},
$$

therefore, taking into account (3′.5), it suffices to show that

$$
(3'.6)\qquad \sup_{x\in\mathbb{R}^3} (1+|x|)^\lambda (f(|\cdot|)*f(|\cdot|))(x) < \infty.
$$

If $f(|\cdot|) \in L^2(\mathbb{R}^3)$ then $f(|\cdot|)*f(|\cdot|) \in L^\infty(\mathbb{R}^3)$. Thus we only need to prove that

$$
(3'.7)\qquad \overline{\lim_{x\to\infty}}\, |x|^\lambda (f(|\cdot|)*f(|\cdot|))(x) < \infty.
$$

Let $x \in \mathbb{R}^3$, $|x| > 2$. We substitute $\kappa = f$ in (2.16) and estimate

$$
\begin{aligned}
&\frac{\lambda-2}{2\pi}(f(|\cdot|)*f(|\cdot|))(x)\\
&\qquad = \frac{1}{|x|}\int\limits_{|x|-1}^{|x|+1} r^{1-\lambda}\,dr + \frac{1}{|x|}\Big(\int\limits_{-\infty}^{|x|-1} + \int\limits_{|x|+1}^{\infty}\Big)|r-|x|\,|^{2-\lambda} r f(|r|)\,dr\\
&\qquad \le \frac{1}{|x|}\int\limits_{|x|-1}^{|x|+1} \left(\frac{|x|}{2}\right)^{1-\lambda} dr + \frac{1}{|x|}\Big(\int\limits_{-\infty}^{0} + \int\limits_{0}^{|x|-1} + \int\limits_{|x|+1}^{\infty}\Big)(\ldots)\,dr
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{2}{|x|}\right)^{\lambda} + \frac{1}{|x|}\Big(- \int_{1}^{\infty} r^{1-\lambda}(r+|x|)^{2-\lambda}\,dr \\
&\quad + \int_{1}^{|x|-1} r^{1-\lambda}(|x|-r)^{2-\lambda}\,dr + \int_{|x|+1}^{\infty} r^{1-\lambda}(r-|x|)^{2-\lambda}\,dr\Big) \\
&\le \left(\frac{2}{|x|}\right)^{\lambda} + \frac{1}{|x|}\Big(- \int_{1}^{|x|/2} r^{1-\lambda}(r+|x|)^{2-\lambda}\,dr \\
&\quad + \int_{1}^{|x|-1} r^{1-\lambda}(|x|-r)^{2-\lambda}\,dr + \int_{|x|+1}^{\infty} |x|^{1-\lambda}(r-|x|)^{2-\lambda}\,dr\Big) \\
&= \left(\frac{2}{|x|}\right)^{\lambda} + \frac{1}{|x|}\Big((\lambda-2) \int_{1}^{|x|/2} r^{1-\lambda} \int_{|x|-r}^{|x|+r} t^{1-\lambda}\,dt\,dr \\
&\quad + \int_{|x|/2}^{|x|-1} r^{1-\lambda}(|x|-r)^{2-\lambda}\,dr + |x|^{1-\lambda} \int_{|x|+1}^{\infty} (r-|x|)^{2-\lambda}\,dr\Big) \\
&\le \left(\frac{2}{|x|}\right)^{\lambda} + \frac{1}{|x|}\Big((\lambda-2) \int_{1}^{|x|/2} r^{1-\lambda} \int_{|x|-r}^{|x|+r} \left(\frac{|x|}{2}\right)^{1-\lambda} dt\,dr \\
&\quad + \int_{|x|/2}^{|x|-1} \left(\frac{|x|}{2}\right)^{1-\lambda} (|x|-r)^{2-\lambda}\,dr + |x|^{1-\lambda} \int_{1}^{\infty} s^{2-\lambda}\,ds\Big) \\
&= \left(\frac{2}{|x|}\right)^{\lambda} + (\lambda-2)\left(\frac{2}{|x|}\right)^{\lambda} \int_{1}^{|x|/2} r^{2-\lambda}\,dr + \frac{1}{2}\left(\frac{2}{|x|}\right)^{\lambda} \int_{1}^{|x|/2} s^{2-\lambda}\,ds + \frac{|x|^{-\lambda}}{\lambda-3} \\
&\le \left(\frac{2}{|x|}\right)^{\lambda} + (\lambda-2)\left(\frac{2}{|x|}\right)^{\lambda} \int_{1}^{\infty} r^{2-\lambda}\,dr + \frac{1}{2}\left(\frac{2}{|x|}\right)^{\lambda} \int_{1}^{\infty} s^{2-\lambda}\,ds + \frac{|x|^{-\lambda}}{\lambda-3} \\
&= |x|^{-\lambda}\left(2^{\lambda} + \frac{\lambda-2}{\lambda-3}\cdot 2^{\lambda} + \frac{2^{\lambda-1}}{\lambda-3} + \frac{1}{\lambda-3}\right) =: |x|^{-\lambda}\cdot R.
\end{aligned}
$$

