la demi-droite l est normale à l'intersection de Σ avec le plan dans lequel l est située, le problème (2.5) devient le troisième problème de Fourier.

§ 3. Nous allons maintenant formuler un critère d'unicité plus général que celui qui est contenu dans le théorème 2.2.

THÉORÈME 3.1. *Supposons que les seconds membres du système d'équations* (0.1) *satisfassent aux hypothèses* 1° *et* 2° *du théorème* 2.1 *et aux inégalités*

$$|f_i(t, X, u_1, \ldots, u_m, p_1, \ldots, p_n, \ldots, r_{jk}, \ldots) - \\ - f_i(t, X, v_1, \ldots, v_m, p_1, \ldots, p_n, \ldots, r_{jk}, \ldots)| \leqslant \sigma(t, \max_j |u_j - v_j|),$$

où la fonction $\sigma(t, y)$ *est continue et non négative pour* $0 < t < T$, $y \geqslant 0$ [1], *et admettons que pour chaque intervalle*

$$0 < t \leqslant t_0, \tag{3.1}$$

où $0 < t_0 < T$, *l'intégrale unique* $y(t)$ *de l'équation*

$$dy/dt = \sigma(t, y),$$

définie dans (3.1) *et satisfaisant à la condition*

$$\lim_{t\to 0} y(t) = 0,$$

soit identiquement nulle. Maintenons enfin l'hypothèse 7° *du théorème* 2.1.

Ceci étant admis, le problème mixte (2.5) *relatif au système d'équations* (0.1) *admet dans* Ω *au plus une solution continue dans la fermeture de* Ω, *possédant une dérivée suivant* l *en tout point de* Σ_a *et toutes les dérivées figurant dans* (0.1) *continues dans* Ω.

La démonstration de ce théorème est tout à fait analogue à celle du théorème 3.1 de la note [1]. Il suffit d'y introduire des modifications analogues à celles dont nous avons parlé dans la démonstration du théorème 2.1 de la présente note.

Travaux cités

[1] J. Szarski, *Sur la limitation et l'unicité des solutions d'un système non-linéaire d'équations paraboliques aux dérivées partielles du second ordre,* Ann. Polon. Math. 2 (1955), p. 237-249.

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(1) La fonction $\sigma(t, y)$ n'est pas supposée continue pour $t = 0$.

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Application of the Nörlund summability to the theory of localization for single and double trigonometric series (I)

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The theory of localization for single trigonometric series developed by Zygmund [1] and that for double trigonometric series developed by Gosselin [2] are restricted to the Cesàro summability. The purpose of this paper is to extend their results by involving a special class of the Nörlund means, containing the Cesàro means as a particular case.

I. On the Nörlund summability

Let $\{B_n\}$ be a sequence with non-zero terms for n large enough. We shall say that the series $\sum_{\nu=0}^{\infty} a_\nu$, or the sequence s_n of its partial sums, is *summable by the Nörlund method* $\mathrm{N}(B_n)$, or $\mathrm{N}(B_n)$ *summable* to the sum s, if the following sequence (determined for n large enough)

$$t_n = \frac{\sum_{\nu=0}^{n} B_{n-\nu} a_\nu}{B_n} = \frac{\sum_{\nu=0}^{n} \Delta B_{n-\nu} s_\nu}{B_n},$$

where $\Delta B_0 = B_0$, $\Delta B_\nu = B_\nu - B_{\nu-1}$ for $\nu \geqslant 1$, converges to s as $n \to \infty$.

In numerous applications we deal with B_n positive and non-decreasing for n large enough. For such a sequence the condition

$$\lim_{n\to\infty} \frac{B_{n-1}}{B_n} = 1$$

is sufficient and necessary for the convergence of the series to involve its $\mathrm{N}(B_n)$ summability to the same sum (the condition of regularity).

Let $\{A_n\}$ be a sequence satisfying the following conditions:

$$A_n > 0 \quad \text{for} \quad n \geqslant n_0, \tag{1.1}$$

$$A_n \geqslant A_{n+1} \quad \text{for} \quad n \geqslant n_0, \tag{1.2}$$

$$\sum_{n=0}^{\infty} A_n = \infty \tag{1.3}$$

and let us write

$$f(z) = \sum_{n=0}^{\infty} A_n z^n.$$

We shall say that $f(z)$ is the *generating function* of the sequence A_n or that the sequence A_n *is generated* by the function $f(z)$. One may easily see that the circle of convergence of this power series and therefore of the series

$$\frac{f(z)}{(1-z)^k} = \sum_{n=0}^{\infty} A_n^{(k)} z^n$$

is the unit circle $|z| = 1$. Since for $k \geqslant 1$ the sequence $A_n^{(k)}$, as we shall prove below, is positive and non-decreasing for n large enough, we have

$$\lim_{n\to\infty} \frac{A_{n-1}^{(k)}}{A_n^{(k)}} = 1,$$

so that the method $\mathrm{N}(A_n^{(k)})$ is regular for $k \geqslant 1$. We shall call it the k-th *Nörlund means* with respect to A_n and write $\mathrm{N}_k(A_n)$. In what follows we shall be dealing only with Nörlund means of this type.

Thus, for example, the binomial coefficients

$$C_n^r = \frac{(r+1)(r+2)\ldots(r+n)}{n!}$$

satisfy the conditions (1.1)-(1.3) for $-1 < r \leqslant 0$. This sequence is generated by the function

$$f(z) = 1/(1-z)^{r+1}$$

and the method $\mathrm{N}_k(C_n^r) = \mathrm{N}(C_n^{r+k})$ is known as the Cesàro means of order $r+k$. The sequence $1/n$ generated by the function

$$f(z) = \log\frac{1}{1-z}$$

gives the harmonic means $\mathrm{N}_1(1/n)$ and the method $\mathrm{N}_k(1/n)$ may be called the *harmonic means of order* k. More general methods are derived from functions of the form

$$\frac{1}{(1-z)^{r+1}}\log^p\frac{e}{1-z},$$

$$\frac{1}{(1-z)^{r+1}}\log^p\frac{e}{1-z}\log^q\left(\log\frac{e^e}{1-z}\right),$$

$$\frac{1}{(1-z)^{r+1}}\log^p\frac{e}{1-z}\log^q\left(\log\frac{e^e}{1-z}\right)\log^s\left(\log\log\frac{e^{e^e}}{1-z}\right),$$

and so on. The sequences $C_n^{r,p}$, $C_n^{r,p,q}$, $C_n^{r,p,q,s}$ etc. generated by such functions satisfy the conditions (1.1)-(1.3) if $-1 < r < 0$, or $r = -1$, $0 < p < 1$, or $r = -1$, $p = 0$, $0 < q < 1$ etc. For such sequences the methods $\mathrm{N}_k(C_n^{r,p})$, $\mathrm{N}_k(C_n^{r,p,q})$, $\mathrm{N}_k(C_n^{r,p,q,s})$ etc. are called the *logarithmic means* of order $(r+k, p)$, $(r+k, p, q)$, $(r+k, p, q, s)$ etc.

In order to study the behaviour of $A_n^{(k)}$ for $k > 0$ and n large enough we start from the remark that

$$\frac{f(z)}{(1-z)^k} = \sum_{n=0}^{\infty} C_n^{k-1} z^n \sum_{n=0}^{\infty} A_n z^n,$$

whence

$$A_n^{(k)} = \sum_{\nu=0}^{n} C_{n-\nu}^{k-1} A_\nu.$$

For the same reason, from

$$\frac{f(z)}{(1-z)^{k+r}} = \frac{1}{(1-z)^r}\cdot\frac{f(z)}{(1-z)^k}$$

follows

$$A_n^{(k+r)} = \sum_{\nu=0}^{n} C_{n-\nu}^{r-1} A_\nu^{(k)}.$$

In particular

$$A_n^{(1)} = A_0 + A_1 + \ldots + A_n, \qquad A_n^{(k+1)} = A_0^{(k)} + A_1^{(k)} + \ldots + A_n^{(k)},$$

and inversely

$$\Delta A_n^{(k+1)} = A_n^{(k)}.$$

Using the well-known asymptotic formula for the binomial coefficients

$$C_n^r \cong \frac{n^r}{\Gamma(r+1)}, \qquad r \neq -1, -2, \ldots$$

we shall prove

LEMMA 1.1. *For $k > 0$ there exist such numbers A and B that for n large enough*

$$0 < A < \frac{A_n^{(k)}}{n^{k-1} A_n^{(1)}} < B.$$

Proof. For $0 < k < 1$ we have

$$A_n^{(k)} = \sum_{\nu=0}^{n_0} C_{n-\nu}^{k-1} A_\nu + \sum_{\nu=n_0+1}^{n} C_{n-\nu}^{k-1} A_\nu$$
$$\geqslant O(n^{k-1}) + C_n^{k-1}(A_n^{(1)} - A_{n_0}^{(1)}) = C_n^{k-1} A_n^{(1)} + O(n^{k-1}),$$

$$A_n^{(k)} = \sum_{\nu=0}^{n_0} C_{n-\nu}^{k-1} A_\nu + \sum_{\nu=n_0+1}^{[n/2]} C_{n-\nu}^{k-1} A_\nu + \sum_{\nu=[n/2]+1}^{n} C_{n-\nu}^{k-1} A_\nu$$
$$\leqslant O(n^{k-1}) + C_{[n/2]}^{k-1}(A_{[n/2]}^{(1)} - A_{n_0}^{(1)}) + A_{[n/2]} C_{[n/2]}^{k} = O(n^{k-1} A_n^{(1)}),$$

because $C_{[n/2]}^{k-1} = O(n^{k-1})$, $(n-n_0)A_n \leqslant A_n^{(1)} - A_{n_0}^{(1)}$.

