Then we have

$$\gamma_E - 2\sigma = \lim_{n\to\infty} \gamma_{a_n} = \lim_{n\to\infty} \int\int K_0(x-y)\, d\mu_{a_n} d\mu_{a_n}$$
$$\geqslant \lim_{n\to\infty} \left[\int\int K_M(x-y)\, d\mu_{a_n} d\mu_{a_n} - \frac{M}{n+1}\right]$$
$$= \int\int K_M(x-y)\, d\mu_0 d\mu_0 \geqslant \int\int K(x-y)\, d\mu_0 d\mu_0 - \sigma \geqslant \gamma_E - \sigma,$$

which is an absurd. Then we must have $\gamma = \gamma_E$.

Turning to the sequence τ_n we shall prove

THEOREM 2. *The limit of τ_n exists and is equal to γ_E.*

Proof. First we shall prove the inequality $\overline{\lim_{n\to\infty}} \tau_n \leqslant \gamma_E$. By the definition of τ_n we have for $x \epsilon E$

$$\sum_{i=1}^{n} K(x-\zeta_i^{(n)}) \geqslant n\tau_n.$$

We integrate both sides and applying the equilibrium theorem we obtain

$$n\tau_n \leqslant \sum_{i=1}^{n} \int K(x-\zeta_i^{(n)})\, d\mu^*(x) \leqslant n\gamma_E.$$

Hence $\overline{\lim_{n\to\infty}} \tau_n \leqslant \gamma_E$. By lemma 2 and theorem 1 we have $\underline{\lim_{n\to\infty}} \tau_n \geqslant \gamma = \gamma_E$. Hence follows the theorem.

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Generalized characteristic directions for a system of differential equations

by Z. KOWALSKI (Kraków)*

1. This paper will be concerned with the asymptotic behavior of integrals for the system of differential equations

$$X^{\cdot} = F(t, X), \tag{1.1}$$

where $X = [x_1, \ldots, x_n]$, $F(t, X) = [f_1(t, X), \ldots, f_n(t, X)]$, $X^{\cdot} = dX/dt$, $F(t, 0) \equiv 0$, and the right-hand member is continuous (for all t) in a neighbourhood of the point $X = 0$.

The characteristic directions play the fundamental role for (1.1) if F is a linear or a perturbed linear dynamical system. If the right-hand member does not contain t explicitly and possesses the Stolz differential at the point $X = 0$, then there is the possibility of employing the characteristic directions (see [2]).

In this paper we give a natural generalization of characteristic directions which holds even for non-differentiable $F(t, X)$. The idea of that generalization for the system $x_1^{\cdot} = f_1(x_1, x_2)$, $x_2^{\cdot} = f_2(x_1, x_2)$ is contained in [1]. We give necessary (theorem 2) and sufficient (theorems 3, 4) conditions for the existence of trajectories tangent to a given generalized characteristic direction at the point $X = 0$. The continuity of tangents to trajectories is also discussed (theorem 2).

2. The letters $X, Y, F, \ldots$ will be systematically reserved to represent vectors or vector-functions, $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$, $F = (f_1, \ldots, f_n)$, $|X| = (x_1^2 + \ldots + x_1^2)^{1/2}$, while $x, y, f, \ldots$ will be used to represent scalars. Denote by S the $(n-1)$-dimensional sphere $|Y| = 1$ with a centre at the origin and a unit radius.

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DEFINITION OF THE SET Σ. Let Σ be a set of such points $Y \in S$ that the following limit exists:

$$G(Y) = \lim_{r\to 0,\, t\to+\infty} \frac{1}{r} F(t, rY), \tag{2.1}$$

and

$$G(Y) \neq 0. \tag{2.2}$$

DEFINITION OF A GENERALIZED CHARACTERISTIC DIRECTION. The direction of a unit vector Y, $|Y| = 1$, will be called the *generalized characteristic direction* if

$$Y \in \Sigma \quad \text{and} \quad G(Y) \text{ parallel to } Y. \tag{2.3}$$

DEFINITION OF A SET Ω. The set of all $Y \in S$ such that the direction of Y is the generalized characteristic direction, will be called Ω.

