Comme

$$H^i(0, \varphi, h^1(0, \varphi), \ldots, h^n(0, \varphi)) = Q^i(A_0)\cos\varphi - P^i(A_0)\sin\varphi$$

et par conséquent

$$\int_0^{2\pi} H^i(0, \varphi, h^1(0, \varphi), \ldots, h^n(0, \varphi))\, d\varphi = 0,$$

nous pouvons écrire (12) sous la forme suivante

$$\frac{h^i(r, 2\pi) - z_0^i}{r^2\pi}$$
$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{r}[H^i(r, \varphi, h^1(r, \varphi), \ldots, h^n(r, \varphi)) - H^i(0, \varphi, h^1(0, \varphi), \ldots, h^n(0, \varphi))]\, d\varphi.$$

Lorsque $r \to 0$, le quotient sous le signe d'intégrale tend vers

$$L^i(\varphi) = \left[\frac{d}{dr} H^i(r, \varphi, h^1(r, \varphi), \ldots, h^n(r, \varphi))\right]_{r=0}$$

uniformément par rapport à φ dans l'intervalle $0 \leqslant \varphi \leqslant 2\pi$. Il s'ensuit que

(13) $$\lim_{r\to 0} \frac{h^i(r, 2\pi) - z_0^i}{r^2\pi} = \frac{1}{\pi}\int_0^{2\pi} L^i(\varphi)\, d\varphi.$$

Mais, d'après (10) et (11), nous avons

(14) $$L^i(\varphi) = Q_x^i(A_0)\cos^2\varphi + Q_y^i(A_0)\cos\varphi\sin\varphi +$$
$$+ \sum_{j=1}^n Q_{z^j}^i(A_0)[P^j(A_0)(\cos\varphi - 1) + Q^j(A_0)\sin\varphi]\cos\varphi -$$
$$- P_x^i(A_0)\cos\varphi\sin\varphi - P_y^i(A_0)\sin^2\varphi -$$
$$- \sum_{j=1}^n P_{z^j}^i(A_0)[P^j(A_0)(\cos\varphi - 1) + Q^j(A_0)\sin\varphi]\sin\varphi.$$

Les relations (13) et (14) impliquent la conclusion (9) de notre théorème.

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Limitations and dependence on parameter of solutions of non-stationary differential operator equations

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The purpose of the present paper is to discuss some properties of the solutions of the differential equation

(∗) $$\frac{dx}{dt} = A(t)x + f(t, x).$$

$A(t)$ is a closed and linear operator defined on a linear subset of the Banach space E. The values of $A(t)$ belong to E. The theory of equation (∗) is a continuation of the theory of one-parameter semi-groups of linear and bounded operators founded by Hille and Yosida (see for instance [2]). Kato in [4] investigated the case of the variable "coefficient" $A(t)$. Krasnoselskiĭ, Kreĭn and Sobolevskiĭ presented in [6], [7] many new results and discussed the case of the non-linear member $f(t, x)$. This paper deals with some general theorems concerning the limitations of the solutions of (∗). We use the epidermic theorem for ordinary differential inequalities. The epidermic theorems have been introduced by T. Ważewski in [13] (see also [8]). We apply the epidermic theorem for the reason that usually the solutions of (∗) do not satisfy the equation at the initial point $t = 0$. The nature of the epidermic effect is explained in [13] and [14]. We present several uniqueness theorems. In § 4 we prove some existence theorems which generalize in a certain sense some results of [6] and [7]. We use the topological method of Leray-Schauder. The a priori limitations needed in this method are ensured by suitable theorems of §§ 2, 3. In the last section we discuss the dependence of the solutions on a real parameter.

§ 1. Notation and definitions. Let E be a real Banach space. The elements of E are denoted by $x, y, z, \ldots$ The functions of the real variable t with values lying in E are denoted by $x(t), y(t), z(t), \ldots$ $|x|$ is the norm of the element x, Θ stands for the zero of E. In the following we investigate the operators which are defined on suitable subsets of E and take on values belonging to E. The operator V is *linear* if it is additive and homogeneous; the set on which V is defined we denote by $D[V]$. V is *closed* if the set of couples $x, Vx)$ $(x \in D[V])$ is closed in the topological product $E\times E$. Take the real λ and suppose that there exists a bounded inverse $(\lambda I - V)^{-1}$ (I being the identity operation), and that $D[(\lambda I-V)^{-1}]$ is dense in E. Write $R(\lambda, V) = (\lambda I - V)^{-1}$. If V is closed then $D[R(\lambda, V)] = E$ and

$$(\lambda I - V)R(\lambda, V)x = x \quad \text{for} \quad x\in E,$$

$$R(\lambda, V)(\lambda I - V)x = x \quad \text{for} \quad x \in D[V].$$

$R(\lambda, V)$ is called the *resolvent* of V. We write

$$D_+x(t) = \lim_{h\to 0+}\frac{x(t+h)-x(t)}{h}, \qquad x'(t) = \lim_{h\to 0}\frac{x(t+h)-x(t)}{h}$$

if the limits on the right exist. By small Greek letters we denote the real valued functions. The right upper derivative of $\varphi(t)$ is defined by

$$\overline{D}_+\varphi(t) = \limsup_{h\to 0+}\frac{\varphi(t+h)-\varphi(t)}{h}.$$

§ 2. For the sake of simplicity we introduce two conditions:

(T) The non-negative function $\sigma(t, u)$ is continuous for $t\in\langle 0, a)$ and $u \geqslant 0$. Denote by $\omega(t, u^0)$ the right maximal solution of the differential equation $u' = \sigma(t, u)$, such that $\omega(0, u^0) = u^0$. We assume that $\omega(t, u^0)$ exists in the interval $\langle 0, a)$ for every $u^0 \geqslant 0$.

(H_Δ) Suppose we are given a one-parameter family $\{A(t)\}$, $t\in\Delta$ of closed linear operators. We assume that for every $t\in\Delta$ there is a $\lambda(t)$ such that for $\lambda\geqslant\lambda(t)$ the operator $A(t)$ possesses the resolvent $R(\lambda, A(t))$.

Now we formulate the epidermic lemma (see [8] and [13]):

LEMMA 1. *Let the function $\sigma(t, u)$ satisfy condition* (T). *Suppose that the function $\varphi(t)\geqslant 0$ is continuous in $\langle 0, a)$ and satisfies the following condition: there exists an at most denumerable set $Z\subset \underset{\tau}{E}\{\tau\in(0, a),\ \omega(\tau, \varphi(0)) < \varphi(\tau)\} = S$ such that the inequality $\overline{D}_+\varphi(\xi)\leqslant\sigma(\xi, \varphi(\xi))$ holds for $\xi\in(S-Z)$. Then the inequality $\varphi(t)\leqslant\omega(t, \varphi(0))$ holds for $t\in\langle 0, a)$.*

The second lemma is the following one:

LEMMA 2. *Suppose we are given two linear operators A_1 and A_2. We assume that $(\lambda I - A_1)^{-1} = \tilde{R}(\lambda, A_1)$, $(\lambda I - A_2)^{-1} = \tilde{R}(\lambda, A_2)$ exist for λ sufficiently large. The functions $x_1(t)$, $x_2(t)$ are defined for $t\in\langle\xi, \xi+\delta)$ $(\delta > 0)$, $x_i(\xi)\in D[A_i]$, and*

(1) $$D_+x_i(\xi) = A_ix_i(\xi)+y_i \qquad (i = 1, 2).$$

Suppose that

(2) $$\lim_{\lambda\to+\infty}\lambda\tilde{R}(\lambda, A_i)A_ix_i(\xi) = A_ix_i(\xi) \qquad (i = 1, 2).$$

Let the inequality

(3) $$\left|\lambda\tilde{R}(\lambda, A_1)x_1(\xi) - \lambda\tilde{R}(\lambda, A_2)x_2(\xi) + \frac{1}{\lambda}[y_1-y_2]\right| \leqslant \frac{1}{\lambda}p + |x_1(\xi)-x_2(\xi)|$$

be satisfied for λ sufficiently large. Our assumptions imply the inequality $\overline{D}_+|x_1(\xi)-x_2(\xi)|\leqslant p$.