Similarly for $k > 1$

$$A_n^{(k)} \geqslant \sum_{\nu=0}^{n_0} C_{n-\nu}^{k-1} A_\nu + \sum_{\nu=n_0+1}^{[n/2]} C_{n-\nu}^{k-1} A_\nu$$
$$\geqslant O(n^{k-1}) + C_{[n/2]}^{k-1}(A_{[n/2]}^{(1)} - A_{n_0}^{(1)}) = C_{[n/2]}^{k-1} A_{[n/2]}^{(1)} + O(n^{k-1}),$$

$$A_n^{(k)} = \sum_{\nu=0}^{n_0} C_{n-\nu}^{k-1} A_\nu + \sum_{\nu=n_0+1}^{n} C_{n-\nu}^{k-1} A_\nu$$
$$\leqslant O(n^{k-1}) + C_n^{k-1}(A_n^{(1)} - A_{n_0}^{(1)}) = C_n^{k-1} A_n^{(1)} + O(n^{k-1}).$$

These inequalities imply our lemma provided $A_n^{(1)} = O(A_{[n/2]}^{(1)})$. But that results from the first part of the proof of the next lemma.

COROLLARY. *For $k \geqslant 1$, $A_n^{(k)} > 0$ and $\Delta A_n^{(k)} = A_n^{(k-1)} > 0$ with n large enough, so that for $k \geqslant 1$ the sequence $A_n^{(k)}$ is positive and non-decreasing for sufficiently large n.*

LEMMA 1.2. *For $k > 0$ $A_{2n}^{(k)} = O(A_n^{(k)})$.*

Proof. For $k = 1$ and n large enough we have

$$A_n^{(1)} \geqslant \sum_{\nu=[n/2]+1}^{n} A_\nu \geqslant \tfrac{1}{2} n A_n,$$

whence

$$\frac{A_{2n}^{(1)}}{A_n^{(1)}} = 1 + \frac{A_{n+1} + \ldots + A_{2n}}{A_n^{(1)}} \leqslant 1 + \frac{nA_n}{\frac{1}{2}nA_n} = 3,$$

so that the lemma is true for $k = 1$. Now, for $k > 0$ the lemma follows from lemma 1.1.

LEMMA 1.3. *For $k > 1$ there exist such numbers A and B that for n large enough*

$$0 < \frac{A}{n} < \frac{\Delta A_n^{(k)}}{A_n^{(k)}} < \frac{B}{n}.$$

This lemma is also a simple corollary to lemma 1.1.

THEOREM 1.1. *If $p > q \geqslant 0$, then the $N_q(A_n)$ summability of the series to the sum s implies its $N_p(A_n)$ summability to the same sum.*

Proof. Letting

$$s_n^{(k)} = \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} s_\nu,$$

we shall demonstrate that

$$s_n^{(p)} = \sum_{\nu=0}^{n} C_{n-\nu}^{p-q-1} s_\nu^{(q)}.$$

Let us denote by $f(z)$ and $g(z)$ the generating functions of the sequences A_n and s_n respectively. The function

$$\frac{f(z)}{(1-z)^{k-1}} g(z)$$

being the generating function of $s_n^{(k)}$, we obtain the desired formula from the relation

$$\frac{f(z)}{(1-z)^{p-1}} g(z) = \frac{1}{(1-z)^{p-q}} \left[\frac{f(z)}{(1-z)^{q-1}} \cdot g(z) \right].$$

Now we suppose that s_n is the partial sum of the series $N_q(A_n)$ summable to s, which means that

$$t_n^{(q)} = \frac{s_n^{(q)}}{A_n^{(q)}} \to s, \quad \text{as} \quad n \to \infty.$$

For to prove that

$$t_n^{(p)} = \frac{s_n^{(p)}}{A_n^{(p)}} = \frac{\sum_{\nu=0}^{n} C_{n-\nu}^{p-q-1} s_\nu^{(q)}}{A_n^{(p)}} = \frac{\sum_{\nu=0}^{n} C_{n-\nu}^{p-q-1} A_\nu^{(q)} + t_\nu^{(q)}}{A_n^{(p)}}$$

converges to s, we need only verify the Toeplitz conditions for the matrix

$$a_{n\nu} = \frac{C_{n-\nu}^{p-q-1} A_\nu^{(q)}}{A_n^{(p)}},$$

namely: (i) $\sum_{\nu=0}^{n} a_{n\nu} = 1$, (ii) $\lim_{n\to\infty} a_{n\nu} = 0$ for $\nu = 0, 1, 2, \ldots$, (iii) $\sum_{\nu=0}^{n} |a_{n\nu}| \leqslant K$ for $n = 0, 1, 2, \ldots$ Condition (i) is obviously satisfied, condition (ii) follows from lemma 1.1. Let $A_\nu^{(q)} > 0$ for $\nu > n_1$. Then

$$\sum_{\nu=0}^{n} |a_{n\nu}| = \sum_{\nu=0}^{n_1} |a_{n\nu}| + \sum_{\nu=n_1+1}^{n} a_{n\nu}$$
$$= 1 + \sum_{\nu=0}^{n_1} (|a_{n\nu}| - a_{n\nu}) = 1 + o(1),$$

so that condition (iii) is also satisfied. Thus the theorem is proved.

It will be noticed that in this theorem we consider also the $\mathrm{N}_k(A_n)$ summability with $0 \leqslant k < 1$, which in general is not regular. This case corresponds to the Cesàro means (C, α) for $-1 < \alpha < 0$ and it seems quite natural to impose such conditions on the sequence A_n in order that some simple properties of the (C, α) summability for $-1 < \alpha < 0$ hold for $\mathrm{N}_k(A_n)$ summability with $0 \leqslant k < 1$. Thus, for example, according to the Hardy-Littlewood theorem from the convergence of the series with terms $o(1/n)$ follows its (C, α) summability for $-1 < \alpha < 0$. It is desirable that an analogous theorem should also hold for Nörlund means.

LEMMA 1.4. *If A_n satisfies, besides* (1.1)-(1.3), *the condition*

$$\frac{\Delta A_n}{A_n} = O\left(\frac{1}{n}\right), \tag{1.4}$$

then also for $0 < k \leqslant 1$ we have

$$\frac{\Delta A^{(k)}_n}{A^{(k)}_n} = O\left(\frac{1}{n}\right).$$

Proof. Since $A^{(1)}_n - A^{(1)}_{n_0} \geqslant (n-n_0) A_n$,

$$\frac{1}{n-n_0} \geqslant \frac{A_n}{A^{(1)}_n - A^{(1)}_{n_0}} = \frac{A_n}{A^{(1)}_n} \cdot \frac{1}{1 - A^{(1)}_{n_0}/A^{(1)}_n} > \frac{2}{3} \cdot \frac{A_n}{A^{(1)}_n}$$

with n large enough, the lemma is true for $k = 1$ even without the restriction 1.4. Now for $0 < k < 1$ we have

$$A^{(k-1)}_n = \sum_{\nu=0}^{n} C^{k-2}_{n-\nu} A_\nu$$

$$= \sum_{\nu=0}^{n_0} C^{k-2}_{n-\nu} A_\nu + \sum_{\nu=n_0+1}^{[n/2]} C^{k-2}_{n-\nu} A_\nu + \sum_{\nu=[n/2]+1}^{n} C^{k-2}_{n-\nu} A_\nu = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where $\mathrm{I} = O(n^{k-2})$ and, in virtue of lemma 1.1,

$$|\mathrm{II}| \leqslant |C^{k-2}_{[n/2]}| \sum_{\nu=n_0+1}^{[n/2]} A_\nu = \frac{1}{n} O(n^{k-1} A^{(1)}_n) = A^{(k)}_n O\left(\frac{1}{n}\right).$$

Applying to III Abel's transformation we obtain

$$\mathrm{III} = \sum_{\nu=[n/2]+1}^{n} C^{k-1}_{n-\nu} \Delta A_\nu + C^{k-1}_{n-[n/2]-1} A_{[n/2]} = \mathrm{III}_1 + \mathrm{III}_2,$$

where, in virtue of lemma 1.1,

$$|\mathrm{III}_1| \leqslant \max_{\nu>n/2} |\Delta A_\nu| C^k_{[n/2]} = A_{[n/2]} O(n^{k-1}) = O\left(A^{(k)}_{[n/2]} \frac{A_{[n/2]}}{A^{(1)}_{[n/2]}}\right) = A^{(k)}_n O\left(\frac{1}{n}\right)$$

and $\mathrm{III}_2 = A^{(k)}_n O(1/n)$ for the same reason. Thus the lemma is established.