It follows that

$$\Omega \subset \Sigma. \tag{2.4}$$

Throughout the rest of the paper we shall use the following assumption H:

ASSUMPTION H. Suppose that Σ is a set on S, open relatively to S, such that

1° the limits (2.1) exist uniformly on every closed domain contained in Σ;

2° all accumulation points of Ω (if they exist) belong to $S-\Sigma$.

3. The following theorem can easily be proved:

THEOREM 1. *Suppose that*

a. *The right-hand member $F(t, X)$ of (1.1) is continuous in the neighbourhood of the point $X = 0$ for $-\infty < t < +\infty$, and there exists a Stolz differential at $X = 0$ for all t.*

b. *There exist finite limits*

$$a_{ij} = \lim_{t\to+\infty} (\partial f_i/\partial x_j)_{X=0} \quad (i, j = 1, 2, \ldots, n). \tag{3.1}$$

We denote by Λ a set of intersection points of S and characteristic directions of the matrix $[a_{ij}]$ that correspond to the real non-vanishing characteristic roots.

Under these assumptions (and the assumption H*) we have*

$$\Omega \subset \Lambda \subset \Omega + (S-\Sigma). \tag{3.2}$$

Remark 1. If $S = \Sigma$, then the set $S-\Sigma$ is empty and from (3.2) follows the equality

$$\Omega = \Lambda,$$

i. e. in that case the sets Ω and Λ determine the same directions. But the set Ω can be considered even when the limits (3.1) do not exist (the right-hand member of (1.1) may not be differentiable at all). Hence the set Ω can be regarded as a set of generalized characteristic directions; they reduce to the classical ones (Λ) under assumption (3.1).

Now we turn to the problem of the existence of tangents to trajectories of (1.1) at the origin. In that problem the essential role is played by generalized characteristic directions for system (1.1).

First we give without proof two lemmas connected with a certain transformation of system (1.1).

LEMMA 1. *Let us introduce the transformation*

$$X = rY, \quad |Y| \equiv 1, \quad r > 0. \tag{3.3}$$

Under the assumption H (3.3) *transforms system* (1.1) *into a system*

$$\begin{aligned} r^{-1}r^{\cdot} &= YG(Y) + \gamma(t, r, Y) \\ Y^{\cdot} &= G(Y) - Y\big(YG(Y)\big) + \Gamma(t, r, Y) \end{aligned} \quad \text{for} \quad Y \in \Sigma, \tag{3.4}$$

where $\gamma(t, r, Y) = o(1)$, $\Gamma(t, r, Y) = o(1)$ uniformly on every closed subset contained in Σ, for $r \to 0$, $t \to +\infty$.

Write

$$\varphi(Y) = YG(Y), \quad \Phi(Y) = G(Y) - Y\big(YG(Y)\big). \tag{3.5}$$

In terms of this notation, system (3.4) takes the simple form

$$\begin{aligned} r^{-1}r^{\cdot} &= \varphi(Y) + \gamma(t, r, Y), \\ Y^{\cdot} &= \Phi(Y) + \Gamma(t, r, Y), \end{aligned} \quad Y \in \Sigma, \tag{3.6}$$

where $\gamma(t, r, Y) = o(1)$, $\Gamma(t, r, Y) = o(1)$ for $r \to 0$, $t \to +\infty$.

LEMMA 2. *Under the assumption* H, *the set of points Y^0, such that $Y^0 \in \Sigma$ and*

$$\Phi(Y^0) = 0, \tag{3.7}$$

is identical with the set Ω.

4. Lemma 1 and 2 permit us to establish the theorem connected with the continuity (at the origin) of tangents to trajectories of (1.1).