Proof. Observe that

(4) $$x_1(\xi+h) - x_2(\xi+h) = [x_1(\xi)+hA_1x_1(\xi)] - [x_2(\xi)+hA_2x_2(\xi)] + h[y_1-y_2] + \varepsilon(h)$$

for $h > 0$ and h sufficiently small. Write $h = 1/\lambda$ and $R_i(h) = \tilde{R}(1/h, A_i)$. In view of the definition of $R_i(h)$ we have

(5) $$\frac{1}{h}R_i(h)x_i(\xi) - h\left[\frac{1}{h}R_i(h)A_ix_i(\xi) - A_ix_i(\xi)\right] = x_i(\xi)+hA_ix_i(\xi) \qquad (i = 1, 2).$$

Relations (3), (4) and (5) imply the inequality

(6) $$\frac{|x_1(\xi+h)-x_2(\xi+h)| - |x_1(\xi)-x_2(\xi)|}{h} \leqslant p + \frac{|\varepsilon(h)|}{h} + \left|\frac{1}{h}R_1(h)A_1x_1(\xi) - A_1x_1(\xi)\right| + \left|\frac{1}{h}R_2(h)A_2x_2(\xi) - A_2x_2(\xi)\right|.$$

By (2), $\lim_{h\to 0+}\left|\frac{1}{h}R_i(h)A_ix_i(\xi) - A_ix_i(\xi)\right| = 0$. From (1), $\lim_{h\to 0+}\frac{|\varepsilon(h)|}{h} = 0$. From (6) we find that $\overline{D}_+|x_1(\xi)-x_2(\xi)|\leqslant p$, q. e. d.

THEOREM 1. *Suppose that the function $\sigma(t, u)$ satisfies* (T). *Let the operators $A_i(t)$ $(i = 1, 2)$ satisfy the condition $(H_{\langle 0,a)})$. Assume that the function $x_i(t)$ $(i = 1, 2)$ is continuous in the interval $\langle 0, a)$ and satisfies the equation $D_+x_i(t) = A_i(t)x_i(t) + f_i(t, x_i(t))$ for $t\in(0, a) - Z_i$; Z_i is an at most denumerable subset of $(0, a)$. Suppose that for every triple (x, y, t) there is a $\lambda(x, y, t)$ such that for $\lambda\geqslant\lambda(x, y, t)$.*

(7) $$\left|\lambda R(\lambda, A_1(t))x - \lambda R(\lambda, A_2(t))y + \frac{1}{\lambda}[f_1(t, x) - f_2(t, y)]\right| \leqslant \frac{1}{\lambda}\sigma(t, |x-y|) + |x-y|.$$

We assume that $\lim_{\lambda\to\infty}\lambda R(\lambda, A_i(t))x = x$ $(i = 1, 2;\ t\in(0, a))$ for every $x\in E$. Our assumptions imply the inequality $|x_1(t)-x_2(t)|\leqslant\omega(t, |x_1(0)-x_2(0)|)$, $t\in\langle 0, a)$.

Proof. Let $\varphi(t) = |x_1(t)-x_2(t)|$. Suppose that

(8) $$\xi \in \underset{\tau}{E}\{\tau\in(0, a),\ \omega(\tau, \varphi(0)) < \varphi(\tau)\} - (Z_1+Z_2).$$

We find that $D_+x_i(\xi) = A_i(\xi)x_i(\xi)+f_i(\xi, x_i(\xi))$ $(i = 1, 2)$. Now write $y_i = f_i(\xi, x_i(\xi))$, $p = \sigma(\xi, \varphi(\xi))$. Because of (7) inequality (3) of lemma 2 holds for λ sufficiently large. Therefore

(9) $$\bar{D}_+\varphi(\xi) \leqslant \sigma(\xi, \varphi(\xi)).$$

We see that (8) implies (9). From lemma 1 we conclude that $|x_1(t)-x_2(t)| \leqslant \omega(t, |x_1(0)-x_2(0)|)$ for $t\in\langle 0, a)$, q. e. d.

Remark. Note that $\lim\limits_{\lambda\to+\infty} \lambda R(\lambda, V)x = x$ for every $x\in E$ if V is closed, $D[V]$ is dense in E and $\limsup\limits_{\lambda\to+\infty} |\lambda R(\lambda, V)| < +\infty$.

Theorem 1 implies the following theorem:

THEOREM 2. *Suppose that the function $\sigma(t, u)$ satisfies the condition* (T). *Let the operators $A(t)$ satisfy the condition* $(\mathrm{H}_{(0,a)})$. *We assume that $D[A(t)]$ is dense in E and $|\lambda R(\lambda, A(t))| \leqslant 1$ for every $t\in(0, a)$. Suppose that $|f_1(t, x)-f_2(t, y)| \leqslant \sigma(t, |x-y|)$. Let the function $x_i(t)$ $(i = 1, 2)$ be continuous in $\langle 0, a)$. Assume that $x_i(t)$ satisfies the equation $D_+x_i(t) = A(t)x_i(t)+f_i(t, x_i(t))$ except an at most denumerable subset of $(0, a)$. Under our assumptions the inequality $|x_1(t)-x_2(t)| \leqslant \omega(t, |x_1(0)-x_2(0)|)$ holds for $t\in\langle 0, a)$.*

Theorem 2 implies the following uniqueness theorem, corresponding to theorem 10 of [12]:

THEOREM 3. *Suppose that $\sigma(t, u)$ satisfies* (T). *Let the operator $A(t)$ satisfy* $(\mathrm{H}_{(0,a)})$. *We assume that $D[A(t)]$ is dense in E and $|\lambda R(\lambda, A(t))| \leqslant 1$ for every $t\in(0, a)$. Suppose that $|f(t, x)-f(t, y)| \leqslant \sigma(t, |x-y|)$. We assume that $\omega(t, 0) \equiv 0$. Then through every point $(0, x_0)$ $(x_0\in E)$ there passes at most one continuous solution satisfying the equation $D_+x(t) = A(t)x(t)+f(t, x(t))$ except an at most denumerable subset of $(0, a)$.*

THEOREM 4. *Let the function $\sigma(t, u)$ satisfy condition* (T). *We assume that $A(t)$ satisfy* $(\mathrm{H}_{(0,a)})$. *Suppose that $D[A(t)]$ is dense in E and $|\lambda R(\lambda, A(t))| \leqslant 1$ for every $t\in(0, a)$. Let the continuous function $x(t)$ satisfy the equation $D_+x(t) = A(t)x(t)+f(t, x(t))$ for $t\in(0, a)-Z$. Z is an at most denumerable subset of $(0, a)$. Suppose that $|f(t, x)| \leqslant \sigma(t, |x|)$. Our assumptions imply the inequality $|x(t)| \leqslant \omega(t, |x(0)|)$, $t\in\langle 0, a)$.*

Proof. Suppose that $t\in \underset{\tau}{E}\{\tau\in(0, a),\ \omega(\tau, |x(0)|) < |x(\tau)|\} - Z$; in lemma 2 we put $\xi = t$, $x_1(\xi) = x(t)$, $x_2(\xi) = \Theta$, $A_1 = A(\xi)$, $A_2 \equiv \Theta$, $y_1 = f(\xi, x(\xi))$, $y_2 = \Theta$, $p = \sigma(\xi, |x(\xi)|)$. Therefore $\bar{D}_+|x(\xi)| \leqslant \sigma(\xi, |x(\xi)|)$. Now we apply lemma 1.