THEOREM 1.2. *If A_n satisfies conditions* (1.1)-(1.4), *then the convergence of the series with terms $o(1/n)$ implies its $\mathrm{N}_k(A_n)$ summability for* $0 < k < 1$.

Proof. Without loss of generality we may suppose that the series $\sum_{n=0}^{\infty} a_n$ with terms $a_n = o(1/n)$ converges to 0, so that its partial sums $s_n = o(1)$. We shall prove that $\sum_{\nu=0}^{n} A^{(k-1)}_{n-\nu} s_\nu = o(A^{(k)}_n)$. Splitting up this sum into two parts,

$$\sum_{\nu=0}^{[n/2]} A^{(k-1)}_{n-\nu} s_\nu + \sum_{\nu=[n/2]+1}^{n} A^{(k-1)}_{n-\nu} s_\nu = \mathrm{I} + \mathrm{II},$$

we have, in virtue of lemma 1.4,

$$|\mathrm{I}| \leqslant \max_{\nu>n/2} |A^{(k-1)}_\nu| \sum_{\nu=0}^{[n/2]} |s_\nu| = O(A^{(k)}_n) \cdot \frac{1}{n} \sum_{\nu=0}^{[n/2]} |s_\nu| = o(A^{(k)}_n)$$

and, applying Abel's transformation to II,

$$\mathrm{II} = \sum_{\nu=[n/2]+1}^{n} A^{(k)}_{n-\nu} a_\nu + A^{(k)}_{n-[n/2]-1} s_{[n/2]} = \mathrm{II}_1 + \mathrm{II}_2,$$

whence $|\mathrm{II}_1| \leqslant \max_{\nu>n/2} |a_\nu| A^{(k+1)}_n = o(A^{(k+1)}_n/n) = o(A^{(k)}_n)$ and $\mathrm{II}_2 = o(A^{(k)})$. The theorem is therefore proved.

Of course, if $A^{(k)}_n \to +\infty$, then the theorem remains true without the restriction $a_n = o(1/n)$. That means that the method $\mathrm{N}_k(A_n)$ may be regular even for $0 < k < 1$. In particular, if $\lim_{n\to\infty} A_n > 0$, then the methods $\mathrm{N}_k(A_n)$ are regular for any $k \geqslant 0$. Similarly, if $\lim_{n\to\infty} A_n = 0$ but there exists such number k_0, $0 < k_0 < 1$, that $\lim_{n\to\infty} A^{(k_0)}_n > 0$, then for any $k < k_0$ $A^{(k)}_n \to 0$ and for any $k > k_0$ $A^{(k)}_n \to +\infty$, so that for any $k \geqslant k_0$ the methods $\mathrm{N}_k(A_n)$ are regular. One may expect that in this case the exceptional method $\mathrm{N}_{k_0}(A_n)$ is equivalent to convergence. In order to explain the situation we remember the well-known conditions of equivalence of two regular Nörlund methods.

Let $f(z)$ and $g(z)$ be the generating functions of the sequences A_n and B_n respectively, where $A_0 \neq 0$ and $B_0 \neq 0$, and let p_n and q_n be the sequences generated by the functions $f(z)/g(z)$ and $g(z)/f(z)$ respectively. Two regular methods, $N(A_n)$ and $N(B_n)$, are equivalent if and only if $\sum|p_n| < \infty$ and $\sum|q_n| < \infty$ (viz. Hardy [3], p. 67). In particular, letting $A_n = 1$, $n = 0, 1, 2, \ldots$, we have $f(z) = 1/(1-z)$, $q_n = \Delta B_n$, whence the corresponding conditions of equivalence of the regular method $N(B_n)$ and the convergence $N(1)$ are: $\sum|p_n| < \infty$ and B_n is of bounded variation. This last condition is satisfied if B_n is monotone and convergent. Of course its limit should be non-zero.

THEOREM 1.3. *If there exists a number k_0, $0 \leqslant k_0 < 1$, such that the method $N_{k_0}(A_n)$ is equivalent to convergence, then for any $k \geqslant k_0$ the method $N_k(A_n)$ is equivalent to $(C, k-k_0)$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} A_n z^n$ and $1/f(z) = \sum_{n=0}^{\infty} p_n z^n$. The corresponding generating functions for the sequences $A_n^{(k)}$ and $C_n^{k-k_0}$ are $f_k(z) = f(z)/(1-z)^k$ and $g_k(z) = 1/(1-z)^{k-k_0+1}$, whence $f_k(z)/g_k(z) = f(z)/(1-z)^{k_0-1} = \sum_{n=0}^{\infty} A_n^{(k_0-1)} z^n$ and $g_k(z)/f_k(z) = 1/f(z)(1-z)^{1-k_0} = \sum_{n=0}^{\infty} p_n^{(1-k_0)} z^n$. We need to prove that the series $\sum_{n=0}^{\infty} A_n^{(k_0-1)}$ and $\sum_{n=0}^{\infty} p_n^{(1-k_0)}$ converge absolutely. Now that follows from the equivalence of $N_{k_0}(A_n)$ and $N(1)$.

In order to deal with the Nörlund methods which are more general than the Cesàro methods we suppose once for all that for any $k \geqslant 0$ the method $N_k(A_n)$ is not equivalent to convergence. Nevertheless even for such methods there exists a number k_0, $0 \leqslant k_0 \leqslant 1$, which separates the regular methods from those not regular. The method $N_{k_0}(A_n)$ may or may not be regular. If, for example, A_n converges to a non-zero limit, then $k_0 = 0$ and $N_{k_0}(A_n)$ is regular. For $A_n = 1/n$ we have $k_0 = 1$ and obviously $N_{k_0}(1/n)$ is regular. For $A_n = 1/\log n$, $k_0 = 0$ and $N_{k_0}(A_n)$ is not regular. We shall call k_0 the *critical order* of the methods $N_k(A_n)$.

LEMMA 1.5. *If the sequence A_n satisfies conditions* (1.1)-(1.4), *then for $0 \leqslant k < k_0$ the sequence $A_n^{(k)}$ is of bounded variation.*

Proof. Let $k < k_1 < k_0$. We have $A_n^{(k_1)} = o(1)$ and, according to lemma 1.4, $A_n^{(k_1-1)} = O(A_n^{(k_1)}/n) = o(1/n)$. In virtue of the Hardy-Littlewood theorem, the sequence $A_n^{(k_1)}$ is (C, a) summable to 0 for any $a > -1$:

$$\sum_{\nu=0}^{n} C_{n-\nu}^{a-1} A_\nu^{(k_1)} = A_n^{(k_1+a)} = o(C_n^a),$$

whence, for $a = k-k_1$, $A_n^{(k)} = o(n^{k-k_1})$. Since

$$A_n^{(k-1)} = O(A_n^{(k)}/n) = o(n^{k-k_1-1}),$$

the series $\sum|A_n^{(k-1)}|$ is convergent, so that the sequence $A_n^{(k)}$ is of bounded variation.

LEMMA 1.6. *If the sequence A_n satisfies conditions* (1.1)-(1.4), *then for any $0 \leqslant k < 1$ $A_n^{(k-2)} = o(1/n)$ and $A_n^{(k-1)}$ is of bounded variation.*

Proof. We have

$$|A_n^{(k-2)}| \leqslant \sum_{\nu=0}^{n} |C_{n-\nu}^{k-2} \Delta A_\nu| = \sum_{\nu=0}^{[n/2]} |C_{n-\nu}^{k-2} \Delta A_\nu| + \sum_{\nu=[n/2]+1}^{n} |C_{n-\nu}^{k-2} \Delta A_\nu|$$

$$\leqslant |C_{[n/2]}^{k-2}| \sum_{\nu=0}^{[n/2]} |\Delta A_\nu| + \max_{\nu>n/2} |\Delta A_\nu| \sum_{\nu=0}^{[n/2]} |C_\nu^{k-2}| = O(n^{k-2}) + O(\Delta A_{[1/2]})$$

so that the convergence of the series $\sum|A_n^{(k-2)}|$ follows from the convergence of the series $\sum n^{k-2}$ and $\sum|\Delta A_n|$, and $A_n^{(k-2)} = o(1/n)$ in virtue of (1.4).