Hence

$$
\frac{\lambda-2}{2\pi} \sup_{|x|>2} |x|^{\lambda}(f(|\cdot|) * f(|\cdot|))(x) \le R,
$$

which, in particular, implies (3′.7). ∎

(3′.8) Proof of Theorem (3.5). Let $(x_\nu) \in (\mathbb{R}^3)^{\mathbb{N}}$ with $x_\nu \to \infty$

(i.e. $|x_\nu| \to \infty$ as $\nu \to \infty$). We decompose

$$
\begin{aligned}
|x_\nu|^\gamma(\varphi * w)(x_\nu) - &\int_{\mathbb{R}^3} \varphi(y)\,dy \cdot w_\infty\Big(\frac{x_\nu}{|x_\nu|}\Big)\\
&= \int_{\mathbb{R}^3} \varphi(y)\Big(|x_\nu|^\gamma w(x_\nu - y) - w_\infty\Big(\frac{x_\nu}{|x_\nu|}\Big)\Big)dy\\
&= \int_{\mathbb{R}^3} \varphi(y)\Big(\Big(\frac{|x_\nu|}{1+|x_\nu - y|}\Big)^\gamma - 1\Big)(1+|x_\nu - y|)^\gamma w(x_\nu - y)\,dy\\
&\quad + \int_{\mathbb{R}^3} \varphi(y)((1+|x_\nu - y|)^\gamma - |x_\nu - y|^\gamma) w(x_\nu - y)\,dy\\
&\quad + \int_{\mathbb{R}^3\setminus\{x_\nu\}} \varphi(y)\Big(|x_\nu - y|^\gamma w(x_\nu - y) - w_\infty\Big(\frac{x_\nu - y}{|x_\nu - y|}\Big)\Big)dy\\
&\quad + \int_{\mathbb{R}^3\setminus\{x_\nu\}} \varphi(y)\Big(w_\infty\Big(\frac{x_\nu - y}{|x_\nu - y|}\Big) - w_\infty\Big(\frac{x_\nu}{|x_\nu|}\Big)\Big)dy\\
&=: I_1(\nu) + I_2(\nu) + I_3(\nu) + I_4(\nu).
\end{aligned}
$$

We will prove that for all $k \in \{1,2,3,4\}$,

$$
\lim_{\nu\to\infty} I_k(\nu) = 0. \tag{3'.9}
$$

In view of Lemma (3.3), for $\Gamma > 3$ and $0 \le \alpha \le \Gamma$,

$$
C_\Gamma(\alpha) := \sup_{x\in\mathbb{R}^3} (1+|x|)^\alpha((1+|\cdot|)^{-\alpha} * (1+|\cdot|)^{-\Gamma})(x) < \infty.
$$

(The letter Γ in the present proof has nothing to do with the Euler function from Example (1.3).) We choose admissible norms in W_1, W_2, W such that $|v_1 v_2| \le |v_1|\cdot|v_2|$ for $(v_1, v_2) \in W_1 \times W_2$. The assumption (3.6) yields

$$
C := \sup_{x\in\mathbb{R}^3} \{(1+|x|)^\lambda|\varphi(x)|, (1+|x|)^\gamma|w(x)|\} < \infty.
$$

First, we will show that (3′.9) holds for $k\in\{2,3,4\}$. After the rearrangement

$$
I_2(\nu) = \int_{\mathbb{R}^3} \varphi(y) \int_{|x_\nu - y|}^{1+|x_\nu - y|} \gamma t^{\gamma-1}\,dt\, w(x_\nu - y)\,dy,
$$

we estimate

$$
|I_2(\nu)| \le \gamma C^2 \int_{\mathbb{R}^3} (1+|y|)^{-\lambda} \int_{|x_\nu - y|}^{1+|x_\nu - y|} t^{\gamma-1}\,dt\,(1+|x_\nu - y|)^{-\gamma}\,dy. \tag{3'.10}
$$

If $\gamma \ge 1$, then $\int_{|x_\nu - y|}^{1+|x_\nu - y|} t^{\gamma-1}\,dt \le (1+|x_\nu - y|)^{\gamma-1}$ and hence

