On Newton's method of approximation

by A. Sharma (Lucknow)*

1. Recently J. Mikusiński [1] has shown that if \( r_0, r_1, r_2, \ldots \) denote the successive convergents for the continued fraction for \( \sqrt{C} \) (\( C > 0 \) and rational and not a perfect square) and if \( x_0, x_1, x_2, \ldots (x_0 = r_0) \) are the successive approximations to \( \sqrt{C} \) by Newton's formula, viz.

\[
x_{n+1} = \frac{1}{2}(x_n + C/x_n)
\]

then

\[
x_n = r_{2n-1}
\]

if and only if

\[
C = \frac{a^2 + 2a}{b}
\]

when \( a \) and \( b \) are integers. For numbers \( C \) not of this kind, he has proved the following results:

(I) If \( x_n = (p-1)\)-th convergent of \( \sqrt{C} \) when \( p \) is the number of terms in a period (not necessarily primitive) the number \( x_{n+1} \) is equal to the \((2p-1)\)-th convergent of \( \sqrt{C} \).

(II) If the primitive period of \( \sqrt{C} \) has \( 2k \) terms then all the iterations that we obtain by Newton's formula on beginning with the \((k-1)\)-th convergent of \( \sqrt{C} \) are also convergents of \( \sqrt{C} \).

Now we know that if \( x_n, x_{n+1} \) are two numbers such that \( x_n < a < x_{n+1} \), \( f(x) \) is continuous and monotonic in \((x_n, x_{n+1})\) and \( a \) is a root of the equation \( f(x) = 0 \), then the number \( x_{n+2} \) given by

\[
x_{n+2} = x_n - \frac{f(x_n)(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}
\]

is also an approximation to the number \( a \). Using this for a quadratic equation \( x^2 + Ax + B = 0 \), we show that if \( x_n < \sqrt{C} + 1 < x_{n+1} \) where

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\( \sqrt{C+1} \) is a real root of this equation, then

\[
\alpha_{n+2} = \frac{\alpha_n \alpha_{n+1} + B}{\alpha_n + \alpha_{n+1} + A}
\]

gives in some cases a better approximation than \( \alpha_n \) and \( \alpha_{n+1} \).

The object of this note is to extend the results (I) and (II) to a class of numbers \( \sqrt{C+1} \) which have a continued fraction development of the type

\[
a(a_1, a_2, \ldots, a_\infty),
\]

the elements in the braces denoting a period. We also obtain similar results for formula (5) by taking \( \alpha_n \) and \( \alpha_{n+1} \) to be two successive convergents of (6), which as we know are approximations from above and below to the irrational number.

2. Consider the number \( \frac{1}{2}(\sqrt{55} - 1) \), which is a root of the equation

\[3a^2 + a = 7\]

and which has the continued fraction

\[1, 2, 1, 2, 1, 2, \ldots\]

The corresponding Newton's formula gives

\[
\alpha_{n+1} = \frac{3\alpha_n^2 + 7}{6\alpha_n + 1}.
\]

Taking \( \alpha_0 = \frac{1}{2} \), we get \( \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{7}{5} \). Again formula (5) becomes

\[
\alpha_{n+2} = \frac{3\alpha_n^2 + 7}{3(\alpha_n + \alpha_{n+1}) + 1}.
\]

Taking \( \alpha_0 = \frac{1}{2}, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2} \), we get \( \alpha_3 = \frac{5}{3} \). Or again taking \( \alpha_0 = \frac{1}{2}, \alpha_1 = \frac{3}{2}, \alpha_2 = \frac{7}{5} \), we get \( \alpha_3 = \frac{11}{7} \) and so on. Again take \( \frac{1}{2}(\sqrt{5} + 2) \), which satisfies the equation \( 9a^2 - 12a - 1 = 0 \). It has the continued fraction

\[1, 2, 1, 1, 2, 1, 1, 2, \ldots\]

and its successive convergents are

\[1, 3, 7, 17, 41, 95, 220, 511, 1225, 2878, 6862, \ldots\]

By Newton's formula, in this case

\[
\alpha_{n+1} = \frac{9\alpha_n^2 + 1}{6(3\alpha_n - 2)}.
\]

Taking \( \alpha_0 = \frac{200}{105} \), we have \( \alpha_1 = \frac{900}{3000} \), and taking \( \alpha_0 = \frac{1}{2} \), we have \( \alpha_1 = \frac{1}{2} \). Similarly formula (5) now becomes

\[
\alpha_{n+1} = \frac{9\alpha_n^2 + 1}{9(\alpha_n + \alpha_{n+1}) - 12},
\]

from which on taking \( \alpha_0 = \frac{1}{2} \) and \( \alpha_1 = \frac{1}{2} \), we get \( \alpha_2 = \frac{17}{12} \).