THEOREM 1.4. *If $k-[k] \neq k_0$, then the method $N_k(A_n)$ may be replaced by a method $N_r(A_n')$ of integer order r with respect to a sequence A_n' satisfying the following conditions:*

(1.5) $$\Delta A_n' = o(1/n),$$

(1.6) *A_n' is of bounded variation,*

(1.7) $$A_n'^{(1)} \to +\infty.$$

Proof. If $k-[k] < k_0$, then the sequence $A_n^{(k-[k])}$, in virtue of lemmas 1.4 and 1.5, satisfies conditions (1.5)-(1.7) and

$$N_k(A_n) = N_{[k]}(A_n^{(k-[k])}).$$

If $k-[k] > k_0$, then the sequence $A_n^{(k-[k]-1)}$, in virtue of lemma 1.6, satisfies conditions (1.5)-(1.7) and

$$N_k(A_n) = N_{[k]+1}(A_n^{(k-[k]-1)}).$$

The last replacement holds also in the case of $k-[k] = k_0$ if $A_n^{(k_0)} \to +\infty$. But in general this case is doubtful, for the sequence $A_n^{(k_0)}$ may be not of bounded variation.

In what follows we consider only the methods $N_k(A_n)$ of integer order, though all results are true for the methods of positive order with the restriction mentioned in the above theorem.

II. On the formal product of two series

In this chapter we shall be dealing with the series of the form

$$\sum_{p=-\infty}^{+\infty} a_p.$$

Such a series is said to be convergent if its symmetric partial sums

$$s_n = \sum_{p=-n}^{n} a_p$$

have a limit as $n \to \infty$.

We define the formal product of two series

$$\sum_{p=-\infty}^{+\infty} a_p \quad \text{and} \quad \sum_{p=-\infty}^{+\infty} b_p$$

by the series

$$\sum_{p=-\infty}^{+\infty} c_p$$

with coefficients

$$c_p = \sum_{q=-\infty}^{+\infty} a_q b_{p-q},$$

if these sums exist. The existence of c_p is ensured if, for example, the coefficients of the former series are bounded and the second series converges absolutely.

LEMMA 2.1 (Rajchman). *If $a_p = o(1)$ and $\sum b_p$ converges absolutely, then the coefficients c_p of the formal product of the series $\sum a_p$ and $\sum b_p$ are also $o(1)$.*

LEMMA 2.2. *Let A_n be a sequence satisfying conditions* (1.1)-(1.4). *If $a_p = o(A_{|p|})$ and $b_p = O(|p|^{-3})$ as $|p| \to \infty$ then the coefficients c_p of the formal product of the series $\sum a_p$ and $\sum b_p$ are also $o(A_{|p|})$.*

Proof. We have for $p > 0$

$$c_p = \sum_{q=-\infty}^{[p/2]} a_q b_{p-q} + \sum_{q=[p/2]+1}^{+\infty} a_q b_{p-q}$$

$$= \sum_{q=p-[p/2]}^{+\infty} a_{-q+p} b_q + \sum_{q=[p/2]+1}^{+\infty} a_q b_{p-q} = c_p' + c_p''.$$

Since $\sum_{q=n}^{+\infty} |b_q| = O(n^{-2})$ and, in virtue of (1.3), $n^2 A_n \to +\infty$,

$$|c_p'| \leqslant \max_q |a_q| \sum_{q=p-[p/2]}^{+\infty} |b_q| = O(p^{-2}) = o(A_p).$$

Writing $a_p = A_p \varepsilon_p$ we have, by hypothesis, $\varepsilon_p = o(1)$ and $\sum |b_p| < \infty$,

whence, in virtue of lemma 2.1, $\sum_{q=-\infty}^{+\infty} |\varepsilon_q b_{p-q}| = o(1)$ and therefore

$$|c_p''| \leqslant \sum_{q=[p/2]+1}^{+\infty} |A_q \varepsilon_q b_{p-q}| \leqslant A_{[p/2]} \sum_{q=-\infty}^{+\infty} |\varepsilon_q b_{p-q}| = o(A_{p/2}).$$

The lemma will be established for $p \to +\infty$ if $A_{p/2} = O(A_p)$. Now the last relation follows from (1.4). In fact, according to (1.4)

$$\frac{A_{n-1} - A_n}{A_n} = \frac{A_{n-1}}{A_n} - 1 = \frac{\eta_n}{n},$$

where $\eta_n = O(1)$ and $\eta_n \geqslant 0$ for n large enough, whence

$$\frac{A_n}{A_{2n}} = \frac{A_n}{A_{n+1}} \cdot \frac{A_{n+1}}{A_{n+2}} \cdots \frac{A_{2n-1}}{A_{2n}} = \prod_{\nu=n+1}^{2n} \left(1 + \frac{\eta_\nu}{\nu}\right) \leqslant \exp\left(\sum_{\nu=n+1}^{2n} \frac{\eta_\nu}{\nu}\right) = O(1).$$

One may easily see that a similar argument holds for $p \to -\infty$. Thus $c_p = o(A_{|p|})$ for $|p| \to +\infty$.

THEOREM 2.1. *Let A_n be a sequence satisfying conditions* (1.1)-(1.4). *If $a_p = o(A_{|p|})$, $b_p = O(p^{-4})$ and $\sum_{p=-\infty}^{+\infty} b_p = 0$, then the formal product of the series $\sum a_p$ and $\sum b_p$ is $\mathrm{N}(A_n)$ summable to* 0.

Proof. We have

$$s_n = \sum_{p=-n}^{n} c_p = \sum_{p=-n}^{n} \sum_{q=-\infty}^{+\infty} a_q b_{p-q} = \sum_{q=-\infty}^{+\infty} a_q \sum_{p=-n}^{n} b_{p-q}$$

$$= \sum_{q=-\infty}^{+\infty} a_q \sum_{p=-n-q}^{n-q} b_p = \sum_{q=-\infty}^{+\infty} a_q \Big(\sum_{p=-n-q}^{+\infty} b_p - \sum_{p=n-q+1}^{+\infty} b_p \Big)$$

$$= \sum_{q=-\infty}^{\infty} a_q (R_{-n-q} - R_{n-q+1}) = \mathrm{I} - \mathrm{II},$$

where $R_n = \sum_{p=n}^{+\infty} b_p$. I and II may be considered as the $-n$-th and the $(n+1)$-th coefficients respectively of the formal product of the series $\sum a_p$ and $\sum R_p$. Since $b_p = O(p^{-4})$, $R_p = O(p^{-3})$ for $p \to +\infty$. From $\sum b_p = 0$ follows $R_{-n} = -\sum_{p=-\infty}^{-n-1} b_p$, whence $R_p = O(|p|^{-3})$ for $p \to -\infty$. The hypotheses of lemma 2.2 being satisfied, I and II are $o(A_n)$, and therefore $s_n = o(A_n)$. Now, for the proof that s_n is $\mathrm{N}(A_n)$ summable to 0, we consider the sum

$$\sum_{\nu=0}^{n} \Delta A_{n-\nu} s_\nu = \sum_{\nu=0}^{[n/2]} \Delta A_{n-\nu} s_\nu + \sum_{\nu=[n/2]+1}^{n} \Delta A_{n-\nu} s_\nu = P + Q.$$

Since $\Delta A_n = O(A_n/n)$ and $s_n = o(1)$, we have

$$|P| \leqslant \max_{\nu>n/2}|\Delta A_\nu| \sum_{\nu=0}^{[n/2]} |s_\nu| = O\Big(A_{[n/2]}\frac{1}{n}\sum_{\nu=0}^{[n/2]}|s_\nu|\Big) = o(A_{[n/2]}),$$

$$|Q| \leqslant \max_{\nu>n/2}|s_\nu| \sum_{\nu=0}^{[n/2]} |\Delta A_\nu| = o(A_{[n/2]}),$$

the series $\sum|\Delta A_\nu|$ being convergent. Thus $P+Q = o(A_{[n/2]}) = o(A_n)$ and the theorem is established.

THEOREM 2.2. *If we suppose in theorem 2.1 that the series $\sum b_p$ converges to the sum λ, then the difference*

$$\sum_{p=-\infty}^{+\infty} c_p - \lambda \sum_{p=-\infty}^{+\infty} a_p$$

is $\mathrm{N}(A_n)$ *summable to* 0.

Proof. We consider the series $\sum a_p$ and $\sum b'_p$, where $b'_0 = b_0 - \lambda$, $b'_p = b_p$ for $p \neq 0$. Since both series satisfy the hypotheses of theorem 2.1, their formal product

$$\sum_{p=-\infty}^{+\infty} c'_p = \sum_{p=-\infty}^{+\infty}\sum_{q=-\infty}^{+\infty} a_q b'_{p-q} = \sum_{p=-\infty}^{+\infty} c_p - \lambda \sum_{p=-\infty}^{+\infty} a_p$$

is $\mathrm{N}(A_n)$ summable to 0 and the theorem is proved.

THEOREM 2.3. *Let d_p be the coefficients of the formal product of the series*

$$\sum_{p=-\infty}^{+\infty}{}' p^{-m} a_p \quad \text{and} \quad \sum_{p=-\infty}^{+\infty} b_p,$$

where m is a non-negative integer and $\sum'$ indicates that the term for $p=0$ is omitted. If $a_p = o(A_{|p|})$, $b_p = 0(|p|^{-m-4})$ and

$$\sum_{p=-\infty}^{+\infty} b_p = \sum_{p=-\infty}^{+\infty} p b_p = \ldots = \sum_{p=-\infty}^{+\infty} p^m b_p = 0,$$

then the series

$$\sum_{p=-\infty}^{+\infty} p^m d_p$$

is $\mathrm{N}(A_n)$ *summable to* 0.