$|I_2(\nu)| \le \gamma C^2((1+|\cdot|)^{-\lambda} * (1+|\cdot|)^{-1})(x_\nu) \le \gamma C^2(1+|x_\nu|)^{-1}\cdot C_\lambda(1) \to 0$

as $\nu \to \infty$. If $\gamma < 1$ and $y \ne x_\nu$, then

$$\int\limits_{|x_\nu - y|}^{1+|x_\nu - y|} t^{\gamma-1}\,dt \le |x_\nu - y|^{\gamma-1} \le (1+|x_\nu - y|)^{\gamma}|x_\nu - y|^{-1}$$

and by Remark (2.1),

$$\begin{aligned}|I_2(x_\nu)| &\le \gamma C^2 \int\limits_{\mathbb{R}^3} (1+|y|)^{-\lambda}|x_\nu - y|^{-1}\,dy\\ &= 4\pi\gamma C^2\bigg(\frac{1}{4\pi|\cdot|} * (1+|\cdot|)^{-\lambda}\bigg)(x_\nu)\\ &\le 4\pi\gamma C^2\bigg(\frac{1}{|x_\nu|}\int\limits_0^\infty \frac{r^2\,dr}{(1+r)^\lambda} + \int\limits_{|x_\nu|}^\infty \frac{r\,dr}{(1+r)^\lambda}\bigg) \to 0 \quad \text{as } \nu \to \infty.\end{aligned}$$

$I_3(\nu)$ can be expressed as an integral over the whole space:

$$I_3(\nu) = \int\limits_{\mathbb{R}^3} \varphi(y)(|x_\nu - y|^\gamma w(x_\nu - y) - v_\nu(y))\,dy, \tag{3'.11}$$

where

$$v_\nu : \mathbb{R}^3 \ni y \mapsto \begin{cases} 0 & \text{for } y = x_\nu,\\ w_\infty((x_\nu - y)/|x_\nu - y|) & \text{for } y \ne x_\nu.\end{cases} \tag{3'.12}$$

According to the assumption (3.7),

$$\forall y \in \mathbb{R}^3 : \lim_{\nu\to\infty}(|x_\nu - y|^\gamma w(x_\nu - y) - v_\nu(y)) = 0.$$

The integrand in (3′.11) is majorized by $|\varphi(y)|\cdot(C + \max_{\zeta\in S_2}|w_\infty(\zeta)|)$, thus by $(\overline{\mathrm{L}})$, $\lim_{\nu\to\infty} I_3(\nu) = 0$.

Keeping the notation (3′.12),we have

$$I_4(\nu) = \int\limits_{\mathbb{R}^3} \varphi(y)(v_\nu(y) - w_\infty(x_\nu/|x_\nu|))\,dy.$$

It is clear that

$$\sup_{y\in\mathbb{R}^3} |v_\nu(y) - w_\infty(x_\nu/|x_\nu|)| \le 2\cdot \max_{\zeta\in S_2}|w_\infty(\zeta)|,$$

therefore, in view of ($\overline{\mathrm{L}}$), it suffices to show that

$$\lim_{\nu\to\infty}(v_\nu(y) - w_\infty(x_\nu/|x_\nu|)) = 0, \qquad \forall y \in \mathbb{R}^3. \tag{3'.13}$$

For a given $y \in \mathbb{R}^3$ there is $\nu_0 \in \mathbb{N}$ such that $y \neq x_\nu$ for $\nu > \nu_0$. Let $\varepsilon > 0$. The function w_∞ is uniformly continuous, thus for some $\delta > 0$,

$$|\zeta_1 - \zeta_2| < \delta \;\Rightarrow\; |w_\infty(\zeta_1) - w_\infty(\zeta_2)| < \varepsilon.$$

We have

$$\lim_{\nu\to\infty}\left|\frac{x_\nu}{|x_\nu|} - \frac{y}{|x_\nu|}\right| = 1, \tag{3'.14}$$

since $1 - |y|/|x_\nu| \le |x_\nu/|x_\nu| - y/|x_\nu|| \le 1 + |y|/|x_\nu|$. Simultaneously

$$\left|\left(\frac{x_\nu}{|x_\nu|}\,\middle|\,\frac{y}{|x_\nu|}\right)\right| \le \frac{|y|}{|x_\nu|} \to 0 \quad \text{as } \nu\to\infty,$$

therefore

$$\left|\frac{x_\nu - y}{|x_\nu - y|} - \frac{x_\nu}{|x_\nu|}\right|^2 = 2\left(1 - \left(1 - \left(\frac{x_\nu}{|x_\nu|}\,\middle|\,\frac{y}{|x_\nu|}\right)\right)\Big/\left|\frac{x_\nu}{|x_\nu|} - \frac{y}{|x_\nu|}\right|\right) \to 0 \quad \text{as } \nu\to\infty.$$

In particular,

$$\exists \nu_1 > \nu_0 \; \forall \nu > \nu_1 : \; \left|\frac{x_\nu - y}{|x_\nu - y|} - \frac{x_\nu}{|x_\nu|}\right|^2 < \delta^2.$$

So

$$\forall \nu > \nu_1 : \; \left|v_\nu(y) - w_\infty\left(\frac{x_\nu}{|x_\nu|}\right)\right| = \left|w_\infty\left(\frac{x_\nu - y}{|x_\nu - y|}\right) - w_\infty\left(\frac{x_\nu}{|x_\nu|}\right)\right| < \varepsilon.$$

In this way we proved (3′.13), and thereby (3′.9) for $k = 4$.