THEOREM 2. *Let us make assumption* H *and let $X = X(t)$ be a trajectory of* (1.1). *Under these assumptions*

1° *there exists no vector* Y^0

$$Y^0 \in \Sigma - \Omega$$

tangent at the origin to trajectories of (1.1) *reaching the singular point as* $t \to +\infty$;

2° *if* $Y^0 \in \Omega$ *and there exists a trajectory tangent to* Y^0 *at the origin, then the tangent is continuous at the origin.*

Proof. First we prove 1°. Suppose the contrary. Then there exists a vector $Y^0 = (y_1^0, \ldots, y_n^0)$, $Y^0 \in \Sigma - \Omega$, tangent to some trajectory $X = X(t)$ at the origin. From the definition of set Ω, relation (2.3) and lemma 2 it follows that there exists such an integer i, $1 \leqslant i \leqslant n$, that

$$\varphi_i(Y) = \varphi_i(Y(t)) \neq 0,$$

for t near $+\infty$, where $\varphi_i(Y)$ is the i-th coordinate of the vector $\Phi(Y)$:

$$\Phi(Y) = [\varphi_1(Y), \varphi_2(Y), \ldots, \varphi_n(Y)].$$

But then (3.6) yields $y_i \neq 0$ for large t, whence y_i can be introduced as a new independent variable. The function $r = r(y_i)$ is then defined for $y_i^0 - \delta \leqslant y_i < y_i^0$ or $y_i^0 < y_i \leqslant y_i + \delta$, $\delta > 0$, and satisfies the equation (see (3.6)):

$$\frac{dr}{dy_i} = r\frac{\varphi(Y) + o(1)}{\varphi_i(Y) + o(1)}.$$

Hence

$$r = r_p \exp \int_{y_{ip}}^{y_i} \frac{\varphi(Y) + o(1)}{\varphi_i(Y) + o(1)} dy_i, \tag{4.1}$$

where r_p and y_{ip} denote initial values for r and y_i respectively. If y_i approaches the finite value y_i^0, then the right-hand member of (4.1) approaches a finite and positive value, which contradicts $r(y_i) \to 0$ as $y_i \to y_i^0$. This completes the proof of the first part of the theorem.

Now we shall show that if $Y^0 \in \Omega$, then the tangent to the trajectory is continuous at the origin, i. e. the direction of a tangent to a trajectory approaches the direction of the tangent Y^0 at the origin as $t \to +\infty$.

There is no loss of generality in assuming that $Y^0 = [1, 0, \ldots, 0]$. Then from (3.7) and (3.5) follows

$$G(Y^0) - Y^0 g_1(Y^0) = 0,$$

where $g_i(Y)$ denotes the i-th coordinate of the vector $G(Y)$:

$$G(Y) = [g_1(Y), g_2(Y), \ldots, g_n(Y)].$$

Hence

$$g_i(Y^0) = 0, \quad i = 2, 3, \ldots, n.$$

On the other hand $G(Y^0) \neq 0$ since $Y^0 \in \Omega$ (see (2.2) and (2.4)), whence

$$g_1(Y^0) \neq 0.$$

The limit (2.1) exists uniformly in a neighbourhood of Y^0; therefore

$$\frac{x_1^{\cdot}}{r} = \frac{f_1(t, rY)}{r} \neq 0 \quad \text{for } t \text{ near } +\infty,$$

where $f_i(t, X)$ denotes the i-th coordinate of the vector $F(t, X)$.

Hence

$$\frac{x_i^{\cdot}}{x_1^{\cdot}} = \frac{f_i(t, rY)}{r} : \frac{f_1(t, rY)}{r} \to \frac{g_i(Y^0)}{g_1(Y^0)} \quad (i = 2, 3, \ldots, n), \tag{4.2}$$

as $t \to +\infty$.

In order to discuss the changes of direction of the tangent vector $X^{\cdot} = [x_1^{\cdot}, \ldots, x_n^{\cdot}]$, we introduce the n-dimensional space and the point A with coordinates $A = (x_1^{\cdot}, x_2^{\cdot}, \ldots, x_n^{\cdot})$ as well as its radius-vector $X^{\cdot}$.

Let us write

$$X^{\cdot} = \sigma U, \quad |U| \equiv 1, \quad \sigma > 0, \tag{4.3}$$

where the unit vector U has the coordinates $U = [u_1, u_2, \ldots, u_n]$. Then from (4.2) and (4.3) follows

$$u_i/u_1 \to 0, \quad i = 2, 3, \ldots, n.$$

Therefore $(u_i/u_1)^2 \to 0$ and

$$\sum_{j=2}^{n} \left(\frac{u_j}{u_1}\right)^2 = \frac{1}{u_1^2} \sum_{j=2}^{n} u_j^2 = \frac{1}{u_1^2}(1 - u_1^2) = \frac{1}{u_1^2} - 1 \to 0,$$

whence $u_1^2 \to 1$.

