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ANNALES
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VI (1959)

On some constants related to the generalized potentials

by A. Szybiak (Kraków)

Let $\mathcal{E}^m$ denote the m-dimensional ($m \geqslant 2$) Euclidean space and $x, y, \ldots$ be the points in this space with the coordinates respectively $x^1, \ldots, x^m$, $y^1, \ldots, y^m$. We denote by $x \pm y$ the point in this space with the coordinates $x^1 \pm y^1, \ldots, x^m \pm y^m$. $|x|$ denotes the Cartesian distance of x from the origin; then $|x| = \big(\sum_{i=1}^{m} (x^i)^2\big)^{1/2}$.

We consider in $\mathcal{E}^m$ a function $K(x)$ which is continuous outside 0 and satisfies the following assumptions:

1. $0 < \lim_{x \to 0} K(x) = K(0) \leqslant +\infty$.

2. $K(x) = K(-x)$.

3. $\int_{|x|<1} K(x)\,dx < \infty$ (dx being the volume element in $\mathcal{E}^m$).

4. (Maximum principle of O. Frostman.) *For every measure $\mu \geqslant 0$ of the carrier $F \subset \mathcal{E}^m$ we have*

$$\sup_{x \in \mathcal{E}^m} \int K(x-y)\,d\mu(y) = \sup_{x \in F} \int K(x-y)\,d\mu(y).$$

The function K will be named the *kernel* and the functions of the shape

$$\int K(x-y)\,d\mu(y)$$

the *generalized potentials*.

These potentials have the following fundamental property:

Equilibrium theorem. *If E is a compact in $\mathcal{E}^m$ such that*

$$\inf_{\mu} \iint K(x-y)\,d\mu(y)\,d\mu(x) \overset{\text{df}}{=} \gamma_E < \infty \qquad (\mu \geqslant 0,\ \mu(E) = 1,\ \mu(\mathcal{E}^m - E) = 0)$$

then there exists a measure μ^* realizing this infimum and such that for $x \in E$ we have

$$\int K(x-y)\,d\mu^*(y) \leqslant \gamma_E; \tag{1}$$

the strong inequality in (1) *holds at most on a subset e of E such that $\mu^*(e) = 0$ and $\gamma_e = \infty$*:

The object of this paper is to prove the equality of the quantity γ_E and some other constants obtained by the generalized method of Fekete.

We fix E and the kernel K and we assume that $\gamma_E < \infty$. n being a positive integer, we choose on E $n+1$ points $\eta_0^{(n)}, \ldots, \eta_n^{(n)}$ for which

$$\sum_{i\neq j} K(\eta_i^{(n)} - \eta_j^{(n)}) = \inf_{\{x_i\}} \sum_{i\neq j} K(x_i - x_j) \qquad (i, j = 0, \ldots, n),$$

the above lower bound being taken over all systems of $n+1$ points $\{x_0, \ldots, x_n\} \subset E$. The points $\eta_i^{(n)}$ are named the *extreme points* of E and the system $\{\eta_0^{(n)}, \ldots, \eta_n^{(n)}\}$ is termed the *n-th extreme system* of E.

Further we choose on E n points $\zeta_0^{(n)}, \ldots, \zeta_n^{(n)}$ such that

$$\min_{x\in E} \sum_{i=1}^{n} K(x - \zeta_i^{(n)}) = \sup_{\{x_i\}} \Big\{ \min_{x\in E} \sum_{i=1}^{n} K(x - x_i) \Big\},$$

the upper bound being taken over all systems of n points $\{x_1, \ldots, x_n\} \subset F$ and F being an arbitrary fixed compact containing E.

In the case of $m = 2$, $F = \mathcal{E}^m$, $K(x) = \log|x|^{-1}$ the function $\exp\{-\sum_{i=1}^{n} K(x - \zeta_i^{(n)})\}$ is the n-th *Čebyšev polynomial* of the set E.

We shall use the notation

$$(n+1)^{-2} \sum_{i\neq j} K(\eta_i^{(n)} - \eta_j^{(n)}) = \gamma_n \qquad (i, j = 0, \ldots, n),$$

$$(n+1)^{-1} \max_{0\leqslant j\leqslant n} \sum_{i\neq j} K(\eta_i^{(n)} - \eta_j^{(n)}) = \delta_n, \qquad n^{-1} \min_{x\in E} \sum_{i=1}^{n} K(x - \zeta_i^{(n)}) = \tau_n$$
$$(i = 0, \ldots, n).$$

G. Pólya and G. Szegö have considered the kernel $K(x) = |x|^{-1}$ for $m = 3$. Transferring their proofs to our more general case we can show that the following lemmas hold:

LEMMA 1. *The sequence*

$$\frac{2}{(n+1)n} \sum_{0\leqslant i<j\leqslant n} K(\eta_i^{(n)} - \eta_j^{(n)}) = \frac{n+1}{n}\gamma_n$$

is not decreasing. Then γ_n converges and its limit — denoted by γ — is finite or ∞.

LEMMA 2. *The following inequality holds:*

$$\gamma_n \leqslant \delta_n \leqslant \frac{n}{n+1}\tau_n \qquad (n = 1, 2, \ldots).$$

We shall prove

THEOREM 1. *If $\gamma_E < \infty$ then $\gamma = \lim_{n\to\infty} \gamma_n$ equals γ_E.*

Proof. Putting

$$K_0(x) = \begin{cases} K(x) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0, \end{cases}$$

we evidently have

$$\iint K(x-y)\,d\mu(y)\,d\mu(x) = \iint K_0(x-y)\,d\mu(y)\,d\mu(x)$$

and

$$(n+1)^{-2} \sum_{\substack{i\neq j \\ i,j=0,\ldots,n}} K(\eta_i^{(n)} - \eta_j^{(n)}) = \iint K_0(x-y)\,d\mu_n(y)\,d\mu_n(x),$$

μ_n being a measure which is equal to $(n+1)^{-1}$ at each point of the n-th extreme system of the set E, and to 0 on the sets disjoint from $\{\eta_0^{(n)}, \ldots, \eta_n^{(n)}\}$.