It remains to prove (3′.9) for $k = 1$. We choose a "buffer" $\Gamma \in\,]\max\{3, \gamma, \lambda - 1\}, \lambda[$, we define

$$B_\nu := \{y \in \mathbb{R}^3 : |x_\nu| > 1 + |x_\nu - y|\},$$

$$b_\nu : \mathbb{R}^3 \ni y \mapsto \begin{cases} 0 & \text{for } y \in B_\nu, \\ 1 - \left(\dfrac{|x_\nu|}{1 + |x_\nu - y|}\right)^\gamma & \text{for } y \in \mathbb{R}^3 \setminus B_\nu, \end{cases}$$

and we estimate

$$|I_1(\nu)| \le C^2 \int\limits_{\mathbb{R}^3} (1 + |y|)^{-\lambda} \left|\left(\frac{|x_\nu|}{1 + |x_\nu - y|}\right)^\gamma - 1\right| dy \tag{3'.15}$$

$$= C^2 \int\limits_{\mathbb{R}^3\setminus B_\nu} (1+|y|)^{-\lambda}\biggl(1-\biggl(\frac{|x_\nu|}{1+|x_\nu-y|}\biggr)^{\gamma}\biggr)dy$$

$$+ C^2 \int\limits_{B_\nu} (1+|y|)^{-\lambda}\biggl(\biggl(\frac{|x_\nu|}{1+|x_\nu-y|}\biggr)^{\Gamma}-1\biggr)dy$$

$$\le C^2 \int\limits_{\mathbb{R}^3} \frac{b_\nu(y)}{(1+|y|)^{\lambda}}\,dy + C^2 \int\limits_{B_\nu} (1+|y|)^{-\lambda}\biggl(\biggl(\frac{|x_\nu|}{1+|x_\nu-y|}\biggr)^{\Gamma}-1\biggr)dy.$$

By ($\overline{\mathrm{L}}$) the integral over $\mathbb{R}^3$ on the right-hand side of (3′.15) tends to zero, because $\|b_\nu\|_{L^\infty} \le 1$ and, by (3′.14), for $y \in \mathbb{R}^3$,

$$|b_\nu(y)| \le \biggl|1-\biggl(\frac{|x_\nu|}{1+|x_\nu-y|}\biggr)^{\gamma}\biggr| \to 0 \quad \text{as } \nu\to\infty.$$

It remains to show that also

(3′.16) the integral over B_ν on the right-hand side of (3′.15) tends to zero as $\nu \to \infty$.

We have

$$(3'.17) \quad |x_\nu| \le |x_\nu - y| + |y| \le (1+|x_\nu-y|) + (1+|y|)$$
$$\Rightarrow |x_\nu| - (1+|x_\nu-y|) \le 1+|y|.$$

Defining $r_\nu(y) := |x_\nu|/(1+|x_\nu-y|)$ for $y \in B_\nu$ and bearing in mind (3′.17) we estimate

$$(3'.18) \quad \biggl(\frac{|x_\nu|}{1+|x_\nu-y|}\biggr)^{\Gamma} - 1$$
$$= r_\nu(y)^{\Gamma} - 1 = \int\limits_1^{r_\nu(y)} \Gamma\cdot t^{\Gamma-1}\,dt \le (r_\nu(y)-1)\cdot \Gamma r_\nu(y)^{\Gamma-1}$$
$$= \Gamma|x_\nu|^{\Gamma-1}(1+|x_\nu-y|)^{-\Gamma}(|x_\nu| - (1+|x_\nu-y|))$$
$$\le \Gamma|x_\nu|^{\Gamma-1}(1+|x_\nu-y|)^{-\Gamma}(1+|y|).$$