It can be seen from this that $U = [u_1, \ldots, u_n] \to [\pm 1, 0, \ldots, 0]$ as $t \to +\infty$, i. e. the direction of the vector $X^{\cdot} = [x_1^{\cdot}, \ldots, x_n^{\cdot}]$ approaches the direction of the vector $Y^0 = [1, 0, \ldots, 0]$. This completes the proof of the second part of the theorem.

Corollary. From theorem 2 it follows immediately that *if there exists tangent* Y^0 (*at the origin*) *to the trajectory*

$$X = X(t), \quad X(t) \to 0 \quad \textit{as} \quad t \to +\infty,$$

then $Y^0 \in S - \Sigma$ *or* $Y^0 \in \Omega$. *In the second case the tangent is continuous at the origin.*

That corollary generalizes a theorem of P. Hartman and A. Wintner [1].

5. Now we shall give a sufficient condition for the existence of trajectories $X = X(t)$, $X(t) \to 0$ as $t \to +\infty$, tangent at the origin to a given generalized characteristic direction determined by a vector $Y^0 \in \Omega$ (see theorems 3 and 4).

For this purpose we introduce the hyperplane

$$ZY^0 - 1 = 0, \tag{5.1}$$

tangent to the unit sphere at the point Y^0, and the transformation

$$X = \varrho Z, \quad \text{where} \quad \varrho > 0, \quad ZY^0 - 1 \equiv 0. \tag{5.2}$$

This transformation determines a correspondence between the radius-vector X and the radius-vector Z of the point on the hyperplane (5.1).

Now we give the following obvious lemma:

Lemma 3. *Let us suppose that the assumption* H *is fulfilled. Then* (5.2) *transforms system* (1.1) *into a system*

$$\begin{aligned} \varrho^{-1}\varrho^{\cdot} &= \psi(Z) + \delta(t, \varrho, Z), \\ Z^{\cdot} &= \Psi(Z) + \Delta(t, \varrho, Z) \end{aligned} \tag{5.3}$$

for Z *belonging to some neighbourhood of* Y^0, *where* $\delta(t, \varrho, Z) = o(1)$, $\Delta(t, \varrho, Z) = o(1)$ *for* $\varrho \to 0$, $t \to +\infty$, *and*

$$\psi(Z) = Y^0 G(Z), \quad \Psi(Z) = G(Z) - Z\big(Y^0 G(Z)\big). \tag{5.4}$$

Remark 2. In the second equation (5.3) the velocity vector $Z^{\cdot}$ of the point moving on the hyperplane (5.1) may be regarded as the sum of two components Ψ and Δ. We shall verify the following conditions of orthogonality:

$$Z^{\cdot} Y^0 \equiv 0, \quad \Psi Y^0 \equiv 0, \quad \Delta Y^0 \equiv 0. \tag{5.5}$$

In fact, the first condition (5.5) follows immediately from (5.1) by differentiation. The second condition can be derived from (5.4) as follows:

$$\Psi Y^0 = G(Z)\, Y^0 - (Y^0 Z)\big(Y^0 G(Z)\big) = \big(Y^0 G(Z)\big)(1 - Y^0 Z) \equiv 0.$$

Finally, the last condition (5.5) results from (5.3), the first and the second condition (5.5).

Remark 3. It can easily be verified that the linear part of the right-hand member of system (1.1) can be transformed into $\psi(Z)$ and $\Psi(Z)$. This means that the system

$$X^{\cdot} = F(t, X) = AX,$$

where A is a constant $n \times n$ matrix, can be transformed into a system (5.3) with $\delta(t, \varrho, Z) \equiv 0$, $\Delta(t, \varrho, Z) \equiv 0$.

Remark 4. The quantity ϱ of the right-hand member of transformation (5.2) can be interpreted as the orthogonal projection of the vector X on the direction of Y^0.

