is the boundary of the closed sphere E, whence EL is not a retract of E.

On the other hand EL is a retract of L. In fact, let us denote by $\hat{P}(\hat{t}, \hat{\varrho}, \hat{Z})$ a variable point on the set L; then $\hat{P}$ satisfies the conditions

$$(6.20) \qquad L: \{|\hat{Z}-Y^0| = \varepsilon_1(\hat{t}),\ 0 < \hat{\varrho} < \varepsilon_2(\hat{t}),\ \hat{t} > T\}.$$

Consider the following transformation $Q = (t^*, \varrho^*, Z^*) = V(P)$:

$$(6.21) \qquad Z^*-Y^0 = \frac{\varepsilon_1(t_1)}{\varepsilon_1(\hat{t})}(\hat{Z}-Y^0), \qquad \varrho^* = \varrho_1, \qquad t^* = t_1.$$

This transformation is continuous on the set L, and

1. if $\hat{P}\epsilon L$, then $V(\hat{P})\epsilon EL$,
2. if $\hat{P}\epsilon EL$, then $V(\hat{P}) = \hat{P}$.

Hence EL is a retract of L. It follows from the theorem of T. Ważewski cited above that there exists a point $P_1(t_1, \varrho_1, Z^{(1)})$, $P_1\epsilon(E-L)$, such that the solution passing through P_1 remains in ω, i. e. the corresponding trajectory remains in the cone C.

There exists at least a one-parameter family of solutions contained in ω (see T. Ważewski [4]), since the quantity ϱ_1 has been arbitrarily chosen in the interval $0 < \varrho < \varepsilon_2(t_1)$. This completes the proof of the theorem.

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On the functional equation $\varphi(x)+\varphi[f(x)] = F(x)$

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§ 1. The object of the present paper is the functional equation

$$(1) \qquad \varphi(x)+\varphi[f(x)] = F(x),$$

where $\varphi(x)$ denotes the required function, and $f(x)$ and $F(x)$ denote known functions.

Equation (1) is a direct generalization of the equation

$$\varphi(x)+\varphi(x^2) = x$$

discussed by H. Steinhaus [6], or of the equation

$$\varphi(x)+\varphi(x^a) = x \qquad (a > 1)$$

solved by G. H. Hardy [3], p. 77. I shall prove that under some natural assumptions equation (1) possesses infinitely many solutions which are continuous for every x that is not a root of the equation

$$(2) \qquad f(x) = x.$$

However, if we require the solution to be continuous for $x = x_0$, satisfying (2), then it turns out that there can exist at most one such solution. In the second part of this paper I shall prove that under further assumptions such a solution exists and is given by an explicit formula.

Of course, further generalizations of equation (1) are possible. R. Raclis [5] discusses equation (1) for complex x and finds meromorphic solutions. N. Gercevanoff [1] solves the equation

$$A(x)\varphi[f(x)]+\varphi(x) = F(x),$$

and M. Ghermanescu [2] solves the equation

$$A_0\varphi+A_1\varphi[f]+A_2\varphi[f(f)]+\ldots+A_n\varphi[f(f\ldots f)\ldots)] = F(x).$$

Nevertheless both these authors assume other hypotheses with regard to the function $f(x)$. Lastly T. Kitamura [4] has shown that the equation

$$F\big(\varphi[f(x,\lambda)], \varphi(x), x, \lambda\big) = 0$$

has, under suitable conditions, a solution containing an arbitrary function, but he does not discuss the regularity of the solutions.

§ 2. Every interval I such that $f(I) = I$ will be called a *modulus-interval* for the function $f(x)$.

LEMMA I. *Suppose that the function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$. In order that the interval $\langle a, b\rangle$ be a modulus-interval for the function $f(x)$ it is necessary and sufficient that a and b be roots of equation* (2).

Proof. Necessity. Since $f(\langle a; b\rangle) = \langle a, b\rangle$, we have

$$\max_{x\epsilon\langle a,b\rangle} f(x) = b, \qquad \min_{x\epsilon\langle a,b\rangle} f(x) = a.$$

The function $f(x)$ is increasing, and consequently

$$\max_{x\epsilon\langle a,b\rangle} f(x) = f(b), \qquad \min_{x\epsilon\langle a,b\rangle} f(x) = f(a),$$

whence $f(b) = b$ and $f(a) = a$.

Sufficiency. Suppose that a and b are roots of equation (2). Since the function $f(x)$ is continuous and increasing, $f(\langle a, b\rangle) = \langle f(a), f(b)\rangle = \langle a, b\rangle$, which completes the proof.

