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# On Newton's method of approximation

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**1.** Recently J. Mikusiński [1] has shown that if $r_0, r_1, r_2, \ldots$ denote the successive convergents for the continued fraction for $\sqrt{C}$ ($C > 0$ and rational and not a perfect square) and if $x_0, x_1, x_2, \ldots$ ($x_0 = r_0$) are the successive approximations to $\sqrt{C}$ by Newton's formula, viz.

$$x_{n+1} = \tfrac{1}{2}(x_n + C/x_n) \tag{1}$$

then

$$x_n = r_{2n-1} \tag{2}$$

if and only if

$$C = a^2 + 2a/b \tag{3}$$

when $a$ and $b$ are integers. For numbers $C$ not of this kind, he has proved the following results:

(I) *If $x_n = (p-1)$-th convergent of $\sqrt{C}$ when $p$ is the number of terms in a period (not necessarily primitive) the number $x_{n+1}$ is equal to the $(2p-1)$-th convergent of $\sqrt{C}$.*

(II) *If the primitive period of $\sqrt{C}$ has $2k$ terms then all the iterations that we obtain by Newton's formula on beginning with the $(k-1)$-th convergent of $\sqrt{C}$ are also convergents of $\sqrt{C}$.*

Now we know that if $x_n, x_{n+1}$ are two numbers such that $x_n < a < x_{n+1}$, $f(x)$ is continuous and monotonic in $\langle x_n, x_{n+1}\rangle$ and $a$ is a root of the equation $f(x) = 0$, then the number $x_{n+2}$ given by

$$x_{n+2} = x_n - \frac{f(x_n)(x_{n+1}-x_n)}{f(x_{n+1})-f(x_n)} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1})-f(x_n)} \tag{4}$$

is also an approximation to the number $a$. Using this for a quadratic equation $x^2 + Ax + B = 0$, we show that if $x_n < \sqrt{C} + l < x_{n+1}$ where

* I am grateful to Professor Mikusiński for several valuable suggestions.

$\sqrt{C}+l$ is a real root of this equation, then

$$x_{n+2} = \frac{x_n x_{n+1} - B}{x_n + x_{n+1} + A} \tag{5}$$

gives in some cases a better approximation than $x_n$ and $x_{n+1}$.

The object of this note is to extend the results (I) and (II) to a class of numbers $\sqrt{C}+l$ which have a continued fraction development of the type

$$a(a_1, a_2, \ldots, a_p), \tag{6}$$

the elements in the braces denoting a period. We also obtain similar results for formula (5) by taking $x_0$ and $x_1$ to be two successive convergents of (6), which as we know are approximations from above and below to the irrational number.

**2.** Consider the number $\frac{1}{6}(\sqrt{85}-1)$, which is a root of the equation $3x^2+x = 7$ and which has the continued fraction $1(2, 1, 2)$. The successive convergents are

$$1, \tfrac{3}{2}, \tfrac{4}{3}, \tfrac{11}{8}, \tfrac{26}{19}, \tfrac{37}{27}, \tfrac{100}{73}, \tfrac{237}{173}, \tfrac{337}{246}, \tfrac{911}{665}, \tfrac{2159}{1576}, \tfrac{3070}{2241}, \ldots$$

The corresponding Newton's formula gives

$$x_{n+1} = \frac{3x_n^2+7}{6x_n+1}.$$

Taking $x_0 = \frac{4}{3}$, we get $x_1 = \frac{37}{27}$, $x_2 = \frac{3070}{2241}$, ... Again formula (5) becomes

$$x'_{n+2} = \frac{3x'_n x'_{n+1}+7}{3(x'_n+x'_{n+1})+1}.$$

Taking $x'_0 = \frac{3}{2}$, $x'_1 = \frac{4}{3}$, we get $x'_2 = \frac{26}{19}$. Or again taking $x'_2 = \frac{26}{19}$, $x'_3 = \frac{37}{27}$, we get $x'_4 = \frac{2159}{1576}$ and so on. Again take $\frac{1}{3}(\sqrt{5}+2)$, which satisfies the equation $9x^2-12x-1 = 0$. It has the continued fraction

$$1(2, 2, 2, 1, 12, 1)$$

and its successive convergents are

$$1, \tfrac{3}{2}, \tfrac{7}{5}, \tfrac{17}{12}, \tfrac{24}{17}, \tfrac{305}{216}, \tfrac{329}{233}, \tfrac{963}{682}, \tfrac{2255}{1597}, \tfrac{5473}{3876}, \tfrac{7728}{5473}, \tfrac{98209}{69552}, \ldots$$