Theorems 1-4 are generalizations of theorem 1 of Kato's paper [4]. By means of these theorems we can prove some theorems concerning the continuous dependence of solutions on the initial point and on the right-hand member of the equation. It is easy to see that there is no necessity to assume that $a = +\infty$. Therefore we can discuss the problem of stability of solutions. Remark that our theorems (except the uniqueness theorem) remain true if instead of the norm one takes a pseudo-norm. One can consider a finite sequence of pseudo-norms just as in [9]. We observe that our method is applicable if the functions $x_i(t)$, $x(t)$ satisfy the suitable differential equations almost everywhere. In this case we assume that the real valued functions $|x_1(t)-x_2(t)|$, $|x(t)|$ are absolutely continuous.

EXAMPLE 1. Suppose that $A(t)$ satisfies $(\mathrm{H}_{(0,a)})$ and that $D[A(t)]$ is dense in E. Let $\varrho(t)$ be continuous and

$$|R(\lambda, A(t))| \leqslant \frac{1}{\lambda}+\frac{\varrho(t)}{\lambda^2}, \qquad \lambda > 0.$$

Assume that $x_i(t)$ $(i = 1, 2)$ is continuous and $x_i'(t) = A(t)x_i(t)$ for $t\in(0, a)$. Then, by theorem 1, $|x_1(t)-x_2(t)| \leqslant |x_1(0)-x_2(0)|\cdot\exp\left(\int_0^t \varrho(s)\,ds\right)$.

EXAMPLE 2. Let $A(t)$ satisfy $(\mathrm{H}_{(0,a)})$ and $|\lambda R(\lambda, A(t))| \leqslant 1$. We assume that $D[A(t)]$ is dense in E and $|f(t, x)| \leqslant \varrho(t)|x|$. Suppose that $x'(t) = A(t)x(t)+f(t, x(t))$. Then, by theorem 4, $|x(t)| \leqslant |x(0)|\exp\left(\int_0^t \varrho(s)\,ds\right)$. If $a = +\infty$ and $\int_0^\infty \varrho(s)\,ds < +\infty$ then the solution $x(t) \equiv \Theta$ is stable.

We shall now prove some uniqueness theorems corresponding to the general theorem of Kamke (see [3]).

THEOREM 5. *Suppose that $A(t)$ satisfy* $(\mathrm{H}_{(0,a)})$. *We assume that $D[A(t)]$ is dense in E and $|\lambda R(\lambda, A(t))| \leqslant 1$ for $\lambda > 0$. Let the following condition be satisfied:*

(S) *The function $\sigma(t, u)$ is continuous for $0 < t < a$ and $u \geqslant 0$. Moreover, for every $\varrho\in(0, a)$ the unique function $\omega(t)$ which satisfies the equation $u' = \sigma(t, u)$ for $0 < t < \varrho$ and the equalities $\lim\limits_{h\to 0+}\omega(h) = \omega'(0) = 0$ is identically equal to zero: $\omega(t) \equiv 0$.*

Suppose that

(10) $$|f(t, x)-f(t, y)| \leqslant \sigma(t, |x-y|)$$

for $0 < t < a$ and $x, y\in E$. Let the continuous functions $x_1(t)$, $x_2(t)$ satisfy the equation $x' = A(t)x+f(t, x)$ for $0 < t < a$. We assume that $|x_1(h)-x_2(h)| = o(h)$ for $h > 0$ and h sufficiently small. Then $x_1(t) \equiv x_2(t)$ for $t\in(0, a)$.

Proof. Suppose that there is a $t_0 \in (0, a)$ such that $x_1(t_0) \neq x_2(t_0)$. Denote by $\tau(t)$ the left minimal solution of the equation $u' = \sigma(t, u)$, such that

$$\tau(t_0) = |x_1(t_0) - x_2(t_0)| > 0. \tag{11}$$

It follows from the inequality $|\lambda R(\lambda, A(t))| \leqslant 1$ and from (10) that

$$\left|\lambda R(\lambda, A(t))x_1(t) - \lambda R(\lambda, A(t))x_2(t) + \frac{1}{\lambda}[f(t, x_1(t) - f(t, x_2(t))]\right|$$
$$\leqslant \frac{1}{\lambda}\sigma(t, |x_1(t) - x_2(t)|) + |x_1(t) - x_2(t)| \qquad (\lambda > 0)$$

for $t \in (0, a)$. By lemma 2 we find that

$$\overline{D}_+ |x_1(t) - x_2(t)| \leqslant \sigma(t, |x_1(t) - x_2(t)|) \quad \text{for} \quad t \in (0, a). \tag{12}$$

Therefore $\tau(t)$ may be extended as a minimal solution to the whole interval $(0, t_0)$. From (12) we find that the extended integral $\tau(t)$ satisfies the inequality $0 \leqslant \tau(t) \leqslant |x_1(t) - x_2(t)|$ for $0 < t \leqslant t_0$. Therefore $\lim\limits_{h \to 0+} \tau(h) = 0 \stackrel{\text{df}}{=} \tau(0)$. On the other hand $0 \leqslant \tau(h) \leqslant |x_1(h) - x_2(h)| = o(h)$. It is clear that $\tau'(0) = 0$. Assumption (S) implies that $\tau(t) \equiv 0$. This contradicts (11).