For each integer k we shall denote by $f^k(x)$ the k-th iteration of the function $f(x)$, i. e. we shall put

$$f^0(x) = x, \qquad \begin{array}{l} f^{k+1}(x) = f\big(f^k(x)\big), \\ f^{k-1}(x) = f^{-1}\big(f^k(x)\big) \end{array} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

LEMMA II. *Let $f(x)$ fulfil the hypotheses of lemma* I *and let $a < b$ be two consecutive roots of equation* (2). *Let us suppose further that $f(x) > x$ for all x in the interval (a, b). Then, for each $x\epsilon(a, b)$, the sequences $\{f^n(x)\}$ and $\{f^{-n}(x)\}$ are monotone and*

$$\lim_{n\to\infty} f^n(x) = b, \tag{3}$$

$$\lim_{n\to\infty} f^{-n}(x) = a. \tag{4}$$

Proof. The monotonity of the sequences $\{f^n(x)\}$ and $\{f^{-n}(x)\}$ follows from the inequality

$$f(x) > x \quad \text{for} \quad x\epsilon(a, b).$$

Thus the limits (3) and (4) exist and, by lemma I, lie in the interval $\langle a, b\rangle$.

Denoting $c \overset{\text{df}}{=} \lim_{n\to\infty} f^n(x)$ and passing to the limit in the relation

$$f\big(f^n(x)\big) = f^{n+1}(x)$$

we obtain $f(c) = c$, whence either $c = a$ or $c = b$. Of course it must be $c = b$.

Relation (4) may be obtained analogically.

§ 3. In what follows we shall restrict ourselves to the treatment of equation (1) in an interval $\langle a, b\rangle$, where a and b are two consecutive roots of equation (2). To be precise, let us assume that $f(x) - x > 0$ for all x in (a, b).

THEOREM I. *If the function $F(x)$ is continuous and the function $f(x)$ is continuous and strictly increasing in the interval $\langle a, b\rangle$, then equation* (1) *has an infinite number of solutions that are continuous in the open interval (a, b).*

Proof. Let us choose an arbitrary point $x_0\epsilon(a, b)$ and let us write $x_n = f^n(x_0)$. Points x_n divide the interval (a, b) into an enumerable number of intervals without common points:

$$(a, b) = \bigcup_{n=-\infty}^{n=+\infty} \langle x_n, x_{n+1}).$$

It can easily be verified that $f\big(\langle x_n, x_{n+1})\big) = \langle x_{n+1}, x_{n+2})$, for $n = 0, \pm 1, \pm 2, \ldots$

Let $g(x)$ be any continuous function defined in the interval $\langle x_0, x_1)$ which fulfils the condition

$$\lim_{x\to x_1 -} g(x) = F(x_0) - g(x_0). \tag{5}$$

We shall define a function $\varphi(x)$ by induction as follows:

$$\begin{aligned} &\varphi(x) = g(x) && \text{for} \quad x\epsilon\langle x_0, x_1), \\ &\varphi(x) = F[f^{-1}(x)] - \varphi[f^{-1}(x)] && \text{for} \quad x\epsilon\langle x_n, x_{n+1}), \quad n > 0, \\ &\varphi(x) = F(x) - \varphi[f(x)] && \text{for} \quad x\epsilon\langle x_n, x_{n+1}), \quad n < 0. \end{aligned} \tag{6}$$

The function $\varphi(x)$ is defined by (6) in the whole interval (a, b). It is obvious that it satisfies equation (1). The continuity of the function $\varphi(x)$ is guaranteed by the continuity of the function $g(x)$ and condition (5).

Taking all possible functions $g(x)$ which are continuous in the interval $\langle x_0, x_1)$ and fulfil condition (5), one can obtain all solutions of equation (1) that are continuous in (a, b). The set of those solutions has the power $\mathfrak{c}$.

Remark. If we do not require the continuity of solutions, their number will grow. Formulae (6) define then a solution of equation (1) for each function $g(x)$ defined in $\langle x_0, x_1)$. The set of those solutions has of course the power $\mathfrak{f}$.

THEOREM II. *Under the hypotheses of theorem* I *equation* (1) *possesses at most one solution that is continuous in the interval* $\langle a, b)$, *and at most one that is continuous in the interval* $(a, b\rangle$.

Proof. The difference of two solutions of equation (1) must fulfil the equation

$$\varphi(x)+\varphi[f(x)] = 0. \tag{7}$$

For the proof of the theorem it is sufficient to show that the unique solution of equation (7) that is continuous in $\langle a, b)$ or $(a, b\rangle$ is the function $\varphi(x) \equiv 0$.