Proof. We have

$$\begin{aligned}
\sigma_n &= \sum_{q=-n}^{n} q^m d_q = \sum_{q=-n}^{n} q^m \sum_{p=-\infty}^{+\infty}{}' p^{-m} a_p b_{q-p} \\
&= \sum_{p=-\infty}^{+\infty}{}' p^{-m} a_p \sum_{q=-n-p}^{n-p} (q+p)^m b_q \\
&= \sum_{p=-\infty}^{+\infty}{}' p^{-m} a_p \sum_{q=-n-p}^{n-p} b_q \sum_{\nu=0}^{m}\binom{m}{\nu} q^\nu p^{m-\nu} \\
&= \sum_{\nu=0}^{m}\binom{m}{\nu}\sum_{p=-\infty}^{+\infty}{}' p^{-\nu} a_p \sum_{q=-n-p}^{n-p} q^\nu b_q \\
&= \sum_{\nu=0}^{m}\binom{m}{\nu}\sum_{q=-n}^{n}\sum_{p=-\infty}^{+\infty}{}' p^{-\nu} a_p (q-p)^\nu b_{q-p} = \sum_{\nu=0}^{m}\binom{m}{\nu}\sigma_n^{(\nu)}.
\end{aligned}$$

The expression

$$\sigma_n^{(\nu)} = \sum_{q=-n}^{n}\sum_{p=-\infty}^{+\infty}{}' p^{-\nu} a_p (q-p)^\nu b_{q-p}$$

may be considered as the n-th symmetric partial sum of the formal product of the series

$$\sum_{p=-\infty}^{+\infty}{}' p^{-\nu} a_p \quad \text{and} \quad \sum_{p=-\infty}^{+\infty} p^\nu b_p.$$

Since $p^{-\nu} a_p = o(A_{|p|})$ and $p^\nu b_p = 0(|p|^{-4})$ for $\nu = 0, 1, \ldots, m$, in virtue of theorem 2.1 all $\sigma_n^{(\nu)}$, and therefore σ_n, are $\mathrm{N}(A_n)$ summable to 0.

The proof of theorem 2.2 is applicable to the following

THEOREM 2.4. *If we suppose in theorem 2.3 that the series $\sum b_p$ converges to the sum λ, then the difference*

$$\sum_{p=-\infty}^{+\infty} p^m d_p - \lambda \sum_{p=-\infty}^{+\infty}{}' a_p$$

is $\mathrm{N}(A_n)$ *summable to* 0.

LEMMA 2.3. *Let B_n be a sequence satisfying the following conditions:*

(2.1) $$B_n > 0 \quad \textit{for} \quad n > n_0,$$

(2.2) $$B_{n+1} \geqslant B_n \quad \textit{for} \quad n > n_0,$$

(2.3) $$B_{2n} = O(B_n).$$

If $a_p = o(B_{|p|})$ and the series $\sum B_{|p|} b_p$ converges absolutely, then the coefficients c_p of the formal product of the series $\sum a_p$ and $\sum b_p$ are also $o(B_{|p|})$.

Proof. We write

$$a_p = B_{|p|}\varepsilon_p, \qquad B_{|p|} b_p = \eta_p.$$

By hypothesis, $\varepsilon_p = o(1)$, $\sum|\eta_p| < \infty$, whence, in virtue of lemma 2.1,

$$\sum_{q=-\infty}^{+\infty} |\varepsilon_q \eta_{p-q}| = o(1).$$

For the proof that

$$c_p = \sum_{q=-\infty}^{+\infty} a_q b_{p-q} = o(B_{|p|}) \quad \text{for} \quad |p| \to +\infty,$$

we suppose first that $p > n_0$ and we split up this sum into six parts. The first part containing the terms with $q < -n_0$ is $o(1)$, because

$$|a_q b_{p-q}| = |\varepsilon_q B_{|q|} b_{p-q}| \leqslant |\varepsilon_q B_{p+|q|} b_{p-q}| = |\varepsilon_q B_{p-q} b_{p-q}| = |\varepsilon_q \eta_{p-q}|.$$

The second part with $|q| \leqslant n_0$ is also $o(1)$, for $b_{p-q} = o(1)$ as $p \to \infty$. Multiplying by B_{n_0+1} the third part, which contains the terms with $n_0+1 \leqslant q \leqslant p-n_0-1$, and taking into account that $n_0+1 \leqslant p-q$, we obtain

$$B_{n_0+1}\Big|\sum_{q=n_0+1}^{p-n_0-1} a_q b_{p-q}\Big| \leqslant \sum_{q=n_0+1}^{p-n_0-1} |\varepsilon_q B_q B_{n_0+1} b_{p-q}|$$

$$\leqslant B_{p-n_0-1}\sum_{q=n_0+1}^{p-n_0-1} |\varepsilon_q B_{p-q} b_{p-q}| \leqslant B_p \sum_{q=-\infty}^{+\infty} |\varepsilon_q \eta_{p-q}| = o(B_p).$$

The fourth part with $|q-p| \leqslant n_0$ is $o(B_p)$, for $a_q = o(B_{q+p}) = o(B_p)$. Multiplying by B_{n_0+1} the fifth part with $p+n_0+1 \leqslant q \leqslant 2p$ and taking into account that $n_0+1 \leqslant q-p$, we obtain

$$B_{n_0+1}\Big|\sum_{q=p+n_0+1}^{2p} a_q b_{p-q}\Big| \leqslant \sum_{q=p+n_0+1}^{2p} |\varepsilon_q B_q B_{|p-q|} b_{p-q}|$$

$$\leqslant B_{2p} \sum_{q=-\infty}^{+\infty} |\varepsilon_q \eta_{p-q}| = o(B_{2p}).$$

Finally, in virtue of condition (2.3) there exists a number K such that for n large enough $B_{2n} \leqslant KB_n$, whence the following estimation of the sixth part with $q > 2p$:

$$\Big|\sum_{q=2p+1}^{+\infty} a_p b_{p-q}\Big| \leqslant \sum_{q=p+1}^{+\infty} |a_{q+p} b_{-q}| = \sum_{q=p+1}^{+\infty} |\varepsilon_{q+p} B_{q+p} b_{-q}|$$

$$\leqslant \sum_{q=p+1}^{+\infty} |\varepsilon_{q+p} b_{-q}| B_{2q} \leqslant K \sum_{q=p+1}^{+\infty} |\varepsilon_{q+p} b_{-q} B_q| \leqslant K \sum_{q=-\infty}^{+\infty} |\varepsilon_{q+p}\eta_{-q}| = o(1).$$

Thus $c_p = o(B_{2p}) = o(B_p)$ for $p \to +\infty$. By a similar argument we prove that $c_p = o(B_{|p|})$ for $p \to -\infty$.

We define for the series $\sum b_p$ the sequence of k-th rests $R_p^{(k)}$ by the formula

$$R_p^{(k)} = \sum_{q=p}^{+\infty} C_{k-1}^{q-p} b_p,$$

if this sum exists. We have for $k = 1$

$$R_p^{(1)} = \sum_{q=p}^{+\infty} b_q$$

and for $k > 1$

$$R_p^{(k)} = \sum_{q=p}^{+\infty} R_q^{(k-1)}.$$

In fact, using the well-known formula

$$\sum_{\nu=0}^{n} C_{k-1}^{n-\nu} = C_k^n$$

we obtain

$$\sum_{q=p}^{+\infty} R_q^{(k-1)} = \sum_{q=p}^{+\infty}\sum_{\nu=q}^{+\infty} C_{k-2}^{\nu-q} b_\nu = \sum_{q=0}^{+\infty}\sum_{\nu=p+q}^{+\infty} C_{k-2}^{\nu-q-p} b_\nu$$

$$= \sum_{\nu=p}^{+\infty} b_\nu \sum_{q=0}^{\nu-p} C_{k-2}^{\nu-p-q} = \sum_{\nu=p}^{+\infty} b_\nu C_{k-1}^{\nu-p} = R_p^{(k)}.$$

From this relation follows

$$R_p^{(k-1)} = R_p^{(k)} - R_{p+1}^{(k)} = -\Delta R_{p+1}^{(k)}.$$

LEMMA 2.4 (Zygmund [1]). *Let m be a positive integer. If $b_p = O(|p|^{-2m-1})$ and*

$$\sum_{p=-\infty}^{+\infty} b_p = \sum_{p=-\infty}^{+\infty} p b_p = \ldots = \sum_{p=-\infty}^{+\infty} p^{m-1} b_p = 0,$$

then

$$\sum_{p=-\infty}^{+\infty} R_p^{(m)} = \frac{1}{m!}\sum_{p=-\infty}^{+\infty} p^m b_p.$$