In fact, from (5.2) follows $XY^0 = \varrho Z Y^0 = \varrho \cdot 1 = \varrho$.

6. Consider the sets

$$C_0\colon \{0 < |Z - Y^0| \leqslant \varepsilon_0,\ 0 < \varrho \leqslant \varrho_0\}$$

and

$$\bar{C}_0\colon \{|Z - Y^0| \leqslant \varepsilon_0,\ 0 < \varrho \leqslant \varrho_0\}.$$

$\bar{C}_0$ in x-coordinates is a circular cone with a vertex at the origin. Denote by D_0 and $\bar{D}_0$ the sets of points Z on the hyperplane (5.1):

$$D_0\colon \{0 < |Z - Y^0| \leqslant \varepsilon_0\},$$

$$\bar{D}_0\colon \{|Z - Y^0| \leqslant \varepsilon_0\}.$$

Assumption K. We assume that the cone C_0 contains only one generalized characteristic direction Y^0 and

$$\psi(Z) < 0 \quad \text{for} \quad Z \in \bar{D}_0. \text{(1)}$$

Theorem 3. *Suppose that the right-hand member of* (1.1) *satisfies assumptions* H, K *and*

$$Z\Psi(Z) < a(Z^2 - 1), \quad a < 0, \quad \textit{for} \quad Z \in D_0, \tag{6.1}$$

where $a = \text{const}$.

Then all trajectories $Z = Z(t)$, $\varrho = \varrho(t)$ *of system* (5.3) *entering into the cone* C_0 *for large* t, *satisfy the conditions*

$$Z(t) \to Y^0, \quad \varrho(t) \to 0 \quad \textit{as} \quad t \to +\infty;$$

(1) There is $\psi(Z) \neq 0$ in some neighbourhood of Y^0, since $\psi(Y^0) = Y^0 G(Y^0) \neq 0$ and $\psi(Z)$ is continuous. If $\psi(Z) > 0$, then $\varrho(t)$ increases for sufficiently large t and small ϱ (see (5.3) and remark 4) and the trajectory cannot reach the origin. Therefore we shall discuss the case $\psi(Z) < 0$.

the corresponding trajectories $X = X(t)$ *of* (1.1) *determined by* (5.2) *reach the singular point as* $t \to +\infty$ *with the tangent* Y^0 *at the origin.*

Proof. We will introduce the moving cone C,

(6.2) $$C: \{|Z-Y^0| \leqslant \varepsilon_1(t),\ 0 < \varrho \leqslant \varepsilon_2(t)\},$$

such that all trajectories enter into C through the lateral surface

(6.3) $$\{|Z-Y^0| = \varepsilon_1(t),\ 0 < \varrho \leqslant \varepsilon_2(t)\},$$

as well as through the rear surface

(6.4) $$\{|Z-Y^0| \leqslant \varepsilon_1(t),\ \varrho = \varepsilon_2(t)\},$$

where $0 < \varepsilon_i(t) \to 0$ as $t \to +\infty$ $(i = 1, 2)$, and $C \subset C_0$.

a. First we shall determine $\varepsilon_2(t)$. If there exists a function $\varepsilon_2(t)$ such that the trajectories enter C through the rear surface, then

(6.5) $$\frac{d}{dt}[\varrho(t)-\varepsilon_2(t)] < 0 \quad \text{along the rear surface.}$$

But

(6.6) $$\frac{d}{dt}[\varrho(t)-\varepsilon_2(t)] = \dot{\varrho}-\dot{\varepsilon}_2 = \varrho[\psi(Z)+\delta(t, \varrho, Z)-\dot{\varepsilon}_2$$
$$= \varepsilon_2[\psi(Z)+\delta(t, \varrho, Z)]-\dot{\varepsilon}_2.$$

The assumption K holds for $Z \epsilon \bar{D}_0$, whence there exists a number $\beta > 0$ such that
$$\psi(Z) < -\beta \quad \text{for} \quad Z \epsilon \bar{D}_0.$$