We can now finish the proof of (3′.16):

$$\int\limits_{B_\nu} (1+|y|)^{-\lambda}\biggl(\biggl(\frac{|x_\nu|}{1+|x_\nu-y|}\biggr)^{\Gamma}-1\biggr)dy$$
$$\le \int\limits_{B_\nu} (1+|y|)^{-\lambda}\Gamma|x_\nu|^{\Gamma-1}(1+|x_\nu-y|)^{-\Gamma}(1+|y|)\,dy$$

$$\begin{aligned}&\le \Gamma|x_\nu|^{\Gamma-1}\int\limits_{\mathbb{R}^3}(1+|y|)^{1-\lambda}(1+|x_\nu-y|)^{-\Gamma}\,dy\\&\le \Gamma(1+|x_\nu|)^{\Gamma-1}((1+|\cdot|)^{-(\lambda-1)}*(1+|\cdot|)^{-\Gamma})(x_\nu)\\&\le \Gamma(1+|x_\nu|)^{\Gamma-1}(1+|x_\nu|)^{1-\lambda}C_\Gamma(\lambda-1)\\&=\Gamma C_\Gamma(\lambda-1)(1+|x_\nu|)^{\Gamma-\lambda}\to 0\quad \text{as } \nu\to\infty.\end{aligned}$$

The proof of Theorem (3.5) is complete. ∎

4′. Proofs

(4′.1) Lemma. *For all* $r\in\mathbb{R}$,

$$\sup_{s\in\mathbb{R}}\frac{s}{a+s^2}=\frac{1}{2\sqrt{a}}, \tag{4′.2}$$

$$\sup_{s\in\mathbb{R}}\frac{a+(r+s)^2}{a+s^2}\le 1+\left|\frac{r}{\sqrt{a}}\right|+\left|\frac{r}{\sqrt{a}}\right|^2, \tag{4′.3}$$

$$\sup_{s\in\mathbb{R}}\frac{(a+s^2)^2}{(a+(s-r)^2)(a+(s+r)^2)}=1+\frac{r^2}{4a}. \tag{4′.4}$$

Proof. (4′.2) follows from

$$\frac{d}{ds}\frac{s}{a+s^2}=\frac{a-s^2}{(a+s^2)^2}.$$

(4′.3) results from (4′.2), because

$$\frac{a+(r+s)^2}{a+s^2}=1+r\left(\frac{r}{a+s^2}+2\cdot\frac{s}{a+s^2}\right)\le 1+|r|\left(\frac{|r|}{a}+2\cdot\frac{1}{2\sqrt{a}}\right).$$

Finally, we prove (4′.4):

$$\frac{(a+s^2)^2}{(a+(s-r)^2)(a+(s+r)^2)}=\frac{(a+s^2)^2}{(a+s^2)^2+r^4+2ar^2-2r^2s^2}=:g(s).$$

It follows that $g(\mathbb{R})=f(\mathbb{R}_+)$, where

$$f(t):=\frac{(a+t)^2}{(a+t)^2+r^4+2ar^2-2r^2t}.$$

Therefore it is sufficient to find $\sup_{t>0}f(t)$. Just note that $f'(r^2+3a)=0$, $f'>0$ on the left of r^2+3a, $f'<0$ on the right of r^2+3a. ∎

(4′.5) Proof of Lemma (4.1). Fix $t>0$ and $x\in\mathbb{R}^3$. In (2.7) we substitute

$$\kappa(r)=(2\nu t)^{-3/2}\exp\left(-\frac{1}{2}\left(\frac{r}{\sqrt{2\nu t}}\right)^2\right)$$

and we estimate, using (4′.4),

$$\begin{aligned}
(2\pi)^{-3/2}\bigg(E(t,\cdot)*\frac{1}{(a+|\cdot|^2)^2}\bigg)(x)
&=(2\pi)^{-3/2}\bigg(\kappa(|\cdot|)*\frac{1}{(a^2+|\cdot|^2)^2}\bigg)(x)\\
&=\bigg(\frac{2}{\pi}\bigg)^{1/2}\int\limits_0^\infty \frac{r^2\kappa(r)\,dr}{(a+(r-|x|)^2)(a+(r+|x|)^2)}\\
&\le\bigg(\frac{2}{\pi}\bigg)^{1/2}\int\limits_0^\infty \frac{r^2\kappa(r)}{(a+|x|^2)^2}\bigg(1+\frac{r^2}{4a}\bigg)dr\\
&=\bigg(\frac{2}{\pi}\bigg)^{1/2}\int\limits_0^\infty \bigg(s^2+\frac{\nu t s^4}{2a}\bigg)\cdot(2\nu t)^{3/2}\kappa(\sqrt{2\nu t}s)\,ds\,\frac{1}{(a+|x|^2)^2}\\
&=\bigg(\frac{2}{\pi}\bigg)^{1/2}\bigg(\int\limits_0^\infty s^2e^{-s^2/2}\,ds+\frac{\nu t}{2a}\int\limits_0^\infty s^4e^{-s^2/2}\,ds\bigg)\frac{1}{(a+|x|^2)^2}.\ \blacksquare
\end{aligned}$$