We give a new proof of this lemma. One may easily see that, by hypothesis, for $\nu = 0, 1, \ldots, m-1$

$$\sum_{q=-n}^{n} b_q q^\nu = O(n^{-2m+\nu}),\qquad \sum_{q=n+1}^{+\infty} b_q q^\nu = O(n^{-2m+\nu}).$$

Since

$$\sum_{p=-n}^{q} C_{m-1}^{q-p} = C_m^{q+n} = \frac{q^m}{m!} + \sum_{\nu=0}^{m-1}\alpha_\nu q^\nu,\qquad \alpha_\nu = O(n^{m-\nu}),$$

$$\sum_{p=-n}^{n} C_{m-1}^{q-p} = \sum_{\nu=0}^{m-1}\beta_\nu q^\nu,\qquad \beta_\nu = O(n^{m-\nu}),$$

we obtain

$$\sum_{p=-n}^{n} R_p^{(k)} = \sum_{p=-n}^{n}\sum_{q=p}^{+\infty} C_{m-1}^{q-p} b_q = \sum_{q=-n}^{n} b_q \sum_{p=-n}^{q} C_{m-1}^{q-p} + \sum_{q=n+1}^{+\infty} b_q \sum_{p=-n}^{n} C_{m-1}^{q-p}$$

$$= \frac{1}{m!}\sum_{q=-n}^{n} q^m b_q + \sum_{\nu=0}^{m-1}\Big(\alpha_\nu \sum_{q=-n}^{n} q^\nu b_q + \beta_\nu \sum_{q=n+1}^{+\infty} q^\nu b_q\Big) = \frac{1}{m!}\sum_{q=-n}^{n} q^m b_q + O(n^{-m}),$$

whence the lemma follows.

THEOREM 2.5. *Let A_n be a sequence satisfying conditions* (1.1)-(1.4). *If $a_p = o(A_{|p|}^{(k)})$, where k is a positive integer, and*

$$\sum_{p=-\infty}^{+\infty} |A_p^{(k)} R_p^{(k+1)}| < \infty,$$

then the formal product of the series $\sum a_p$ and $\sum b_p$ is $N_k(A_n)$ summable to 0.

Proof. To begin with, we notice that the last hypothesis of the theorem involves the absolute convergence of the series

$$\sum_{p=-\infty}^{+\infty} a_p R_p^{(j+1)},\qquad j = 0, 1, \ldots, k.$$

In fact, from the formula

$$R_p^{(j+1)} = (-1)^{k-j}\Delta^{k-j} R_{p+k-j}^{(k+1)} = (-1)^{k-j}\sum_{\nu=0}^{k-j}\binom{k-j}{\nu}(-1)^\nu R_{p+k-j-\nu}^{(k+1)}$$

we have

$$\sum_{p=-\infty}^{+\infty}|a_p R_p^{(j+1)}| \leqslant \sum_{\nu=0}^{k-j}\binom{k-j}{\nu}\sum_{p=-\infty}^{+\infty}|O(A_p^{(k)})R_{p+k-j-\nu}^{(k+1)}|$$

$$= \sum_{\nu=0}^{k-j}\binom{k-j}{\nu}\sum_{p=-\infty}^{+\infty}|O(A_p^{(k)})R_p^{(k+1)}| < \infty,$$

for $A_p^{(k)} = O(A_{p+k-j-\nu}^{(k)})$.

We have to prove that the sequence

$$s_\nu = \sum_{p=-\nu}^{\nu} c_p = \sum_{p=-\nu}^{\nu}\sum_{q=-\infty}^{+\infty} a_q b_{p-q} = \sum_{q=-\infty}^{+\infty} a_q \sum_{p=-\nu}^{\nu} b_{p-q} = \sum_{q=-\infty}^{+\infty} a_q \sum_{p=-\nu-q}^{\nu-q} b_p$$

$$= \sum_{q=-\infty}^{+\infty} a_q\Big(\sum_{p=-\nu-q}^{+\infty} b_p - \sum_{p=\nu-q+1}^{+\infty} b_p\Big) = \sum_{q=-\infty}^{+\infty} a_q R_{-\nu-q}^{(1)} - \sum_{q=-\infty}^{+\infty} a_q R_{\nu-q+1}^{(1)}$$

is $N_k(A_n)$ summable to 0, i. e. that $t_n = o(A_n^{(k)})$, where

$$t_n = \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} s_\nu$$

$$= \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}\sum_{q=-\infty}^{+\infty} a_q R_{-\nu-q}^{(1)} - \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}\sum_{q=-\infty}^{+\infty} a_q R_{\nu-q+1}^{(1)} = t_n' - t_n''.$$

Applying Abel's transformation k times we obtain

$$\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} R_{-\nu-q}^{(1)} = (-1)^k\sum_{\nu=0}^{n} \Delta A_{n+k-\nu}\, R_{-\nu-q}^{(k+1)} -$$

$$- \sum_{j=1}^{k} (-1)^j A_{n+j}^{(k-j)} R_{-q}^{(j+1)} + \sum_{j=1}^{k} (-1)^j A_{j-1}^{(k-j)} R_{-n+j-q}^{(j+1)},$$

whence

$$t_n' = (-1)^k \sum_{\nu=0}^{n} \Delta A_{n+k-\nu}\sum_{q=-\infty}^{+\infty} a_q R_{-\nu-q}^{(k+1)} + \sum_{j=1}^{k} (-1)^{j+1} A_{n+j}^{(k-j)}\sum_{q=-\infty}^{+\infty} a_q R_{-q}^{(j+1)}$$

$$+ \sum_{j=1}^{k} (-1)^j A_{j-1}^{(k-j)} \sum_{q=-\infty}^{+\infty} a_q R_{-n+j-q}^{(j+1)} = \mathrm{I}+\mathrm{II}+\mathrm{III}.$$

Since the sequence $A_n^{(k)}$ satisfies for $k \geqslant 1$ conditions (2.1)-(2.3), the hypotheses of the theorem coincide with the hypotheses of lemma 2.3 for the series $\sum a_p$ and $\sum R_p^{(k+1)}$, whence

$$\sum_{q=-\infty}^{+\infty} a_p R_{-\nu-q}^{(k+1)} = A_\nu^{(k)}\varepsilon_\nu,\qquad \varepsilon_\nu = o(1).$$

Now we have

$$|\mathrm{I}| \leqslant \sum_{\nu=0}^{n} |\Delta A_{n+k-\nu}\, A_\nu^{(k)} \varepsilon_\nu|$$

$$\leqslant A_n^{(k)} \Big[\sum_{\nu=0}^{[n/2]} |\Delta A_{n+k-\nu}\, \varepsilon_\nu| + \sum_{\nu=[n/2]+1}^{n} |\Delta A_{n+k-\nu}\, \varepsilon_\nu|\Big]$$

$$\leqslant A_n^{(k)} \Big[\max_{\nu>n/2+k} |\Delta A_\nu| \sum_{\nu=0}^{n} |\varepsilon_\nu| + \max_{\nu>n/2} |\varepsilon_\nu| \sum_{\nu=0}^{n} |\Delta A_{k+\nu}|\Big]$$

$$= A_n^{(k)} \Big[O\Big(A_n \frac{1}{n} \sum_{\nu=0}^{n} |\varepsilon_\nu|\Big) + \max_{\nu>n/2} |\varepsilon_\nu| \cdot O(1)\Big] = o(A_n^{(k)}).$$

From the convergence of the series $\sum a_q R_{-q}^{(j+1)}$ for $j = 1, 2, \ldots, k$, follows $\mathrm{II} = O(A_n^{(k-1)})$ and therefore $\mathrm{II} = o(A_n^{(k)})$. Finally, in virtue of lemma 2.3 applied to the series $\sum a_p$ and $\sum R_p^{(j+1)}$ we have

$$\sum_{q=-\infty}^{+\infty} a_q R_{-n+j-q}^{(j+1)} = o(A_{-n+j}^{(k)}) = o(A_n^{(k)}), \qquad j = 1, 2, \ldots, k,$$

whence $\mathrm{III} = O(A^{(k)})$. Thus we have $t_n' = o(A_n^{(k)})$.

By similar arguments it follows that $t_n'' = o(A_n^{(k)})$. Hence we have. $t_n = o(A_n^{(k)})$ and the theorem is proved.

In conclusion we observe that from the hypotheses of the theorem concerning the series $\sum b_p$ follows

$$\sum_{p=-\infty}^{+\infty} b_p = \sum_{p=-\infty}^{+\infty} R_p^{(1)} = \ldots = \sum_{p=-\infty}^{+\infty} R_p^{(k)} = 0, \quad \text{provided} \quad |a_p| \geqslant \varepsilon > 0.$$

In fact, the convergence of the series $\sum a_p R_p^{(j+1)}$, $j = 0, 1, \ldots, k$, implies $R_p^{(j+1)} = o(1)$ for $|p| \to +\infty$, whence

$$0 = \lim_{p\to-\infty} R_p^{(j+1)} = \lim_{p\to-\infty} \sum_{q=p}^{+\infty} R_q^{(j)} = \sum_{q=-\infty}^{+\infty} R_q^{(j)}.$$

THEOREM 2.6. *Let A_n be a sequence satisfying conditions* (1.1)-(1.4). *If $a_p = o(A_{|p|}^{(k)})$, $b_p = O(|p|^{-2k-3})$, where k is a positive integer, and*

$$\sum_{p=-\infty}^{+\infty} b_p = \sum_{p=-\infty}^{+\infty} p b_p = \ldots = \sum_{p=-\infty}^{+\infty} p^k b_p = 0,$$

then the formal product of the series $\sum a_p$ and $\sum b_p$ is $\mathrm{N}_k(A_n)$ summable to 0.