We may now state the general results:

(I') If \( \alpha_n \) is the \((p-1)\)-th convergent of a given by (6) \( p \) being the number of terms in a period not necessarily primitive \( \), then the number \( \alpha_{n+1} \) given by

\[
\alpha_{n+1} = \frac{\alpha_n^2 + 7}{2(\alpha_n + \alpha_{n+1})^2 + 12}
\]

is equal to the \((2p-1)\)-th convergent of \( \alpha \), where \( \alpha \) is a root of the equation

\[a^2 + ax + B = 0.
\]

(II') If the number \( a \) has a period of length 2\( n \) and is given by

\[a = a(a_1, a_2, \ldots, a_{n-1}, b, a_{n-1}, \ldots, a_1, a),
\]

so that the period is symmetric except for the last term, then all iterations given by Newton's formula on taking \( \alpha_n = (i-1) \)-th convergent of \( a \) are also convergents of \( a \).

It may be remarked that an increase in the non-periodic part disturbs the rules very much, as is seen from the following examples:

Consider the number \( 4 - \sqrt{3} \), whose continued fraction is \( 2, 3, 1, 2 \) and successive convergents are

\[2, 7, 21, 65, 191, 577, 1762, 5247, 15751, 47252, 139807, 418099, \ldots\]

Newton's rule gives

\[
\alpha_{n+1} = \frac{\alpha_n^2 - 13}{2(\alpha_n + 4)},
\]

so that taking \( \alpha_0 = 2 \), we get \( \alpha_1 = \frac{3}{2} = r_1, \alpha_2 = \frac{13}{5} = r_2, \alpha_3 = \frac{21}{8} = r_3, \alpha_4 = \frac{347}{128} = r_4, \alpha_5 = \frac{7724}{2895} = r_5, \alpha_6 = \frac{47252}{17625} = r_6.

Taking \( \alpha_0 = \frac{1}{2} \), we get \( \alpha_1 = \frac{1}{2} = r_1, \alpha_2 = \frac{1}{2} = r_2, \alpha_3 = \frac{1}{2} = r_3, \alpha_4 = \frac{1}{2} = r_4, \alpha_5 = \frac{1}{2} = r_5, \alpha_6 = \frac{1}{2} = r_6.

On the other hand the number \( \frac{1}{2}(\sqrt{17} + \sqrt{3}) \), which satisfies the equation

\[3a^2 - 4ax + 22 = 0,
\]

has the continued fraction \( 1, 2, 3, 1, 2 \), and its successive convergents are

\[1, 3, 7, 17, 41, 95, 220, 511, 1225, 2878, 6862, \ldots\]
By Newton's rule
\[ x_{n+1} = \frac{13x_n^2 - 22}{2(x_n - 17)} \]

On putting successively the first 4 values of the above convergents we obtain
\[ \frac{9}{2}, \frac{23}{5}, \frac{113}{25}, \frac{423}{94}, \frac{18125}{4029}, \frac{445503}{100507}, \frac{181291125}{40299034} \]

none of which agrees with any of its convergents. It is the same with
\[ \frac{1}{17} + \frac{1}{17} \]

3. Proof of (I). Observe that \( \sqrt{C} + l = f(a_{k+1} + a - a) = f(m_k + \sqrt{C}) \) where \( m_k = a_{k+1} + l - a \)
\[ \frac{(m_k + \sqrt{C})P_{k+1} + P_k}{(m_k + \sqrt{C})Q_{k+1} + Q_k} \]

and let \( P_k/Q_k \) denote the \( k \)-th convergent. Then
\[ \sqrt{C} + l = f(a_{k+1} + a - a) = f(m_k + \sqrt{C}) \]

which, on simplifying and since \( \sqrt{C} \) is irrational, gives the following:
\[ P_k = (l + m_k)Q_{k+1} + Q_k, \quad l(m_kQ_{k+1} + Q_k) = m_kP_k + P_{k+1} \]

We now have
\[ \frac{P_k + a}{Q_k} = f(q_k - a) \]

Again, eliminating \( m_k \) with the help of (8), we have
\[ \frac{P_k + a}{Q_k} = \frac{P_k + (C - a)Q_k}{2Q_kP_k} \]

Now \( \sqrt{C} + l \), being a root of \( x^2 + Ax + B = 0 \), gives
\[ B = \overline{C}, \quad A = -2l, \]

and putting \( x_n = P_n/Q_n \), we have by (7)
\[ x_{n+1} = \frac{P_k + a}{Q_k} \]

which reduces to (9) on using (10), which proves the result (I).

It is easy to show in an analogous way the following theorem for formula (5):

1. In (5) we take for \( \frac{x}{x} \) the \((p-2)\)-th convergent and for \( x \), the \((p-1)\)-th convergent for \( \sqrt{C} + l \) having a continued fraction given by (6), then \( x_n \) is the \((2p-2)\)-th convergent.