We shall show that for each $n = 1, 2, \ldots$ we have $\gamma_n \leqslant \gamma_E$. $\{x_0, \ldots, x_n\}$ being an arbitrary system of points of the set E, we have

$$(n+1)^2 \gamma_n \leqslant \sum_{\substack{i\neq j \\ i,j=0,\ldots,n}} K(x_i - x_j). \tag{2}$$

Applying to both sides of (2) the operator

$$\int \ldots \int d\mu^*(x_0) \ldots d\mu^*(x_n)$$

and considering that $\mu^*(E) = \mu^*(\mathcal{E}^n) = 1$, we obtain the inequality

$$(n+1)^2 \gamma_n \leqslant \sum_{i\neq j} \int \ldots \int K(x_i - x_j)\,d\mu^*(x_1) \ldots d\mu^*(x_n)$$
$$= \sum_{i\neq j} \iint K(x_i - x_j)\,d\mu^*(x_i)\,d\mu^*(x_j) = (n+1)^2 \gamma_E \qquad (i, j = 0, \ldots, n)$$

which gives directly $\gamma \leqslant \gamma_E$.

In order to show the equality we suppose that $\gamma_E - \gamma = 2\sigma > 0$. We choose from the sequence of measures μ_n a subsequence μ_{a_n} $(a_{n+1} > a_n)$ which converges to some measure μ_0. In order to operate with a continuous and everywhere bounded kernel, we put for $M \in (0, \infty)$

$$K_M(x) = \min\{K(x), M\}.$$

Taking M large enough we have

$$\iint K_M(x-y)\,d\mu_0(y)\,d\mu_0(x) \geqslant \iint K(x-y)\,d\mu_0(y)\,d\mu_0(x) - \sigma.$$

Then we have

$$\gamma_E - 2\sigma = \lim_{n\to\infty} \gamma_{a_n} = \lim_{n\to\infty} \iint K_0(x-y)\, d\mu_{a_n} d\mu_{a_n}$$
$$\geqslant \lim_{n\to\infty} \left[\iint K_M(x-y)\, d\mu_{a_n} d\mu_{a_n} - \frac{M}{n+1} \right]$$
$$= \iint K_M(x-y)\, d\mu_0 d\mu_0 \geqslant \iint K(x-y)\, d\mu_0 d\mu_0 - \sigma \geqslant \gamma_E - \sigma,$$

which is an absurd. Then we must have $\gamma = \gamma_E$.

Turning to the sequence τ_n we shall prove

THEOREM 2. *The limit of τ_n exists and is equal to γ_E.*

Proof. First we shall prove the inequality $\overline{\lim_{n\to\infty}} \tau_n \leqslant \gamma_E$. By the definition of τ_n we have for $x \epsilon E$

$$\sum_{i=1}^{n} K(x - \zeta_i^{(n)}) \geqslant n\tau_n.$$

We integrate both sides and applying the equilibrium theorem we obtain

$$n\tau_n \leqslant \sum_{i=1}^{n} \int K(x - \zeta_i^{(n)})\, d\mu^*(x) \leqslant n\gamma_E.$$

Hence $\overline{\lim_{n\to\infty}} \tau_n \leqslant \gamma_E$. By lemma 2 and theorem 1 we have $\underline{\lim_{n\to\infty}} \tau_n \geqslant \gamma = \gamma_E$. Hence follows the theorem.

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ANNALES
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Generalized characteristic directions for a system of differential equations

by Z. KOWALSKI (Kraków)*

1. This paper will be concerned with the asymptotic behavior of integrals for the system of differential equations

$$X^{\cdot} = F(t, X), \tag{1.1}$$

where $X = [x_1, \ldots, x_n]$, $F(t, X) = [f_1(t, X), \ldots, f_n(t, X)]$, $X^{\cdot} = dX/dt$, $F(t, 0) \equiv 0$, and the right-hand member is continuous (for all t) in a neighbourhood of the point $X = 0$.

The characteristic directions play the fundamental role for (1.1) if F is a linear or a perturbed linear dynamical system. If the right-hand member does not contain t explicitly and possesses the Stolz differential at the point $X = 0$, then there is the possibility of employing the characteristic directions (see [2]).

In this paper we give a natural generalization of characteristic directions which holds even for non-differentiable $F(t, X)$. The idea of that generalization for the system $x_1^{\cdot} = f_1(x_1, x_2)$, $x_2^{\cdot} = f_2(x_1, x_2)$ is contained in [1]. We give necessary (theorem 2) and sufficient (theorems 3, 4) conditions for the existence of trajectories tangent to a given generalized characteristic direction at the point $X = 0$. The continuity of tangents to trajectories is also discussed (theorem 2).

2. The letters $X, Y, F, \ldots$ will be systematically reserved to represent vectors or vector-functions, $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$, $F = (f_1, \ldots, f_n)$, $|X| = (x_1^2 + \ldots + x_1^2)^{1/2}$, while $x, y, f, \ldots$ will be used to represent scalars. Denote by S the $(n-1)$-dimensional sphere $|Y| = 1$ with a centre at the origin and a unit radius.

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