(4′.6) P r o o f o f L e m m a (4.2). Fix $\tau>0$ and $x\in\mathbb{R}^3$. $F(\tau,\cdot)$ is a radius function; namely, $F(\tau,y)=(2\nu\tau)^{-5/2}\kappa(|y|)$, where

$$\kappa(r):=\begin{cases} r & \text{for } r<\sqrt{2\nu\tau},\\ 0 & \text{for } r\ge\sqrt{2\nu\tau}.\end{cases}$$

We differentiate the equality (obtained with the aid of (2.7))

$$\frac{1}{4\pi}((a+|\cdot|^2)^{-2}*F(\tau,\cdot))(x)=(2\nu\tau)^{-5/2}\int\limits_0^\infty\frac{r^2\kappa(r)\,dr}{(a+(r-|x|)^2)(a+(r+|x|)^2)}$$

twice with respect to a and we get

$$\text{(4′.7)}\qquad ((a+|\cdot|^2)^{-4}*F(\tau,\cdot))(x) = \frac{2\pi}{3}(2\nu\tau)^{-5/2}\int\limits_0^{\sqrt{2\nu\tau}} r^3\frac{d^2}{da^2}\frac{1}{(a+(r-|x|)^2)(a+(r+|x|)^2)}\,dr.$$

For $z_1,z_2\in\mathbb{R}_+$ we compute

$$\text{(4′.8)}\qquad \frac{d^2}{da^2}\frac{1}{(a+z_1)(a+z_2)}=2\cdot\frac{3a\cdot(a+z_1+z_2)+z_1^2+z_2^2+z_1z_2}{(a+z_1)^3(a+z_2)^3}.$$

In particular, if, for a fixed $r>0$, $z_1:=(r-|x|)^2$, $z_2:=(r+|x|)^2$ then

$$\text{(4′.9)}\qquad \begin{gathered} z_1+z_2=2r^2+2|x|^2,\qquad z_1^2+z_2^2=2r^4+2|x|^4+12r^2|x|^2,\\ z_1z_2=r^4-2r^2|x|^2+|x|^4.\end{gathered}$$

After substituting (4′.9) in (4′.8) we estimate

$$(4'.10)\qquad \frac{d^2}{da^2}\frac{1}{(a+(r-|x|)^2)(a+(r+|x|)^2)}$$
$$= 2\cdot\frac{3(a+r^2+|x|^2)^2+4r^2|x|^2}{(a+(r-|x|)^2)^3(a+(r+|x|)^2)^3}$$
$$= \frac{2}{(a+(r-|x|)^2)^3(a+(r+|x|)^2)}$$
$$\cdot\frac{3(a+r^2+|x|^2)^2+4r^2|x|^2}{(a+r^2+|x|^2)^2+4r^2|x|^2+4r|x|(a+r^2+|x|^2)}$$
$$\le \frac{2}{(a+(r-|x|)^2)^3(a+(r+|x|)^2)}\cdot 3.$$

If, additionally, $r<\sqrt{2\nu\tau}$ then after applying (4′.3) and (4′.4) we have

$$(4'.11)\qquad \frac{d^2}{da^2}\frac{1}{(a+(r-|x|)^2)(a+(r+|x|)^2)}$$
$$\le \frac{6}{(a+|x|^2)^4}\left(\frac{a+(r+|x|)^2}{a+|x|^2}\right)^2\left(\frac{(a+|x|^2)^2}{(a+(|x|-r)^2)(a+(|x|+r)^2)}\right)^3$$
$$\le \frac{6}{(a+|x|^2)^4}\left(1+\frac{r}{\sqrt{a}}+\left(\frac{r}{\sqrt{a}}\right)^2\right)^2\left(1+\frac{r^2}{4a}\right)^3$$
$$\le \frac{6}{(a+|x|^2)^4}\left(1+\sqrt{\frac{2\nu\tau}{a}}+\frac{2\nu\tau}{a}\right)^2\left(1+\frac{\nu\tau}{2a}\right)^3.$$

Using (4′.11) we estimate the right-hand side of (4′.7) to get

$$((a+|\cdot|^2)^{-4}*F(\tau,\cdot))(x)$$
$$\le (2\nu\tau)^{-1/2}\pi\left(1+\sqrt{\frac{2\nu\tau}{a}}+\frac{2\nu\tau}{a}\right)^2\left(1+\frac{\nu\tau}{2a}\right)^3(a+|x|^2)^{-4}.$$