Theorem 5 implies the following theorem:

THEOREM 6. *Suppose that $A(t)$ satisfy $H_{\langle 0,a)}$. We assume that for every $t \in (0, a)$ $D[A(t)]$ is dense in E and $|\lambda R(\lambda, A(t))| \leqslant 1$. Let the function $\sigma(t, u)$ satisfy condition* (S). *Suppose that $|f(t, x) - f(t, y)| \leqslant \sigma(t, |x-y|)$. Then through each point $(0, x_0)$ $(x_0 \in E)$ there passes at most one solution of the equation $x' = A(t)x + f(t, x)$ satisfying this equation for $t \in \langle 0, a)$.*

A trivial modification of the proof of theorem 5 shows the validity of the following theorem:

THEOREM 7. *Let $A(t)$ satisfy $(H_{(0,a)})$. We assume that $D[A(t)]$ is dense in E for every $t \in (0, a)$ and $|\lambda R(\lambda, A(t))| \leqslant 1$. Suppose that the function $\sigma(t, u)$ satisfies the following condition: $\sigma(t, u)$ is continuous for $0 < t < a$ and $u \geqslant 0$, and for every $\varrho \in (0, a)$ the unique continuous function $\omega(t)$, which satisfies the equation $u' = \sigma(t, u)$ for $0 < t < \varrho$ and which satisfies the equality $\omega(0) = 0$, is identically equal to zero: $\omega(t) \equiv 0$. Assume that $|f(t, x) - f(t, y)| \leqslant \sigma(t, |x-y|)$ for $t \in (0, a)$ and $x, y \in E$. Then through every point $(0, x)$ $(x \in E)$ there passes at most one solution (in $(0, a)$) of the equation $x' = A(t)x + f(t, x)$.*

THEOREM 8. *Suppose that $A(t)$ satisfy $(H_{(0,a)})$. We assume that $D[A(t)]$ are dense in E and $\limsup\limits_{\lambda \to +\infty} |\lambda R(\lambda, A(t))| < +\infty$. The functional $\Phi(x)$ possesses the Fréchet differential $L(x, y)$ and $L(x, f(t, x)) \leqslant \sigma(t, \Phi(x))$ and $L(x, \lambda R(\lambda, A(t))x) \leqslant L(x, x)$. Suppose that $\sigma(t, u)$ satisfies* (T). *Let $x(t)$ be the continuous solution of the equation $x' = A(t)x + f(t, x)$. Then $\Phi[x(t)] \leqslant \omega(t, \Phi[x(0)])$ for $t \in \langle 0, a)$.*

Proof. Suppose that $t \in \underset{\tau}{E}\{\tau \in (0, a), \omega(\tau, \Phi[x(0)]) < \Phi[x(\tau)]\}$. We find that

$$x(t+h) = x(t) + hA(t)x(t) + hf(t, x(t)) + o(h)$$
$$= \frac{1}{h} R\left(\frac{1}{h}, A(t)\right) x(t) + h\left[A(t)x(t) - \frac{1}{h} R\left(\frac{1}{h}, A(t)\right) A(t)x(t)\right] +$$
$$+ hf(t, x(t)) + o(h).$$

Therefore

$$L(x(t), x(t+h)) \leqslant L(x(t), x(t)) + hL(x(t), A(t)x(t)) -$$
$$- \frac{1}{h} R\left(\frac{1}{h}, A(t)\right) A(t)x(t) + h\sigma(t, \Phi[x(t)]) + L(x(t), o(h)). \tag{13}$$

Write $\varphi(t) = \Phi[x(t)]$. We have $\varphi'(t) = L(x(t), x'(t))$. (13) implies the inequality $\varphi'(t) = L(x(t), x'(t)) \leqslant \sigma(t, \varphi(t))$. Now we apply lemma 1.

§ 3. LEMMA 3 (see [11], theorem 1). *Assume that the function $\sigma(t, u)$ satisfies* (T). *Suppose that $\sigma(t, u)$ increases in u. Let the continuous function $\varphi(t) \geqslant 0$ satisfy the inequality*

$$\varphi(t) \leqslant \eta + \int_0^t \sigma(s, \varphi(s))\,ds, \quad t \in (0, a). \tag{14}$$

Then $\varphi(t) \leqslant \omega(t, \eta)$ for $t \in \langle 0, a)$.

Suppose we are given a family $U(t, s)$ of linear and bounded operators. Let $U(t, s)$ satisfy the following conditions:

$$U(t, s) \text{ is strongly continuous for } 0 \leqslant s \leqslant t \leqslant a, \tag{15}$$

$$\lim_{t \to 0+} U(t, 0)x = x \quad \text{for} \quad x \in E, \tag{16}$$

$$|U(t, s)| \leqslant 1 \quad \text{for} \quad 0 \leqslant s \leqslant t \leqslant a. \tag{17}$$

THEOREM 9. *Suppose that $\sigma(t, u)$ satisfies* (T). *We assume that $\sigma(t, u)$ increases in u. Let the function $f_i(t, x)$ be continuous for $0 \leqslant t \leqslant a$ and $x \in E$. Suppose that the operator function $U(t, s)$ satisfies* (15)-(17). *Assume that $x_i(t)$ $(i = 1, 2)$ is continuous and satisfies the equation*

$$x_i(t) = U(t, 0)\overline{x}_i + \int_0^t U(t, s) f_i(s, x_i(s))\,ds \quad (0 \leqslant t < a). \tag{18}$$

Suppose that $|f_1(t,x)-f_2(t,y)| \leqslant \sigma(t,|x-y|)$. *Then* $|x_1(t)-x_2(t)| \leqslant \omega(t,|\bar{x}_1-\bar{x}_2|)$ *for* $t \in \langle 0, a)$.

Proof. Write $\varphi(t) = |x_1(t)-x_2(t)|$. From (18) we find that

$$\varphi(t) \leqslant |\bar{x}_1-\bar{x}_2| + \int_0^t \big|U(t,s)[f_1(s,x_1(s))-f_2(s,x_2(s))]\big|\,ds$$

$$\leqslant |\bar{x}_1-\bar{x}_2| + \int_0^t \big|f_1(s,x_1(s))-f_2(s,x_2(s))\big|\,ds.$$

On the other hand $|f_1(s,x_1(s))-f_2(s,x_2(s))| \leqslant \sigma(s,\varphi(s))$. Therefore $\varphi(t)$ satisfies (14) with $\eta = |\bar{x}_1-\bar{x}_2|$. The assertion of our theorem now follows from lemma 3.

One easily proves the following theorem:

THEOREM 10. *Suppose that* $\sigma(t,u)$ *satisfies* (T). *We assume that* $\sigma(t,u)$ *increases in* u. *Let the operator function* $U(t,s)$ *satisfy* (15)-(17). *Assume that the functions* $x(t), f(t,x)$ *are continuous and*

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))\,ds. \tag{19}$$

Suppose that $|f(t,x)| \leqslant \sigma(t,|x|)$. *Then* $|x(t)| \leqslant \omega(t,|x_0|)$ *for* $t \in \langle 0, a)$.

Equation (19) has been discussed in [6] in connection with the differential equation $x' = A(t)x+f(t,x)$: in this case $U(t,s)$ is the general solution of the homogeneous equation $x' = A(t)x$. The existence of the general solution is ensured by some assumptions given by Kato in [4] (see also [6]). These assumptions are formulated in the last section of this paper. The solutions of (19) may be interpreted as the generalized solutions of the corresponding differential equation. Theorem 9 implies the following uniqueness theorem for generalized solutions:

THEOREM 11. *Suppose that* $\sigma(t,u)$ *satisfies* (T) *and increases in* u. *Assume that* $\omega(t,0)=0$. *We suppose that* $|f(t,x)-f(t,y)| \leqslant \sigma(t,|x-y|)$. *Then equation* (19) *has at most one solution.*

§ 4. In this section we shall consider some existence theorems. By $C_E\langle 0, a\rangle$ we denote the space of functions $x(t)$ continuous in $\langle 0, a\rangle$ with the traditional norm $\|u\| = \|x(\cdot)\| = \max_{0\leqslant t\leqslant a}|x(t)|$. We say that $g(t,x)$ is *completely continuous* if it is continuous and compact.