On the other hand there exist such numbers $\varrho_0 > 0$ and T that
$$|\delta(t, \varrho, Z)| < \tfrac{1}{2}\beta \quad \text{for} \quad t \geqslant T, \quad \varrho \leqslant \varrho_0,$$

whence

(6.7) $$\psi(Z)+\delta(t, \varrho, Z) < -\tfrac{1}{2}\beta.$$

(6.6) and (6.7) imply that
$$\frac{d}{dt}[\varrho(t)-\varepsilon_2(t)] < -\frac{1}{2}\beta\varepsilon_2-\dot{\varepsilon}_2.$$

If
$$-\tfrac{1}{2}\beta\varepsilon_2-\dot{\varepsilon}_2 = 0,$$

then (6.5) holds; therefore we take
$$\varepsilon_2(t) = e^{-\beta t/2}.$$

The function $\varepsilon_2(t)$ just determined satisfies condition (6.5) and $0 < \varepsilon_2(t) \to 0$ as $t \to +\infty$.

b. Now we shall find $\varepsilon_1(t)$. If there exist $\varepsilon_1(t)$ such that the trajectories enter C through the lateral surface, then

(6.8) $$\frac{d}{dt}[(Z-Y^0)^2-\varepsilon_1^2(t)] < 0 \quad \text{along the lateral surface.}$$

Let us note that there exists such a function $\mu(t)$, $0 < \mu(t) \to 0$ as $t \to +\infty$, $\dot{\mu}(t) < 0$, that

(6.9) $$|Z\Delta(t, \varrho, Z)| < \mu(t)\sqrt{Z^2-1},$$

along the lateral surface (6.3). In fact,
$$|Z\Delta(t, \varrho, Z)| = |Z|\cdot|\Delta(t, \varrho, Z)|\cdot|\cos(Z, \Delta)| = |Z|\cdot|\Delta|\frac{|Z-Y^0|}{|Z|}$$
$$= |\Delta|\sqrt{Z^2-1}.$$

On the other hand, if $\Theta(t)$ denotes the cone
$$\Theta(t)\colon \{Z \epsilon \bar{D}_0,\ 0 < \varrho \leqslant \varepsilon_2(t)\}$$

and $t \to +\infty$, then
$$|\Delta(t, \varrho, Z)| \to 0.$$

That ensures that the function $|\Delta(t, \varrho, Z)|$ has the finite upper bound $m(t)$ on the set $\Theta(t)$
$$m(t) = \sup_{\Theta(t)}|\Delta(t, \varrho, Z)| < +\infty,$$

and $0 \leqslant m(t) \to 0$ as $t \to +\infty$. The function $m(t)$ may not possess the derivative $m'(t)$ for some t, but we can find a new function $\mu(t)$ such that there exist $\mu'(t)$, and for large t
$$0 \leqslant m(t) < \mu(t) \to 0, \quad \text{as} \quad t \to +\infty, \quad \mu'(t) < 0.$$

Hence (6.9) holds.

Now we turn to the left-hand member of (6.8). We obtain successively
$$\frac{d}{dt}[(Z-Y^0)^2-\varepsilon_1^2(t)] = 2(Z-Y^0)\dot{Z}-2\varepsilon_1\dot{\varepsilon}_1 = 2Z\dot{Z}-2\varepsilon_1\dot{\varepsilon}_1$$
$$= 2Z\Psi(Z)+2Z\Delta(t, \varrho, Z)-2\varepsilon_1\dot{\varepsilon}_1$$
$$< 2a(Z^2-1)+2|Z\Delta|-2\varepsilon_1\dot{\varepsilon}_1$$
$$< 2a\varepsilon_1^2+2\mu(t)\varepsilon_1-2\varepsilon_1\dot{\varepsilon}_1$$
$$= 2\varepsilon_1(a\varepsilon_1+\mu(t)-\dot{\varepsilon}_1).$$

We put

$$a\varepsilon_1 + \mu(t) - \varepsilon_1^{\cdot} = 0. \tag{6.10}$$

The function

$$\varepsilon_1(t) = e^{at}\int_0^t e^{-a\tau}\mu(\tau)d\tau \tag{6.11}$$

satisfies equation (6.10) and the condition

$$0 < \varepsilon_1(t) \to 0 \quad \text{as} \quad t \to +\infty,$$

which can be verified by means of de l'Hospital's theorem. Hence $\varepsilon_1(t)$ just determined satisfies (6.8).

c. From the parts a and b of the proof it follows that all trajectories $X = X(t)$ of (1.1) entering the cone C remain in its interior, i. e. they reach the singular point as $t \to +\infty$ and possess the common tangent Y^0 at the origin.