We put

$$(4'.12)\qquad M(\varrho)=2^{-1/2}\pi(1+\sqrt{2\varrho}+2\varrho)^2(1+\tfrac12\varrho)^3.\ \blacksquare$$

(4′.13) P r o o f of L e m m a (4.3). Fix $\tau>0$ and $x\in\mathbb{R}^3$. Then $G(\tau,\cdot) = \kappa(|\cdot|)$, where

$$\kappa(r):=\begin{cases}0 & \text{if } r<\sqrt{2\nu\tau},\\ r^{-4} & \text{if } r\ge\sqrt{2\nu\tau}.\end{cases}$$

We differentiate the equality (obtained from (2.7))

$$\frac{1}{4\pi}((a+|\cdot|^2)^{-2}*G(\tau,\cdot))(x)=\int_0^\infty \frac{r^2\kappa(r)\,dr}{(a+(r-|x|)^2)(a+(r+|x|)^2)}$$

twice with respect to a and next, using (4′.10) and (4′.4), we estimate

$$(4'.14)\quad ((a+|\cdot|^2)^{-4} * G(\tau,\cdot))(x)$$
$$= \frac{2\pi}{3}\int_{\sqrt{2\nu\tau}}^{\infty} r^{-2}\frac{d^2}{da^2}\frac{1}{(a+(r-|x|)^2)(a+(r+|x|)^2)}\,dr$$
$$\le \frac{4\pi}{(a+|x|^2)^2}\int_{\sqrt{2\nu\tau}}^{\infty}\frac{r^{-2}}{(a+(r-|x|)^2)^2}\cdot\frac{(a+|x|^2)^2}{(a+(|x|-r)^2)(a+(|x|+r)^2)}\,dr$$
$$\le \frac{4\pi}{(a+|x|^2)^2}\Bigg(\int_{\sqrt{2\nu\tau}}^{\infty}\frac{r^{-2}\,dr}{(a+(r-|x|)^2)^2}+\frac{1}{4a}\int_{\sqrt{2\nu\tau}}^{\infty}\frac{dr}{(a+(r-|x|)^2)^2}\Bigg).$$

We differentiate the formula

$$\int (a+(r-|x|)^2)^{-1}\,dr = a^{-1/2}\arctan((r-|x|)\cdot a^{-1/2})$$

with respect to a and we get

$$\int (a+(r-|x|)^2)^{-2}\,dr = \frac{1}{2}\cdot a^{-3/2}\Bigg(\arctan\frac{r-|x|}{\sqrt{a}}+\sqrt{a}\cdot\frac{r-|x|}{a+(r-|x|)^2}\Bigg).$$

Hence, after applying (4′.2), we obtain

$$(4'.15)\quad \int_{\sqrt{2\nu\tau}}^{\infty}\frac{dr}{(a+(r-|x|)^2)^2}$$
$$= \frac{1}{2}\cdot a^{-3/2}\Bigg(\frac{\pi}{2}+\arctan\frac{|x|-\sqrt{2\nu\tau}}{\sqrt{a}}+a^{1/2}\cdot\frac{|x|-\sqrt{2\nu\tau}}{a+(|x|-\sqrt{2\nu\tau})^2}\Bigg)$$
$$\le \frac{1}{2}a^{-3/2}\Bigg(\frac{\pi}{2}+\frac{\pi}{2}+\sqrt{a}\cdot\frac{1}{2\sqrt{a}}\Bigg)=\frac{1}{4}\cdot(2\pi+1)\cdot a^{-3/2}.$$

With the aid of the formula

$$\int\frac{r^{-2}\,dr}{a+(r-|x|)^2}$$
$$= -\frac{1}{a+|x|^2}\Bigg(\frac{|x|}{a+|x|^2}\ln\frac{a+(r-|x|)^2}{r^2}+\frac{1}{r}+\frac{a-|x|^2}{a+|x|^2}\frac{1}{\sqrt{a}}\arctan\frac{r-|x|}{\sqrt{a}}\Bigg)$$

we compute

$$(4'.16)\quad \int_{\sqrt{2\nu\tau}}^{\infty}\frac{r^{-2}\,dr}{a+(r-|x|)^2}=(2\nu\tau)^{-1/2}(a+|x|^2)^{-1}$$
$$+(a+|x|^2)^{-2}\Bigg(|x|\ln\frac{a+(|x|-\sqrt{2\nu\tau})^2}{2\nu\tau}-\frac{a-|x|^2}{\sqrt{a}}\Bigg(\frac{\pi}{2}+\arctan\frac{|x|-\sqrt{2\nu\tau}}{\sqrt{a}}\Bigg)\Bigg).$$