One easily proves the following lemma:

LEMMA 4. *Let* $U(t,s)$ $(0 \leqslant s \leqslant t \leqslant a)$ *be a strongly continuous operator function. Suppose that* $g(t,x)$ *is completely continuous. Then the transformation* G *defined by the formula*

$$x(t) \to \int_0^t U(t,s)g(s,x(s))\,ds$$

is completely continuous when considered in the space $C_E\langle 0, a\rangle$.

Let the function $h(t,x)$ satisfy the Lipschitz condition of the form

$$|h(t,x)-h(t,y)| \leqslant L(t)|x-y|. \tag{20}$$

$L(t)$ is supposed to be summable in $\langle 0, a\rangle$. Owing to the idea of A. Bielecki (see [1]) we introduce in $C_E\langle 0, a\rangle$ the norm $\|x(\cdot)\|_\varkappa$ by means of the formula

$$\|x(\cdot)\|_\varkappa = \sup_{0\leqslant t\leqslant a}\Big|x(t)\exp\Big(-\varkappa\int_0^t L(s)\,ds\Big)\Big| \quad (\varkappa > 0).$$

This norm is equivalent to the traditional one.

LEMMA 5 ([1]). *Suppose that* $h(t,x)$ *is continuous and satisfies* (20) *with summable* $L(t)$. *Let the operator function* $U(t,s)$ *be strongly continuous and* $|U(t,s)| \leqslant 1$. *Then the transformation* H *defined by the formula*

$$x(t) \to U(t,0)x + \int_0^t U(t,s)h(s,x(s))\,ds$$

satisfies the inequality

$$\|Hx(\cdot)-Hy(\cdot)\|_\varkappa \leqslant \frac{1}{\varkappa}\|x(\cdot)-y(\cdot)\|_\varkappa. \tag{21}$$

Proof. We have

$$\Big|\exp\Big(-\varkappa\int_0^t L(s)\,ds\Big)\cdot\int_0^t U(t,s)[h(s,x(s))-h(s,y(s))]\,ds\Big|$$

$$\leqslant \frac{1}{\varkappa}\int_0^t \varkappa L(u)\cdot\exp\Big(-\varkappa\int_u^t L(s)\,ds\Big)\cdot\exp\Big(-\varkappa\int_0^u L(s)\,ds\Big)\cdot|x(u)-y(u)|\,du$$

$$\leqslant \frac{1}{\varkappa}\|x(\cdot)-y(\cdot)\|_\varkappa\Big(1-\exp\Big(-\varkappa\int_0^t L(s)\,ds\Big)\Big) \leqslant \frac{1}{\varkappa}\|x(\cdot)-y(\cdot)\|_\varkappa.$$

Therefore

$$\|Hx(\cdot)-Hy(\cdot)\|_\varkappa = \sup_{0\leqslant t\leqslant a}\Big|\exp\Big(-\varkappa\int_0^t L(s)\,ds\Big)\times$$

$$\times\int_0^t U(t,s)[h(s,x(s))-h(s,y(s))]\,ds\Big| \leqslant \frac{1}{\varkappa}\|x(\cdot)-y(\cdot)\|_\varkappa, \quad \text{q. e. d.}$$

We shall now prove the non-local version of theorem 5 of [6]. In the proof we use some properties of a resolvent of the non-linear operator. These properties are discussed in [5], p. 148.

THEOREM 12. *Let the function $\sigma(t,u)$ be continuous for $t\in\langle 0,a\rangle$ and $u\geqslant 0$. We assume that for every $u^0\geqslant 0$ the right maximal solution $\omega(t,u^0)$ of the equation $u'=\sigma(t,u)$ exists in the whole interval $\langle 0,a\rangle$. Suppose that $\sigma(t,u)$ increases in u. Assume that the operator function $U(t,s)$ satisfies* (15)-(17). *The function $g(t,x)$ is completely continuous; the function $h(t,x)$ is continuous and satisfies* (20) *with summable $L(t)$. Suppose that $f(t,x)=g(t,x)+h(t,x)$ satisfies the inequality $|f(t,x)|\leqslant\sigma(t,|x|)$ $(t\in\langle 0,a\rangle, x\in E)$. Take an arbitrary $x_0\in E$. Then the equation*

$$x(t)=U(t,0)x_0+\int_0^t U(t,s)f\big(s,x(s)\big)ds \tag{22}$$

has at least one solution belonging to $C_E\langle 0,a\rangle$.

Proof. Let us take a family of equations

$$x(t)=\lambda U(t,0)x_0+\lambda\int_0^t U(t,s)f\big(s,x(s)\big)ds \qquad (0\leqslant\lambda\leqslant 1). \tag{23}$$

Using the notation introduced previously we write (23) in the form

$$u=\lambda Gu+\lambda Hu \tag{24}$$

where $u=x(\cdot)$. From lemma 4 we deduce that G is completely continuous. By lemma 5 we find that H satisfies (21). Suppose that $\varkappa>1$. With the help of some theorems of Krasnoselskiĭ (see [5], p. 148) we conclude that the inverse $(I-\lambda H)^{-1}$ exists for $0\leqslant\lambda\leqslant 1$ and depends continuously on λ. It is thus seen that (24) may be written in an equivalent form as follows:

$$u=(I-\lambda H)^{-1}\lambda Gu. \tag{25}$$

Suppose that $u=x(\cdot)\in C_E\langle 0,a\rangle$ satisfies (25). Then $x(t)$ satisfies (23). From theorem 10 we find that $|x(t)|\leqslant\omega(t,\lambda|x_0|)\leqslant\omega(t,|x_0|)\leqslant\varrho=\max_{0\leqslant t\leqslant a}\omega(t,|x_0|)$. Therefore

$$(I-\lambda H)^{-1}\lambda Gu\neq u \quad \text{for} \quad \|u\|_\varkappa=\varrho+\varepsilon \qquad (\varepsilon>0;\ 0\leqslant\lambda\leqslant 1). \tag{26}$$

The transformation $T_\lambda=(I-\lambda H)^{-1}\lambda Gu$ is completely continuous and $T_0u\equiv\Theta$. From (26) and from the Leray-Schauder principle (see [10] and [5]) we find that there is a $u^0=x^0(\cdot)\in C_E\langle 0,a\rangle$ such that $u^0=(I-H)^{-1}Gu^0$. Therefore $x^0(t)$ satisfies (22) for $t\in\langle 0,a\rangle$.

Suppose now that E is a Hilbert space. Let A be a self-adjoint operator and suppose that

$$(Ax,x)\leqslant -(x,x) \quad \text{for} \quad x\in D[A]. \tag{27}$$

Then $R(\lambda,A)$ exists for $\lambda>-1$ and $|R(\lambda,A)|\leqslant 1/(\lambda+1)$. Denote by $U(t)$ the semi-group of linear, bounded operators generated by A. We introduce the following definition: the function $z(t)$ is said to satisfy the *local Hölders condition* in $(0,a)$ if for every $t\in(0,a\rangle$ there exist constants $\delta>0$, $\beta\in(0,1\rangle$, $K>0$ such that $|z(\bar t)-z(\bar{\bar t})|\leqslant K|\bar t-\bar{\bar t}|^\beta$ for $\bar t,\bar{\bar t}\in(t-\delta, t+\delta)$.