This completes the proof of theorem 3.

THEOREM 4. *If the right-hand member of* (1.1) *satisfies the assumptions* H, K *and*

$$Z\Psi(Z) > b(Z^2-1), \quad b > 0, \quad Z \in D_0, \tag{6.12}$$

where $b = \text{const}$, *then there exists a one-parameter family of integrals* $Z = Z(t)$, $\varrho = \varrho(t)$ *of system* (5.3) *such that*

$$Z(t) \to Y^0, \quad \varrho(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

The corresponding trajectories $X = X(t)$ *of* (1.1) *determined by* (5.2), *reach the singular point as* $t \to +\infty$ *with the tangent* Y^0 *at the origin.*

Proof. We shall determine the cone (6.2) such that the trajectories leave C through the lateral surface (6.3) and enter C through the rear surface (6.4), where $0 < \varepsilon_i(t) \to 0$ as $t \to +\infty$ $(i = 1, 2)$. We may determine

$$\varepsilon_2(t) = e^{-\beta t/2}$$

β satisfying (6.7)

Now we shall detemine $\varepsilon_1(t)$. If there exist $\varepsilon_1(t)$ such that the trajectories leave C through the lateral surface, then

$$\frac{d}{dt}[(Z-Y^0)^2 - \varepsilon_1^2(t)] > 0 \quad \text{along the lateral surface.} \tag{6.14}$$

Let us note that as in the part b of the proof of theorem 3, there exists a differentiable function $\mu(t)$ such that

$$0 < \mu(t) \to 0 \quad \text{as} \quad t \to +\infty, \quad \mu^{\cdot}(t) < 0,$$

and

$$|Z\Delta(t, \varrho, Z)| < \mu(t)\sqrt{Z^2-1}, \tag{6.9}$$

along the lateral surface (6.3).

Hence along the lateral surface we obtain

$$\begin{aligned}\frac{d}{dt}[(Z-Y^0)^2 - \varepsilon_1^2(t)] &= 2(Z-Y^0)Z^{\cdot} - 2\varepsilon_1\varepsilon_1^{\cdot} = 2ZZ^{\cdot} - 2\varepsilon_1\varepsilon_1^{\cdot} \\ &= 2Z\Psi(Z) + 2Z\Delta(t, \varrho, Z) - 2\varepsilon_1\varepsilon_1^{\cdot} \\ &> 2b(Z^2-1) - 2|Z\Delta| - 2\varepsilon_1\varepsilon_1^{\cdot} \\ &> 2b(Z^2-1) - 2\mu(t)\sqrt{Z^2-1} - 2\varepsilon_1\varepsilon_1^{\cdot} \\ &= 2b\varepsilon_1^2 - 2\mu(t)\varepsilon_1 - 2\varepsilon_1\varepsilon_1^{\cdot} \\ &= 2\varepsilon_1(b\varepsilon_1 - \mu(t) - \varepsilon_1^{\cdot}).\end{aligned}$$

Let us consider the condition

$$b\varepsilon_1 - \mu(t) - \varepsilon_1^{\cdot} > 0. \tag{6.15}$$

The function

$$\varepsilon_1(t) = \frac{1}{b}\mu(t) \tag{6.16}$$

satisfies (6.15) and $0 < \varepsilon_1(t) \to 0$ as $t \to +\infty$. Hence $\varepsilon_1(t)$ just determined satisfies (6.14).