Let $\varphi(x)$ be a solution of equation (7) and let us suppose that $\varphi(x) \not\equiv 0$. Consequently, there exists a point x_0 such that $\varphi(x_0) = c \neq 0$. Let us write $x_n = f^n(x_0)$. On account of (7) we have

$$\varphi(x_n)+\varphi(x_{n+1}) = 0, \qquad \varphi(x_n) = -\varphi(x_{n+1})$$

whence

$$\varphi(x_n) = (-1)^n c.$$

Consequently the limits $\lim\limits_{n\to\infty}\varphi(x_n)$ and $\lim\limits_{n\to-\infty}\varphi(x_n)$ and hence also $\lim\limits_{x\to b}\varphi(x)$ and $\lim\limits_{x\to a}\varphi(x)$ do not exist. Then the function $\varphi(x)$ cannot be continuous in $(a, b\rangle$ or $\langle a, b)$, which completes the proof.

THEOREM III. *If the functions* $f(x)$ *and* $F(x)$ *fulfil the assumptions of theorem* I, *and if there exist functions* $\varphi(x)$ *and* $\psi(x)$ *which satisfy equation* (1) *and are continuous in the intervals* $(a, b\rangle$ *and* $\langle a, b)$ *respectively, then*

$$\varphi(x) = \tfrac{1}{2}F(b)+\sum_{\nu=0}^{\infty}(-1)^\nu[F[f^\nu(x)]-F(b)], \tag{8}$$

$$\psi(x) = \tfrac{1}{2}F(a)-\sum_{\nu=1}^{\infty}(-1)^\nu[F[f^{-\nu}(x)]-F(a)]. \tag{9}$$

Proof. At first let us assume $F(b) = 0$. Let $\varphi(x)$ be the solution of equation (1) that is continuous in $(a, b\rangle$. Putting in equation (1) $x = b$ we get $\varphi(b) = 0$. Since $\varphi(x)$ is continuous for $x = b$, we must have $\lim\limits_{x\to b}\varphi(x) = 0$, and hence

$$\lim_{n\to\infty}\varphi[f^n(x)] = 0. \tag{10}$$

From relation (1) we have

$$\varphi(x) = F(x)-\varphi[f(x)]. \tag{11}$$

Next

$$\varphi[f(x)] = F[f(x)]-\varphi[f^2(x)]. \tag{12}$$

From (11) and (12) we obtain

$$\varphi(x) = F(x)-F[f(x)]+\varphi[f^2(x)].$$

By induction one can obtain the relation

$$\varphi(x) = \sum_{\nu=0}^{n}(-1)^\nu F[f^\nu(x)]+(-1)^{n+1}\varphi[f^{n+1}(x)],$$

i. e.

$$\varphi(x)-(-1)^{n+1}\varphi[f^{n+1}(x)] = \sum_{\nu=0}^{n}(-1)^\nu F[f^\nu(x)].$$

Passing to the limit as $n\to\infty$, we obtain, according to (10)

$$\varphi(x) = \sum_{\nu=0}^{\infty}(-1)^\nu F[f^\nu(x)].$$

Now let $F(b)$ be arbitrary. $\varphi(x)$ being the solution of equation (1) that is continuous in $(a, b\rangle$, the function

$$\gamma(x) \overset{\text{df}}{=} \varphi(x)-\tfrac{1}{2}F(b) \tag{13}$$

is the solution of the equation

$$\gamma(x)+\gamma[f(x)] = F(x)-F(b)$$

that is continuous in $(a, b\rangle$. From what has just been proved, the function $\gamma(x)$ must be expressible by the formula

$$\gamma(x) = \sum_{\nu=0}^{\infty}(-1)^\nu[F[f^\nu(x)]-F(b)]$$

whence, according to (13), we obtain formula (8). Formula (9) can be obtained in a similar manner.

§ 4. Of course, a solution of equation (1) that is continuous for $x = a$ or $x = b$ may be non-existent. It depends upon the function $F(x)$. If we assume some simple hypotheses regarding the function $F(x)$, we shall show that such a solution necessarily exists.

THEOREM IV. *If the functions* $f(x)$ *and* $F(x)$ *fulfil the assumptions of theorem* I, *and if moreover the function* $F(x)$ *is monotone in an interval* $(b-\eta, b\rangle$ *or* $\langle a, a+\eta)$, *where* η *is a positive number, then a solution of equation* (1) *that is continuous in* $(a, b\rangle$ *or* $\langle a, b)$ *necessarily exists.*

Proof. Let us suppose that the function $F(x)$ is increasing in an interval $(b-\eta, b\rangle$. We shall show that the series

$$\sum_{\nu=0}^{\infty}(-1)^\nu[F[f^\nu(x)]-F(b)] \tag{14}$$

uniformly converges in the interval $\langle h, b\rangle$ for every $a < h < b$.