Applying theorem 3 of [7] one can prove the following lemma:

LEMMA 6. *Suppose that A is self-adjoint and satisfies* (27). *Let $z(t)$ be continuous in $\langle 0,a\rangle$ and let $z(t)$ satisfy the local Hölders condition. Take an arbitrary $x_0\in E$. Then the function $x(t)=U(t)x_0+\int_0^t U(t-s)z(s)ds$ satisfies the equation $x'=Ax+z(t)$ for $t\in(0,a\rangle$ and $x(0)=x_0$.*

Lemma 6 is a slight improvement of theorem 4 of [7]. From theorems 5 and 6 of [7] we deduce the following lemma:

LEMMA 7. *Suppose that A is self-adjoint and satisfies* (27). *Assume that $z(\cdot)\in C_E\langle 0,a\rangle$. Then*

$$\Big|\int_0^{\bar t}U(\bar t-s)z(s)ds-\int_0^{\bar{\bar t}}U(\bar{\bar t}-s)z(s)ds\Big|\leqslant K_1|\bar t-\bar{\bar t}|\big|\ln|\bar t-\bar{\bar t}|\big|\,\|z(\cdot)\| \tag{28}$$

K_1 being a suitable constant. If A^{-1} is completely continuous then the transformation

$$x(t)\to\int_0^t U(t-s)x(s)ds$$

is completely continuous when considered in the space $C_E\langle 0,a\rangle$.

We say that the function $f(t,x)$ satisfies the *local Hölders condition* if for every $(t,x)\in(0,a\rangle\times E$ there exist a neighbourhood $N(t,x)$ of (t,x) and a constant $M>0$, $\gamma\in(0,1\rangle$ such that $|f(\bar t,\bar x)-f(\bar{\bar t},\bar{\bar x})|\leqslant M[|\bar x-\bar{\bar x}|^\gamma+|\bar t-\bar{\bar t}|^\gamma]$ when $(\bar t,\bar x),(\bar{\bar t},\bar{\bar x})\in N(t,x)$.

LEMMA 8 (see [7], theorem 7). *Suppose that A is self-adjoint and satisfies* (27). *Assume that $f(t,x)$ is continuous and satisfies the local Hölders condition. Let $x(\cdot)\in C_E\langle 0,a\rangle$ and*

$$x(t)=U(t)x_0+\int_0^t U(t-s)f\big(s,x(s)\big)ds \quad \text{for} \quad t\in\langle 0,a\rangle. \tag{29}$$

Then $x'(t)=Ax(t)+f\big(t,x(t)\big)$ for $0<t\leqslant a$.

Proof. Write $y(t) = \int_0^t U(t-s)f(s, x(s))ds$. From (28) we find that

$$|y(\bar{t})-y(\bar{\bar{t}})| \leqslant K_1|\bar{t}-\bar{\bar{t}}|^{1/2}|\bar{t}-\bar{\bar{t}}|^{1/2}\big|\ln|\bar{t}-\bar{\bar{t}}|\big|\cdot \sup_{0\leqslant s\leqslant a}\big|f(s, x(s))\big|.$$

On the other hand $\lim_{\bar{t},\bar{\bar{t}}\to t}|\bar{t}-\bar{\bar{t}}|^{1/2}\big|\ln|\bar{t}-\bar{\bar{t}}|\big| = 0$. Therefore $y(t)$ satisfies the local Hölders condition. The function $z(\tau) = U(\tau)x_0$ satisfies the equation $z'(\tau) = AU(\tau)x_0$ for $\tau > 0$. Take an arbitrary $\tau\in(0, a\rangle$. Then $z'(\tau) = U(\tau-h)AU(h)x_0$ for $0 < h < \tau$. Take a fixed $h < \tau$. Then $|z'(\tau)| \leqslant |AU(h)x_0|$. Therefore $z(\tau)$ satisfies in $\langle h, a\rangle$ the Lipschitz condition. From the previous discussion we conclude that $x(t)$ satisfies the local Hölders condition. Hence the function $f(s, x(s))$ satisfies the local Hölders condition. The assertion of our lemma now follows from lemma 6.

We shall now prove a non-local generalization of theorem 7 of [7].

THEOREM 13. *Suppose that A is self-adjoint and satisfies* (27). *Let A^{-1} be completely continuous. Assume that $f(t, x)$ is continuous and satisfies the local Hölders condition. Suppose that $\sigma(t, u) \geqslant 0$ is continuous for $t\in\langle 0, a\rangle$ and $u \geqslant 0$. We assume that for every $u^0 \geqslant 0$ the right maximal solution $\omega(t, u^0)$ of the differential equation $u' = \sigma(t, u)$ exists in the whole interval $\langle 0, a\rangle$. Suppose that $|f(t, x)| \leqslant \sigma(t, |x|)$ $(t\in\langle 0, a\rangle; x\in E)$. Take an arbitrary $x_0\in E$. Then there exists at least one function $x(t)\in C_E\langle 0, a\rangle$ which satisfies the equation $x' = Ax+f(t, x)$ for $t\in(0, a\rangle$ and $x(0) = x_0$.*

Proof. Denote by F_μ the following transformation:

$$x(t) \to \mu U(t)x_0+\mu\int_0^t U(t-s)f(s, x(s))ds \qquad (0 \leqslant \mu \leqslant 1).$$

Observe that

$$|f(s, x(s))| \leqslant \sup_{0\leqslant t\leqslant a,\, 0\leqslant u\leqslant \|x(\cdot)\|} \sigma(t, u).$$

By lemma 7 we conclude that F_μ is completely continuous. Suppose that $F_\mu u = u$, $u = x(\cdot)$. Then $x(t) = \mu U(t)x_0+\mu\int_0^t U(t-s)f(s, x(s))ds$. From lemma 8 we find that $x'(t) = Ax(t)+\mu f(t, x(t))$ for $0 < t \leqslant a$, $x(0) = \mu x_0$. From theorem 4 we conclude that

$$|x(t)| \leqslant \omega(t, \mu|x_0|) \leqslant \varrho = \max_{0\leqslant t\leqslant a}\omega(t, |x_0|).$$

Therefore $F_\mu u \neq u$ if $\|u\| = \varrho+\varepsilon$ $(\varepsilon > 0)$. Hence F_1 is homotopic to zero on the sphere $\|u\| = \varrho+\varepsilon$. From the Leray-Schauder's principle we deduce that there is at least one $u^0 = x^0(\cdot)\in C_E\langle 0, a\rangle$ such that $F_1u^0 = u^0$. Now we apply lemma 8.

Remark. Write

$$Z = \mathop{E}_{x(\cdot)}\{|x(t)| \leqslant \omega(t, |x_0|),\ t\in\langle 0, a\rangle\}.$$

If $\sigma(t, u)$ increases in u then $F_1Z\subset Z$. Z is convex, bounded and closed in $C_E\langle 0, a\rangle$. In this case the existence of the solution of the equation $F_1v = v$ follows from Schauder's fixed-point theorem. The use of the Leray-Schauder method is superfluous.

§ 5. Let us formulate the assumptions given by Kato in [4] (see also [6]):

(K) For every $t\in\langle 0, a\rangle$ $A(t)$ is a closed and linear operator. The operators $A(t)$ are defined on a linear set D. D is dense in E. For $t\in\langle 0, a\rangle$ the inequality

$$|R(\lambda, A(t))| \leqslant 1/(\lambda+1) \qquad (\lambda > -1)$$

holds. The derivative

$$\frac{\partial}{\partial t}[A(t)A^{-1}(s)x] = C(t, s: x)$$

exists for every $x\in E$ and $t, s\in\langle 0, a\rangle$. $C(t, s: x)$ is continuous with respect to t for fixed s and x.