Proof. From the hypotheses of the theorem follows $R_p^{(j+1)} = O(|p|^{-2k+j-2})$, $j = 0, 1, \ldots, k$. In fact, we have for $p > 0$

$$R_p^{(1)} = \sum_{q=p}^{+\infty} b_q = \sum_{q=p}^{+\infty} O(|q|^{-2k-3}) = O(|q|^{-2k-2})$$

and for $p < 0$, since $\sum b_p = 0$,

$$R_p^{(1)} = -\sum_{q=p-1}^{-\infty} b_q = O(|p|^{-2k-2}).$$

Supposing that $R_p^{(j)} = O(|p|^{-2k+j-3})$ we obtain for $p > 0$

$$R_p^{(j+1)} = \sum_{q=p}^{+\infty} R_q^{(j)} = O(|p|^{-2k+j-2}).$$

Now, in virtue of lemma 2.4,

$$\sum_{q=-\infty}^{+\infty} R_q^{(j)} = \frac{1}{j!} \sum_{q=-\infty}^{+\infty} q^j b_q,$$

whence, the last series converging to 0, we have for $p < 0$

$$R_p^{(j+1)} = -\sum_{q=p-1}^{-\infty} R_q^{(j)} = O(|p|^{-2k+j-2}).$$

From these estimations and taking into account that, in virtue of lemma 1.1, $A_n^{(k)} = O(n^k)$ we have

$$\sum_{p=-\infty}^{+\infty} |A_{|p|}^{(k)} R_p^{(k+1)}| = \sum_{p=-\infty}^{+\infty} O(|p|^{-2}) < \infty.$$

Thus, the hypotheses of theorem 2.5 are satisfied and our theorem follows.

THEOREM 2.7 *If we suppose in theorem 2.6 that the series $\sum b_p$ converges to the sum λ, then the difference*

$$\sum_{p=-\infty}^{+\infty} c_p - \lambda \sum_{p=-\infty}^{+\infty} a_p$$

is $\mathrm{N}_k(A_n)$ summable to 0.

For the proof we need only recall the proof of theorem 2.2.

THEOREM 2.8. *Let A_n be a sequence satisfying conditions* (1.1)-(1.4), *$a_p = o(A_{|p|}^{(k)})$, $b_p = O(|p|^{-m-4})$, where $m \geqslant 2k+1$, and*

$$\sum_{p=-\infty}^{+\infty} b_p = \sum_{p=-\infty}^{+\infty} p b_p = \ldots = \sum_{p=-\infty}^{+\infty} p^m b_p = 0.$$

If $\sum d_p$ denotes the formal product of the series

$$\sum_{p=-\infty}^{+\infty}{}' p^{-m} a_p \quad \textit{and} \quad \sum_{p=-\infty}^{+\infty} b_p$$

where $\sum'$ indicates that the term for $p=0$ is omitted, then the series

$$\sum_{p=-\infty}^{+\infty} p^m d_p$$

is $\mathrm{N}_k(A_n)$ summable to 0.

Proof. From the proof of theorem 2.3 it follows that our theorem will be established if the formal products of the series

$$\sum_{p=-\infty}^{+\infty}{}' p^{-\nu} a_p \quad \text{and} \quad \sum_{p=-\infty}^{+\infty} p^{\nu} b_p$$

for $\nu=0,1,\ldots,m$ are $\mathrm{N}_k(A_n)$ summable to 0.

For $\nu=0,1,\ldots,k-1$ we have $p^{-\nu}a_p=o(p^{-\nu}A_{|p|}^{(k)})=o(A_{|p|}^{(k-\nu)})$ and $p^{\nu}b_p=O(|p|^{-(m-\nu)-4})=O(|p|^{-2(k-\nu)-3})$, so that, according to theorem 2.6, the corresponding formal products are $\mathrm{N}_{k-\nu}(A_n)$ summable, and therefore $\mathrm{N}_k(A_n)$ summable, to 0.

For $\nu=k,k+1,\ldots,m$ we have $p^{-\nu}a_p=o(1)$ and $p^{\nu}b_p=O(p^{-4})$, so that, in virtue of theorem 2.1, the corresponding formal products are $\mathrm{N}(1)$ summable (convergent), and therefore $\mathrm{N}_k(A_n)$ summable, to 0.

THEOREM 2.9. *If we suppose in theorem 2.8 that the series $\sum b_p$ converges to sum λ, then the difference*

$$\sum_{p=-\infty}^{+\infty} p^m d_p - \lambda \sum_{p=-\infty}^{+\infty}{}' a_p$$

is $\mathrm{N}_k(A_n)$ summable to 0.

For the proof viz. theorem 2.2.

Concluding this chapter we remark that in the sequel we shall be dealing with series whose coefficients depend upon a parameter. Now, if we suppose that the coefficients a_p and b_p are functions defined on a set E and satisfy uniformly in this set all conditions of lemmas and theorems we have proved, then the conclusions of these lemmas and theorems hold also uniformly in this set. For the sake of brevity we omit the proof of this statement.

III. The theory of localization for single trigonometric series

Using the results of the last chapter we may prove theorems concerning the theory of localization for single trigonometric series. We shall

use the complex form of these series:

$$\sum_{p=-\infty}^{+\infty} a_p e^{ipx},$$

where $\bar{a}_p=a_{-p}$, $Ja_0=0$. Convergence and summability of such series are always defined by means of their symmetric partial sums.

It is easy to see that the formal product of two trigonometric series

$$\sum_{p=-\infty}^{+\infty} a_p e^{ipx} \quad \text{and} \quad \sum_{p=-\infty}^{+\infty} b_p e^{ipx}$$

is also a trigonometric series with coefficients

$$c_p=\sum_{q=-\infty}^{+\infty} a_q b_{p-q}.$$

Let us suppose that the coefficients of the former series are $o(A_{|p|}^{(k)})$, where A_n is a sequence satisfying conditions (1.1)-(1.4) and k is a non--negative integer. Integrating formally this series m times with respect to x we obtain the series

$$\frac{a_0x^m}{m!}+u_1x^{m-1}+\ldots+u_{m-1}x+u_m+\sum_{p=-\infty}^{+\infty}{}' \frac{a_p}{(ip)^m}e^{ipx}$$

with coefficients $o(A_{|p|}^{(k)}/|p|^m)=o(|p|^{k-m})$.

If m is large enough, then the series integrated converges absolutely and is the Fourier series of its sum. Let us denote by $F(x)$, $G(x)$ and $P(x)$ the sum of the series integrated, the sum of its periodical part and its polynomial part respectively. Of course,

$$F(x)=P(x)+G(x).$$

Let $-\pi\leqslant a<a'<b'<b\leqslant\pi$. A periodic function $\lambda(x)$ with the period 2π is called a *localizing function* if

$$\lambda(x)=\begin{cases}1, & x\in\langle a',b'\rangle,\\ 0, & x \text{ outside } \langle a,b\rangle\end{cases} \pmod{2\pi}$$

and $\lambda(x)$ has sufficiently many continuous derivatives.

Denoting by

$$\sum_{p=-\infty}^{+\infty} b_p e^{ipx}$$

the Fourier series of the localizing function $\lambda(x)$ with m continuous derivatives we have $b_p=O(|p|^{-m})$ (viz. Zygmund [2], p. 39).

As a matter of notation we let $D_n(t)$ be the Dirichlet kernel of order n:

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}.$$

THEOREM 3.1. *Let A_n be a sequence satisfying conditions* (1.1)-(1.4). *If $a_p = o(A_{|p|})$, $m \geqslant 2$ and the localizing function $\lambda(x)$ has $m+4$ continuous derivatives, then the difference*

$$\frac{(-1)^m}{\pi}\int_a^b F(t)\lambda(t)\frac{d^m}{dt^m}D_n(x-t)\,dt - \sum_{p=-n}^{n} a_p e^{ipx}$$

is uniformly $\mathrm{N}(A_n)$ *summable to* 0.

The proof of this theorem follows the lines of the proof of the next one.

THEOREM 3.2. *Let A_n be a sequence satisfying conditions* (1.1)-(1.4). *If $a_p = o(A_{|p|}^{(k)})$, $m \geqslant 2k+1$ and the localizing function $\lambda(x)$ has $m+4$ continuous derivatives, then the difference*

$$\frac{(-1)^m}{\pi}\int_a^b F(t)\lambda(t)\frac{d^m}{dt^m}D_n(x-t)\,dt - \sum_{p=-n}^{n} a_p e^{ipx}$$

is uniformly $\mathrm{N}_k(A_n)$ *summable to* 0.