We differentiate (4′.16) with respect to a, and next, using the inequalities

$\ln s < s^{1/2}$ for $s > 0$ (because $e > 2$), (4′.2) and (4′.3), we estimate

$$
\begin{aligned}
(4'.17)\qquad & \int\limits_{\sqrt{2\nu\tau}}^{\infty} \frac{r^{-2}\,dr}{(a+(r-|x|)^2)^2} \\
&= \frac{2}{\sqrt{a}} \cdot \frac{1}{(a+|x|^2)^2} \cdot \frac{|x|^2-a}{|x|^2+a}\left(\frac{\pi}{2} + \arctan\frac{|x|-\sqrt{2\nu\tau}}{\sqrt{a}}\right) \\
&\quad + \frac{a^{-3/2}}{2} \cdot \frac{1}{a+|x|^2} \cdot \left(\frac{\pi}{2} + \arctan\frac{|x|-\sqrt{2\nu\tau}}{\sqrt{a}}\right) \\
&\quad + \frac{1}{2a} \cdot \frac{1}{a+|x|^2} \cdot \frac{|x|^2-a}{|x|^2+a} \cdot \frac{|x|-\sqrt{2\nu\tau}}{a+(|x|-\sqrt{2\nu\tau})^2} \\
&\quad + \frac{(2\nu\tau)^{-1/2}}{(a+|x|^2)^2} + \frac{2|x|}{(a+|x|^2)^3}\ln\frac{a+(|x|-\sqrt{2\nu\tau})^2}{2\nu\tau} \\
&\quad - \frac{|x|}{(a+|x|^2)^2} \cdot \frac{1}{a+(|x|-\sqrt{2\nu\tau})^2} \\
&\le \frac{2}{\sqrt{a}} \cdot \frac{1}{a^2} \cdot 1 \cdot \pi + \frac{a^{-3/2}}{2} \cdot \frac{1}{a} \cdot \pi \\
&\quad + \frac{1}{2a} \cdot \frac{1}{a+|x|^2} \cdot \left|\frac{|x|^2-a}{|x|^2+a}\right| \cdot \left|\frac{|x|-\sqrt{2\nu\tau}}{a+(|x|-\sqrt{2\nu\tau})^2}\right| \\
&\quad + \frac{(2\nu\tau)^{-1/2}}{(a+|x|^2)^2} + \frac{2}{(a+|x|^2)^2} \cdot \frac{|x|}{\sqrt{a+|x|^2}} \cdot \left(\frac{1}{a+|x|^2}\right)^{1/2} \\
&\quad \cdot \left(\frac{a+(|x|-\sqrt{2\nu\tau})^2}{2\nu\tau}\right)^{1/2} + 0 \\
&\le \frac{5}{2}\pi a^{-5/2} + \frac{1}{2a} \cdot \frac{1}{a} \cdot 1 \cdot \frac{1}{2\sqrt{a}} + \frac{(2\nu\tau)^{-1/2}}{(a+|x|^2)^2} \\
&\quad + \frac{2}{(a+|x|^2)^2} \cdot 1 \cdot \left(\frac{1}{2\nu\tau}\right)^{1/2}\left(\frac{a+(|x|-\sqrt{2\nu\tau})^2}{a+|x|^2}\right)^{1/2} \\
&\le \frac{1}{4} \cdot (10\pi+1) \cdot a^{-5/2} \\
&\quad + (2\nu\tau)^{-1/2}\left(1 + 2\left(1 + \sqrt{\frac{2\nu\tau}{a}} + \frac{2\nu\tau}{a}\right)^{1/2}\right)(a+|x|^2)^{-2}.
\end{aligned}
$$

Combining (4′.17) and (4′.15) allows us to conclude the estimate (4′.14):

$$
\begin{aligned}
((a+|\cdot|^2)^{-4} * G(\tau,\cdot))(x) &\le \frac{\pi}{4} \cdot (42\pi+5) \cdot a^{-5/2}(a+|x|^2)^{-2} \\
&+ (\nu\tau)^{-1/2} \cdot 2\pi\sqrt{2}\left(1 + 2\left(1 + \sqrt{\frac{2\nu\tau}{a}} + \frac{2\nu\tau}{a}\right)^{1/2}\right)(a+|x|^2)^{-4}.
\end{aligned}
$$

We put

$$C = \frac{\pi}{4} \cdot (42\pi + 5), \tag{4'.18}$$

$$M(\varrho) = 2\pi\sqrt{2}(1 + 2(1 + \sqrt{2\varrho} + 2\varrho)^{1/2}). \ \blacksquare \tag{4'.19}$$

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INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
REYMONTA 4
30-059 KRAKÓW, POLAND

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