The assumption (K) implies the existence of the general solution $U(t, s)$ of the differential equation $x' = A(t)x$. $U(t, s)$ satisfies (15)-(17). If $x\in D$ then $x(t) = U(t, s)x$ is continuously differentiable and $x'(t) = A(t)x(t)$ for $0 \leqslant s \leqslant t \leqslant a$. $U(t, s)$ will be called the *Kato function* corresponding to $A(t)$.

We introduce the following assumption:

(P) We assume that for each fixed $\mu\in\langle\bar{\mu}, \bar{\bar{\mu}}\rangle$ $(\bar{\mu} < \bar{\bar{\mu}})$ the operator $A(t, \mu)$ satisfies the assumption (K). The set $D[A(t, \mu)]$ does not depend on t and μ, i. e. $D[A(t, \mu)] = D = \text{const}$. We suppose that for every $x\in E$, $t\in\langle 0, a\rangle$ and $\mu, \lambda\in\langle\bar{\mu}, \bar{\bar{\mu}}\rangle$ the limit

$$\lim_{h\to 0}\frac{A(t, \mu+h)A^{-1}(t, \lambda)x-A(t, \mu)A^{-1}(t, \lambda)x}{h} = B(t, \mu, \lambda)x$$

exists and $|B(t, \mu, \lambda)| \leqslant M$ $(M = \text{const} < +\infty)$ for $t\in\langle 0, a\rangle$. We assume that $B(t, \mu, \lambda)$ is strongly continuous in t.

Denote by $U(t, s: \lambda)$ the Kato function corresponding to $A(t, \lambda)$. We have the following theorem:

THEOREM 14. *Suppose that the assumption* (P) *holds. Take $x\in E$. Then $\lim_{h\to 0}U(t, s: \lambda+h)x = U(t, s: \lambda)x$ uniformly with regard to t for* *fixed s. If $x \in D$ then there exists the derivative $\partial U(t, s: \lambda)x/\partial \lambda$ and*

$$\frac{\partial U(t, s: \lambda)x}{\partial \lambda} = \int_s^t U(t, \tau; \lambda)B(\tau, \lambda, \lambda)A(\tau, \lambda)U(\tau, s; \lambda)x\,d\tau.$$

Proof. Write $\varphi_\mu(\lambda) = A(\tau, \lambda)A^{-1}(\tau, \mu)x$. Then $|d\varphi_\mu(\lambda)/d\lambda| \leqslant M|x|$. Therefore

$$\left|\frac{\varphi_\mu(\lambda+h)-\varphi_\mu(\lambda)}{h}\right| \leqslant M|x|. \tag{30}$$

On the other hand

$$A(t, \lambda+h)A^{-1}(t, \lambda)x = x+hB(t, \lambda, \lambda)x+\varepsilon(x, h) \tag{31}$$

and

$$\lim_{h\to 0}\varepsilon(x, h)/h = \Theta. \tag{32}$$

From (30) and (31) we get

$$|\varepsilon(x, h)/h| \leqslant 2M|x|. \tag{33}$$

Suppose that $x \in D$ and put $U(t, s: \lambda)x = y(t, \lambda)$, $z_h(t) = \big(y(t, \lambda+h)-y(t, \lambda)\big)/h$. We define $B(t, \lambda, \lambda) = B(t, \lambda)$. By (31) we find that

$$U(t, \tau: \lambda+h)z_h'(\tau) = U(t, \tau: \lambda+h)A(\tau, \lambda+h)z_h(\tau)+ \\ + U(t, \tau: \lambda+h)B(\tau, \lambda)y'(\tau, \lambda)+U(t, \tau: \lambda+h)\frac{\varepsilon\big(y'(\tau, \lambda), h\big)}{h}. \tag{34}$$

Remark that (see [6] formula (12))

$$\frac{d}{d\tau}[U(t, \tau: \lambda+h)x] = -U(t, \tau: \lambda+h)A(\tau, \lambda+h)x \quad \text{for} \quad x \in D. \tag{35}$$

On the other hand

$$\frac{d}{d\tau}[U(t, \tau: \lambda+h)z_h(\tau)] = U(t, \tau: \lambda+h)z_h'(\tau)+\frac{d}{ds}[U(t, s: \lambda+h)x]_{\substack{x=z_h(\tau)\\ s=\tau}}. \tag{36}$$

From (34), (35) and (36) it follows that

$$\frac{d}{d\tau}[U(t, \tau: \lambda+h)z_h(\tau)] = U(t, \tau: \lambda+h)B(\tau, \lambda)y'(\tau, \lambda)+U(t, \tau: \lambda+h)\frac{\varepsilon\big(y'(\tau, \lambda), h\big)}{h}. \tag{37}$$

The functions $y'(\tau, \lambda)$, $\varepsilon\big(y'(\tau, \lambda), h\big)$ are continuous with regard to τ. $B(\tau, \lambda)$ is strongly continuous with respect to τ. Hence the right-hand member of (37) is continuous with regard to τ. Observe now that $z_h(s) = \Theta$. From (37) we get by integrating

$$z_h(t) = \int_s^t U(t, \tau: \lambda+h)B(\tau, \lambda)y'(\tau, \lambda)d\tau+\int_s^t U(t, \tau: \lambda+h)\frac{\varepsilon\big(y'(\tau, \lambda), h\big)}{h}d\tau. \tag{38}$$

Therefore, because of (33)

$$|U(t, s: \lambda+h)x-U(t, s: \lambda)x| \leqslant 3MK|h|\cdot|t-s| \tag{39}$$

where $K = \sup_{s\leqslant t\leqslant a}|y'(t, \lambda)|$. From (39) we find that

$$\lim_{h\to 0}U(t, s: \lambda+h)x = U(t, s: \lambda)x \quad \text{for} \quad x \in D \tag{40}$$

uniformly with regard to t, for a fixed s. But D is dense and $|U(t, s: \lambda)| \leqslant 1$. Hence (40) holds for every $x \in E$. Because of (32) and (33) we get

$$\lim_{h\to 0}U(t, s: \lambda+h)\frac{\varepsilon\big(y'(s, \lambda), h\big)}{h} = \Theta, \tag{41}$$

$$\left|U(t, s: \lambda+h)\frac{\varepsilon\big(y'(s, \lambda), h\big)}{h}\right| \leqslant 2MK. \tag{42}$$

By (40) we find that

$$\lim_{h\to 0}U(t, \tau: \lambda+h)B(\tau, \lambda)y'(\tau, \lambda) = U(t, \tau: \lambda)B(\tau, \lambda)y'(\tau, \lambda). \tag{43}$$

Obviously

$$|U(t, \tau: \lambda+h)B(\tau, \lambda)y'(\tau, \lambda)| \leqslant MK. \tag{44}$$

By (38), (41)-(44), applying the Lebesgue theorem we conclude that the limit $\lim_{h\to 0}z_h(t)$ exists and

$$\lim_{h\to 0}z_h(t) = \int_s^t U(t, \tau: \lambda)B(\tau, \lambda)y'(\tau, \lambda)d\tau, \qquad \text{q. e. d.}$$