Now we shall apply the theorem of T. Ważewski given in [3]. Let us consider the set ω of points $P(t, \varrho, Z)$:

$$\omega\colon \{|Z-Y^0| < \varepsilon_1(t),\ 0 < \varrho < \varepsilon_2(t),\ t > T\},$$

and let us denote by L a part of the boundary of ω such that the solutions of (5.3) leave ω through the points of L. If $P(t, \varrho, Z)$ denote the variable point on L, then

$$L\colon \{|Z-Y^0| = \varepsilon_1(t),\ 0 < \varrho < \varepsilon_2(t),\ t > T\}. \tag{6.17}$$

We introduce the set E,

$$E\colon \{|Z-Y^0| \leqslant \varepsilon_1(t),\ \varrho = \varrho_1,\ t = t_1\}, \tag{6.18}$$

where t_1 is a fixed value $t_1 > T$, and ϱ_1 is arbitrarily chosen so as to satisfy

$$0 < \varrho_1 < \varepsilon_2(t_1).$$

Then the set EL

$$EL\colon \{|Z-Y^0| = \varepsilon_1(t_1),\ \varrho = \varrho_1,\ t = t_1\} \tag{6.19}$$

is the boundary of the closed sphere E, whence EL is not a retract of E.

On the other hand EL is a retract of L. In fact, let us denote by $\hat{P}(\hat{t}, \hat{\varrho}, \hat{Z})$ a variable point on the set L; then $\hat{P}$ satisfies the conditions

(6.20) $$L\colon \{|\hat{Z}-Y^0| = \varepsilon_1(\hat{t}),\ 0 < \hat{\varrho} < \varepsilon_2(\hat{t}),\ \hat{t} > T\}.$$

Consider the following transformation $Q = (t^*, \varrho^*, Z^*) = V(P)$:

(6.21) $$Z^* - Y^0 = \frac{\varepsilon_1(t_1)}{\varepsilon_1(\hat{t})}(\hat{Z}-Y^0), \qquad \varrho^* = \varrho_1, \qquad t^* = t_1.$$

This transformation is continuous on the set L, and

1. if $\hat{P} \epsilon L$, then $V(\hat{P}) \epsilon EL$,
2. if $\hat{P} \epsilon EL$, then $V(\hat{P}) = \hat{P}$.

Hence EL is a retract of L. It follows from the theorem of T. Ważewski cited above that there exists a point $P_1(t_1, \varrho_1, Z^{(1)})$, $P_1 \epsilon (E-L)$, such that the solution passing through P_1 remains in ω, i. e. the corresponding trajectory remains in the cone C.

There exists at least a one-parameter family of solutions contained in ω (see T. Ważewski [4]), since the quantity ϱ_1 has been arbitrarily chosen in the interval $0 < \varrho < \varepsilon_2(t_1)$. This completes the proof of the theorem.

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On the functional equation $\varphi(x)+\varphi[f(x)] = F(x)$

by M. Kuczma (Kraków)

§ 1. The object of the present paper is the functional equation

(1) $$\varphi(x)+\varphi[f(x)] = F(x),$$

where $\varphi(x)$ denotes the required function, and $f(x)$ and $F(x)$ denote known functions.

Equation (1) is a direct generalization of the equation

$$\varphi(x)+\varphi(x^2) = x$$

discussed by H. Steinhaus [6], or of the equation

$$\varphi(x)+\varphi(x^a) = x \qquad (a > 1)$$

solved by G. H. Hardy [3], p. 77. I shall prove that under some natural assumptions equation (1) possesses infinitely many solutions which are continuous for every x that is not a root of the equation

(2) $$f(x) = x.$$

However, if we require the solution to be continuous for $x = x_0$, satisfying (2), then it turns out that there can exist at most one such solution. In the second part of this paper I shall prove that under further assumptions such a solution exists and is given by an explicit formula.

Of course, further generalizations of equation (1) are possible. R. Raclis [5] discusses equation (1) for complex x and finds meromorphic solutions. N. Gercevanoff [1] solves the equation

$$A(x)\varphi[f(x)]+\varphi(x) = F(x),$$

and M. Ghermanescu [2] solves the equation

$$A_0\varphi+A_1\varphi[f]+A_2\varphi[f(f)]+\ldots+A_n\varphi[f(f\ldots f)\ldots)] = F(x).$$

Nevertheless both these authors assume other hypotheses with regard to the function $f(x)$. Lastly T. Kitamura [4] has shown that the