Put $h_n = f^n(h)$. Since $\lim_{n\to\infty} h_n = b$, there exists an integer N_1 such that for $n > N_1$, $h_n \epsilon (b-\eta, b\rangle$. Further, for any given number $\varepsilon > 0$ one can choose an index $N > N_1$ such that for $n > N$

$$F(b) - F(h_n) < \varepsilon.$$

Let us now take an arbitrary $x \epsilon \langle h, b\rangle$ and let us write $x_n = f^n(x)$. For every n we have $x_n \geqslant h_n$, whence for $n > N$, $x_n \epsilon (b-\eta, b\rangle$ and

$$F(x_n) \geqslant F(h_n).$$

The series

$$\sum_{\nu=N+1}^{\infty} (-1)^{\nu} [F[f^{\nu}(x)] - F(b)]$$

obviously converges. Moreover the following inequalities hold (for $n > N$):

$$\Big|\sum_{\nu=n}^{\infty} (-1)^{\nu} [F(x_\nu) - F(b)]\Big| \leqslant F(b) - F(x_n) \leqslant F(b) - F(h_n) < \varepsilon,$$

whence the uniform convergence of series (14) follows immediately. Consequently the function $\varphi(x)$ defined by formula (8) is continuous in $(a, b\rangle$. It is obvious that it satisfies equation (1).

In the remaining cases the proof may be made out in a similar manner.

THEOREM V. *If the functions $f(x)$ and $F(x)$ fulfil the assumptions of theorem* I, *and if moreover for the function $F(x)$ in the interval $\langle a, b\rangle$ we have either the inequality $|F(x) - F(b)| \leqslant G(x)$ or the inequality $|F(x) - F(a)| \leqslant G(x)$, where $G(x)$ is any bounded function such that*

$$(15) \qquad G[f(x)]/G(x) < \vartheta < 1 \quad \textit{for} \quad x \epsilon (b-\eta, b)$$

or

$$(16) \qquad G(x)/G[f(x)] < \vartheta < 1 \quad \textit{for} \quad x \epsilon (a, a+\eta),$$

then a solution of equation (1) *that is continuous in $(a, b\rangle$ or $\langle a, b)$ exists.*

Proof. Supposing that formula (15) is fulfilled, we shall show that series (14) converges uniformly in an interval $\langle h, b\rangle$ for every $a < h < b$.

Let us write $h_n = f^n(h)$. There exists an integer N such that for $n > N$, $h_n \epsilon (b-\eta, b\rangle$. Let us put

$$A_n = \begin{cases} \sup\limits_{\langle h,b\rangle} G(x) & \text{for} \quad n \leqslant N, \\ \sup\limits_{\langle h_n,h_{n+1})} G(x) & \text{for} \quad n > N. \end{cases}$$

The sequence $\{A_n\}$ is decreasing, and moreover the series $\sum_{n=0}^{\infty} A_n$ converges. In fact, for every $x \epsilon \langle h_{n+1}, h_{n+2})$, $f^{-1}(x) \epsilon \langle h_n, h_{n+1})$, whence, according to (15), we have for $n > N$

$$G(x) < \vartheta G[f^{-1}(x)] \leqslant \vartheta \sup_{\langle h_n,h_{n+1})} G(x) = \vartheta A_n.$$

Hence, for $n > N$

$$A_{n+1} = \sup_{\langle h_{n+1},h_{n+2})} G(x) \leqslant \vartheta A_n,$$

whence the convergence of the series $\sum_{n=0}^{\infty} A_n$ follows immediately.

Now let us take an arbitrary $x \epsilon \langle h, b\rangle$ and let us write $x_n = f^n(x)$. We have

$$|F(x_n) - F(b)| \leqslant G(x_n).$$

As $x_n \geqslant h_n$, there exists an integer $k \geqslant 0$ such that $x_n \epsilon \langle h_{n+k}, h_{n+k+1})$. Hence $G(x_n) \leqslant A_{n+k} \leqslant A_n$. Consequently

$$|F(x_n) - F(b)| \leqslant A_n \quad \text{for} \quad x \epsilon \langle h, b\rangle,$$

whence the uniform convergence of series (14) follows immediately. Consequently the function $\varphi(x)$ defined by formula (8) is continuous in $(a, b\rangle$. It is obvious that it satisfies equation (1).

If we assume relation (16), the proof is analogous.

§ 5. All the above theorems will remain valid if one or both ends of the modulus-interval are infinite. If, for example, $b = \infty$, then by $F(b)$ we shall understand $\lim_{x\to\infty} F(x)$; the function $\varphi(x)$ will be called *continuous at infinity*, if there exists a finite limit $\lim_{x\to\infty} \varphi(x)$. Nevertheless, if $\lim_{x\to b} F(x) = \infty$ (b finite or infinite), then the solutions for which $\lim_{x\to b} \varphi(x)$ exists (equal to infinity of course) will not be unique.

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