THEOREM 15. *Suppose that assumption* (P) *holds. We assume that the continuous function $f(t, y, \lambda)$ possesses the bounded and continuous derivative $\partial f/\partial\lambda$ and the bounded and continuous Fréchet differential $L(t, x, \lambda)$ taken with respect to y. Let $y(t, \lambda)$ be the continuously differentiable solution of the differential equation $x' = A(t, \lambda)x+f(t, x, \lambda)$. Assume that $y(0, \lambda)$ $=$ const $= x \in D$ for $\lambda \in \langle\underline{\mu}, \bar{\mu}\rangle$. Under our assumptions there exists a deriv-*

ative $\partial y/\partial\lambda$, continuous in t and satisfying the integral equation

$$\frac{\partial y(t,\lambda)}{\partial\lambda} = \int_0^t U(t,\tau:\lambda)B(\tau,\lambda,\lambda)\left[\frac{\partial y(s,\lambda)}{\partial s}\right]_{s=\tau} d\tau + \int_0^t U(t,\tau:\lambda)\left[\frac{\partial f(\tau,z,\lambda)}{\partial\lambda}\right]_{z=y(\tau,\lambda)} d\tau + \int_0^t U(t,\tau:\lambda)L\big(\tau,y(\tau,\lambda),\lambda\big)\frac{\partial y(\tau,\lambda)}{\partial\lambda}\,d\tau.$$

Proof. We shall prove that $\lim_{h\to 0} y(t,\lambda+h) = y(t,\lambda)$ uniformly in the interval $\langle 0,a\rangle$. Our hypotheses imply that

(45) $$f(t,x,\lambda)-f(t,y,\mu) = L(t,x,\lambda)(x-y)+\frac{\partial f(t,x,\lambda)}{\partial\lambda}(\lambda-\mu)+\delta(t,x,\lambda: x-y,\lambda-\mu)$$

where

(46) $$\lim_{y\to x,\mu\to\lambda}\frac{\delta(t,x,\lambda: x-y,\lambda-\mu)}{|x-y|+|\lambda-\mu|} = 0.$$

Furthermore there exist such constants R and F that $|L(t,x,\lambda)|\leqslant R$ $|\partial f(t,x,\lambda)/\partial\lambda|\leqslant F$ for $0\leqslant t\leqslant a$, $x\in E$, $\lambda\in\langle\underline{\mu},\overline{\mu}\rangle$. Now write $\eta(\tau,h,\lambda) = \delta\big(\tau,y(\tau,\lambda),\lambda: y(\tau,\lambda+h)-y(\tau,\lambda),h\big)$. Applying the mean value theorem one shows that

(47) $$|\eta(\tau,h,\lambda)|\leqslant 2[R|y(\tau,\lambda+h)-y(\tau,\lambda)|+|hF|].$$

One easily proves the following formula

(48) $$y(t,\lambda+h)-y(t,\lambda) = h\int_0^t U(t,\tau:\lambda+h)B(\tau,\lambda,\lambda)y'(\tau,\lambda)\,d\tau + \int_0^t U(t,\tau:\lambda+h)L\big(\tau,y(\tau,\lambda),\lambda\big)[y(\tau,\lambda+h)-y(\tau,\lambda)]\,d\tau + h\int_0^t U(t,\tau:\lambda+h)\left[\frac{\partial f\big(\tau,y(\tau,\mu),\lambda\big)}{\partial\lambda}\right]_{\mu=\lambda} d\tau + \int_0^t U(t,\tau:\lambda+h)\varepsilon\big(y'(\tau,\lambda),h\big)\,d\tau + \int_0^t U(t,\tau:\lambda+h)\eta(\tau,h,\lambda)\,d\tau$$

(the function ε is defined by (31)). We write $K = \sup_{0\leqslant t\leqslant a}|y'(t,\lambda)|$. By (33) we find that

(49) $$\big|\varepsilon\big(y'(\tau,\lambda),h\big)\big|\leqslant 2MK|h|.$$

Now let $Q = MK+F+Fa+2MK$ and $\varphi_h(t) = |y(t,\lambda+h)-y(t,\lambda)|$. By (48) and (49) we find that

(50) $$0\leqslant\varphi_h(t)\leqslant Q|h|+\int_0^t 3R\varphi_h(s)\,ds.$$

Applying lemma 3 we conclude that

(51) $$0\leqslant\varphi_h(t)\leqslant Q|h|\cdot\exp(3Rt).$$

Hence $\lim_{h\to 0}\varphi_h(t) = 0$ uniformly in $\langle 0,a\rangle$, q. e. d. By (46) we conclude therefore that

(52) $$\lim_{h\to 0}\eta(\tau,h,\lambda)/h = 0.$$

On the other hand it follows from (47), (51) that

(53) $$|\eta(\tau,h,\lambda)/h|\leqslant 2[RQ\exp(3Ra)+F].$$

Now write the integral equation

(54) $$z(t) = \int_0^t U(t,\tau:\lambda)B(\tau,\lambda,\lambda)y'(\tau,\lambda)\,d\tau + \int_0^t U(t,\tau:\lambda)L\big(\tau,y(\tau,\lambda),\lambda\big)z(\tau)\,d\tau + \int_0^t U(t,\tau:\lambda)\left[\frac{\partial f(\tau,x,\lambda)}{\partial\lambda}\right]_{x=y(\tau,\lambda)} d\tau$$

with an unknown function $z(t)$. Equation (54) has a unique solution $z(t)$. This solution may be constructed by means of the method of successive approximations. We write $z_h(t) = \big(y(t,\lambda+h)-y(t,\lambda)\big)/h$, $\varrho_h(t) = |z_h(t)-z(t)|$ and

$$\xi_h(t) = \left|\int_0^t [U(t,\tau:\lambda+h)-U(t,\tau:\lambda)]\left[B(\tau,\lambda,\lambda)y'(\tau,\lambda)+\frac{\partial f}{\partial\lambda}+L\big(\tau,y(\tau,\lambda),\lambda\big)z(\tau)\right]d\tau + \int_0^t U(t,\tau:\lambda+h)\frac{\varepsilon\big(y'(\tau,\lambda),h\big)}{h}\,d\tau + \int_0^t U(t,\tau:\lambda+h)\frac{\eta(\tau,h,\lambda)}{h}\,d\tau\right|.$$

By (49), (52) and (53) we conclude that

(55) $$\lim_{h\to 0}\xi_h(t) = 0 \quad\text{for the fixed } t\in\langle 0,a\rangle.$$

Obviously there is a C such that

(56) $|\xi_h(t)| \leqslant C$ for $t \in \langle 0, a\rangle$ and h sufficiently small.

By (48) and (54) we find that

$$0 \leqslant \varrho_h(t) \leqslant \xi_h(t) + R\int_0^t \varrho_h(s)\,ds. \tag{57}$$

Therefore

$$0 \leqslant \int_0^t \varrho_h(s)\,ds \leqslant \exp(Rt)\cdot \int_0^t \xi_h(s)\cdot\exp(-Rs)\,ds. \tag{58}$$

From (55)-(58) it follows that $\lim_{h\to 0} \varrho_h(t) = 0$, q. e. d.

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