Proof. We let the series

$$\sum_{p=-\infty}^{+\infty} d_p e^{ipx}$$

be the formal product of the series

$$\sum_{p=-\infty}^{+\infty}{}' \frac{a_p e^{ipx}}{(ip)^m} \quad \text{and} \quad \sum_{p=-\infty}^{+\infty} b_p e^{ipx}.$$

The first series is absolutely convergent, for its coefficients are $o(|p|^{k-m})$ and $k-m \leqslant k-(2k+1) \leqslant -k-1 \leqslant -2$. The second series being the Fourier series of the localizing function with $m+4$ continuous derivatives, its coefficients are $O(|p|^{-m-4}) = O(p^{-4})$. Since both series satisfy the hypotheses of theorem 2.2 for $A_n = 1$, their formal product is uniformly $\mathrm{N}(1)$ summable or, simply, uniformly convergent, namely to the sum $\lambda(x)G(x)$, where $G(x)$ denotes the periodic part of the function $F(x)$. From the uniformity of convergence it follows that this formal product

is the Fourier series of the function $\lambda(x)G(x)$, whence

$$\sum_{p=-n}^{n} d_p e^{ipx} = \frac{1}{2\pi}\int_0^{2\pi}\lambda(t)G(x)D_n(x-t)\,dt$$

and

$$\sum_{p=-n}^{n}(ip)^m d_p e^{ipx} = \frac{1}{2\pi}\int_0^{2\pi}\lambda(x)G(x)\frac{d^m}{dx^m}D_n(x-t)\,dt.$$

Now the series $\sum a_p e^{ipx}$ and $\sum b_p e^{ipx}$ verify uniformly in the interval $\langle a', b'\rangle$ the conditions of theorem 2.9. In fact, according to the definition of the localizing function we have for $x \in \langle a', b'\rangle$

$$\lambda(x) = 1, \quad \lambda'(x) = \lambda''(x) = \ldots = \lambda^{(m)}(x) = 0,$$

which means that

$$\sum_{p=-\infty}^{+\infty} b_p e^{ipx} = 1,$$

$$\sum_{p=-\infty}^{+\infty} ipb_p e^{ipx} = \sum_{p=-\infty}^{+\infty}(ip)^2 b_p e^{ipx} = \ldots = \sum_{p=-\infty}^{+\infty}(ip)^m b_p e^{ipx} = 0$$

uniformly in $\langle a', b'\rangle$. Also $a_p e^{ipx} = o(A_{|p|}^{(k)})$ and $b_p e^{ipx} = O(|p|^{-m-4})$ uniformly. In virtue of theorem 2.9 the difference

$$\sum_{p=-n}^{n}(ip)^m d_p e^{ipx} - 1\cdot\sum_{p=-n}^{n}{}' a_p e^{ipx}$$

is uniformly $\mathrm{N}_k(A_n)$ summable to 0. Taking into account the formula concerning the former term of this difference and the fact of the vanishing of $\lambda(x)$ outside $\langle a, b\rangle (\mathrm{mod}\,2\pi)$ we obtain the difference

$$\frac{(-1)^m}{2\pi}\int_a^b \lambda(t)G(t)\frac{d^m}{dt^m}D_n(x-t)\,dt - \sum_{p=-n}^{n}{}' a_p e^{ipx},$$

which is uniformly $\mathrm{N}_k(A_n)$ summable to 0.

Our theorem will be established if also the difference

$$p_n(x) = \frac{(-1)^m}{2\pi}\int_a^b \lambda(t)P(t)\frac{d^m}{dt^m}D_n(x-t)\,dt - a_0$$

is $\mathrm{N}_k(A_n)$ summable to 0 uniformly in $\langle a', b'\rangle$. Integrating m times by parts we obtain

$$p_n(x) = \frac{1}{2\pi}\int_a^b \frac{d^m}{dt^m}[\lambda(t)P(t)]D_n(x-t)\,dt - a_0$$

$$= \frac{1}{2\pi}\int_a^b \sum_{\nu=0}^{m}\binom{m}{\nu}\lambda^{(\nu)}(t)P^{(m-\nu)}(t)D_n(x-t)\,dt - a_0$$

$$= \sum_{\nu=1}^{m}\binom{m}{\nu}\frac{1}{2\pi}\int_a^b \lambda^{(\nu)}(t)P^{(m-\nu)}(t)D_n(x-t)\,dt -$$

$$-\frac{a_0}{2\pi}\int_0^{2\pi}[1-\lambda(t)]D_n(x-t)\,dt.$$

Since for $x\in\langle a', b'\rangle$ $1-\lambda(x) = \lambda'(x) = \ldots = \lambda^{(m)}(x) = 0$, the sequence $p_n(x)$ uniformly converges to 0 and therefore is uniformly $N_k(A_n)$ summable to 0. Thus the theorem is proved.

Theorems 3.1 and 3.2 contain the theory of localization for trigonometric series.

THEOREM 3.3. *If $a_p = o(A_{|p|}^{(k)})$, $m \geqslant 2k+1$ and the function*

$$F(x) = \frac{a_0 x^m}{m!} + u_1 x^{m-1} + \ldots + u_m + \sum_{p=-\infty}^{+\infty}\frac{a_p}{(ip)^m}e^{ipx}$$

vanishes in an interval $\langle a, b\rangle$, then in any interval $\langle a', b'\rangle$ interior to $\langle a, b\rangle$ the trigonometric series $\sum a_p e^{ipx}$ is uniformly $N_k(A_n)$ summable to 0.

THEOREM 3.4. *Let a_p and a_p' be $o(A_{|p|}^{(k)})$, $m \geqslant 2k+1$ and let $F(x)$ and $\tilde{F}(x)$ be the sums of the series obtained by integrating m times the series $\sum a_p e^{ipx}$ and $\sum a_p' e^{ipx}$ respectively. If $F(x) = \tilde{F}(x)$ in an interval $\langle a, b\rangle$ or, more generally, if $F(x)-\tilde{F}(x)$ is equal to a polynomial of degree m in this interval, then in any interval $\langle a', b'\rangle$ interior to $\langle a, b\rangle$ the series $\sum a_p e^{ipx}$ and $\sum a_p' e^{ipx}$ are uniformly $N_k(A_n)$ equisummable, which means that the difference of these two series is uniformly $N_k(A_n)$ summable to 0.*

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Remarque sur la régularité des intégrales des équations différentielles hyperboliques du second ordre

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Considérons le problème de Darboux dans lequel il s'agit de trouver une intégrale de l'équation

$$u_{xy}(x, y) = f(x, y, u(x, y), u_x(x, y), u_y(x, y)) \tag{1}$$

prenant des valeurs données le long des caractéristiques $x = 0$ et $y = 0$. Cette solution peut être cherchée parmi les fonctions appartenant à diverses classes de régularité.

DÉFINITION 1. Une fonction φ définie dans un ensemble Z sera appelée dans la suite *fonction de classe C^n* ($n \geqslant 1$), lorsqu'elle possède des dérivées partielles continues d'ordre n dans l'ensemble Z.

La plupart des théorèmes concernant le problème de Darboux relatif à l'équation (1) assure l'existence d'une solution dans l'ensemble des fonctions $u(x, y)$ continues avec leurs dérivées u_x, u_y, u_{xy} ou, ce qui revient au même, en vertu de la continuité de la fonction f qui y intervient, dans l'ensemble des fonctions de classe C^1. L'importance des solutions de classe C^2 a été indiquée par E. Kamke (cf. [1], p. 402, renvoi (1)). Dans l'hypothèse que $f(x, y, u, p, q)$ est une fonction de classe C^1 dans le parallélépipède: $|x|, |y| < d$, $|u|, |p|, |q| < M$ ($d > 0$, $M > 0$), H. Schaefer (cf. [2]) a démontré que l'équation (1) admet dans le rectangle suffisamment petit: $|x|, |y| < d_1$ ($d_1 \leqslant d$) une solution unique de classe C^2 s'annulant le long des caractéristiques $x = 0$ et $y = 0$.

L'une des deux démonstrations de ce théorème, données dans [2], est basée sur le résultat bien connu concernant l'existence et l'unicité d'une solution $u(x, y)$ de classe C^1 du problème considéré. Ainsi cette démonstration se ramène à prouver l'existence et la continuité des dérivées $u_{xx}(x, y)$ et $u_{yy}(x, y)$ (cf. [2], 2). Dans la suite nous donnons une autre démonstration, plus simple, de cette dernière propriété et même dans un cas plus général, à savoir dans le cas où $u(x, y)$ est une solution arbitraire de l'équation (1), de classe C^1 dans le rectangle R,

$$R\colon\quad -a < x < a, \quad -\beta < y < \beta \quad \text{où} \quad 0 < a < \infty, \quad 0 < \beta < \infty$$