4. Proof of (II'). Take \( a \) to be given by
\[ a = a(a_1, a_2, \ldots, a_{n-1}, b, a_{n-1}, \ldots, a_1, a), \]

so that the period is symmetric except for the last term, and put
\[ f(x) = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{b + x} + \frac{1}{Q_{n-1}Q_n}x \]

\[ g(x) = \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}} + \cdots + \frac{1}{b + x} + \frac{1}{Q_{n-1}Q_n}x \]

Then the number \( a \) given by the continued fraction satisfies the equation
\[ x = f(g[1/(x + d - a)]) \]

Simplifying and putting
\[ P_{n-1} + P_{n-2}Q_{n-1} = a_1, \quad P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1} = y_1, \]

we get from (12),
\[ x^2 + x[\delta_1(d - 2a) + \beta_1 - a_1] = (d - 2a)a_1 + y_1. \]

It is easy to see that \( \beta_1 = a_1. \) Newton's formula now gives
\[ \delta_1 = \delta_1 \frac{a_1^2 + (d - 2a)a_1 + y_1}{2a_1(d - 2a)} \]

Also
\[ P_{n-1}/Q_{n-1} = f(g[9]) = a_1/\delta_1. \]

It then remains to show that the right side of (15) = \( a_1/\delta_1. \) Simplifying the right side of (15) with the help of (13), we must show that
\[ Q_{n-1}(P_{n-1} + P_{n-2}Q_{n-2} - 2P_{n-1}Q_{n-2}) = 0, \]

and this is easily verified if we recall that \( a_1 = \beta_1. \) This proves (II').

5. A theorem analogous to (II') for formula (5) does not seem to hold true, as the following example shows:
\[ \sqrt{19} = 4(2, 1, 3, 1, 2, 8) \]
and the successive convergents are
\[ \frac{4}{3}, \frac{11}{8}, \frac{13}{10}, \frac{170}{121}, \frac{433}{322}, \frac{1631}{1210}, \ldots \]
Taking \( a_3 = \frac{5}{3}, a_4 = \frac{8}{5} \), we have \( a_5 = \frac{23}{14} \), which lies between \( \frac{170}{121} \) and \( \frac{433}{322} \). But on taking \( a_5 = \frac{433}{322}, a_6 = \frac{1631}{1210} \), we get \( a_7 = \frac{5974}{4157} \). In the first case we get what is called by Weber [2] a Nebenbruch, and it appears that in order to obtain similar relations between Newton's formula and formula (5) and the successive convergents one must take into consideration the Nebenbrüche also. This is clear on examining the example of \( \sqrt{59} \) and \( \sqrt{13} \) considered by Mikusiński. To this problem we propose to return later.

References

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Interprétation géométrique des conditions d’intégrabilité
d’un système d’équations aux différentielles totales

par J. Szarski et T. Ważewski (Kraków)

Considérons un système d’équations aux différentielles totales

(1) \( dx^i = P^i(x, y, z^1, \ldots, z^n)dx + Q^i(x, y, z^1, \ldots, z^n)dy \)

\( (i = 1, 2, \ldots, n) \),

où les fonctions \( P^i(x, y, z^1, \ldots, z^n) \) et \( Q^i(x, y, z^1, \ldots, z^n) \) sont de classe \( C^r \) dans un domaine \( \Omega \). Le système (1) est dit complètement intégrable dans \( \Omega \) lorsque on a dans \( \Omega \)

(2) \( Q_x^i - P_y^i + \sum_{j=1}^{n}(Q_j^i P^j - P_j^i Q^j) = 0 \)

\( (i = 1, 2, \ldots, n) \).

Nous nous proposons de donner une interprétation géométrique des premiers membres des identités (2). L'idée de cette interprétation est la suivante.

Le système (1) définit en chaque point \((ξ, η, ξ^1, \ldots, ξ^n)\) du domaine \(\Omega\) un plan à deux dimensions

(3) \( dx^i - ξ^i = P^i(ξ, η, ξ^1, \ldots, ξ^n)(x - ξ) + Q^i(ξ, η, ξ^1, \ldots, ξ^n)(y - η) \)

\( (i = 1, 2, \ldots, n) \).

Soit \((x_1, y_1, z_1, \ldots, z_n)\) un point du domaine \(\Omega\) et considérons la surface

cylindrique à \(n+1\) dimensions dont les équations paramétriques sont

(4) \( x = a_i + r \cos Ψ, \quad y = y_0 + r \sin Ψ, \quad z^i = h^i \) \( (i = 1, 2, \ldots, n) \),
où \( r > 0 \) est fixé et suffisamment petit et \( \Psi, h_1, \ldots, h_n \) sont des paramètres. Définissons par \( Σ_r \) la partie de la surface (4) contenue dans \(\Omega\) et soit

(5) \( ξ = a_i + r \cos Ψ, \quad η = y_0 + r \sin Ψ, \quad ξ^i = h^i \) \( (i = 1, 2, \ldots, n) \)

un point quelconque appartenant à \( Σ_r \). Le plan (3) passant par le point

(6) coupe la surface (4) le long de la courbe dont l'équation paramétrique,