By Newton's formula, in this case

$$x_{n+1} = \frac{9x_n^2+1}{6(3x_n-2)};$$



taking $x_0 = \frac{305}{216}$ we have $x_1 = \frac{98209}{69552}$, and taking $x_0 = \frac{24}{17}$ we have $x_1 = \frac{5473}{3876}$. Similarly formula (5) now becomes

$$x'_{n+2} = \frac{9x'_n x'_{n+1}+1}{9(x'_n+x'_{n+1})-12},$$

from which on taking $x'_0 = \frac{24}{17}$ and $x'_1 = \frac{305}{216}$, we get $x'_2 = \frac{7728}{5473}$.

We may now state the general results:

(I′) *If $x_n$ is the $(p-1)$-th convergent of $a$ given by* (6) *($p$ being the number of terms in a period not necessarily primitive) then the number $x_{n+1}$ given by*

$$x_{n+1} = \frac{x_n^2-B}{2x_n+A} \tag{7}$$

*is equal to the $(2p-1)$-th convergent of $a$, where $a$ is a root of the equation $x^2+Ax+B = 0$.*

(II′) *If the number $a$ has a period of length $2i$ and is given by*

$$a = a(a_1, a_2, \ldots, a_{i-1}, b, a_{i-1}, \ldots, a_1, d),$$

*so that the period is symmetric except for the last term, then all iterations given by Newton's formula on taking $x_n = (i-1)$-th convergent of $a$ are also convergents of $a$.*

It may be remarked that an increase in the non-periodic part disturbs the rules very much, as is seen from the following examples:

Consider the number $4-\sqrt{3}$, whose continued fraction is 2, 3 (1, 2) and successive convergents are

$$2, \tfrac{7}{3}, \tfrac{9}{4}, \tfrac{25}{11}, \tfrac{34}{15}, \tfrac{93}{41}, \tfrac{127}{56}, \tfrac{347}{153}, \tfrac{474}{209}, \tfrac{1295}{571}, \tfrac{1769}{780}, \ldots$$

Newton's rule gives

$$x_{n+1} = \frac{x_n^2-13}{2(x_n-4)},$$

so that taking $x_0 = 2$, we get $x_1 = \frac{9}{4} = r_2$, $x_2 = \frac{127}{56} = r_6$, $x_3 = \frac{24639}{10864} = r_{14}$. Taking $x_0 = \frac{7}{3} = r_1$ we get $x_1 = \frac{34}{15} = r_4$, $x_2 = \frac{1769}{780} = r_{10}$, $x_3 = \frac{4779839}{2107560} = r_{22}$.

On the other hand the number $\frac{1}{13}(17+\sqrt{3})$, which satisfies the equation $3x^2-34x+22 = 0$, has the continued fraction $1, 2, 3(1, 2)$, and its successive convergents are

$$1, \tfrac{3}{2}, \tfrac{10}{7}, \tfrac{13}{9}, \tfrac{36}{25}, \tfrac{49}{34}, \tfrac{134}{93}, \tfrac{183}{127}, \tfrac{500}{347}, \tfrac{683}{474}, \ldots$$

By Newton's rule

$$x_{n+1} = \frac{13x_n^2-22}{2(13x_n-17)}.$$

On putting successively the first 4 values of the above convergents we obtain

$$\tfrac{9}{8},\ \tfrac{29}{20},\ \tfrac{111}{77},\ \tfrac{415}{288},$$

none of which agrees with any of its convergents. It is the same with $\frac{1}{13}(17-\sqrt{3}) = 1, 5\,(1, 2)$.

**3. Proof of (I′).** Observe that $a = \sqrt{C}+l$ where $C$ and $l$ are rational ($C>0$) and $C$ is not a perfect square. Put

$$f(x) = a+\frac{1|}{|a_1}+\frac{1|}{|a_2}+\ldots+\frac{1|}{|a_i}+\frac{1}{x} = \frac{xP_i+P_{i-1}}{xQ_i+Q_{i-1}}$$

and let $P_k/Q_k$ denote the $k$-th convergent. Then

$$\sqrt{C}+l = f(a_{i+1}+a-a) = f(m_i+\sqrt{C}) \quad \text{where} \quad m_i = a_{i+1}+l-a$$
$$= \frac{(m_i+\sqrt{C})P_i+P_{i-1}}{(m_i+\sqrt{C})Q_i+Q_{i-1}},$$

which, on simplifying and since $\sqrt{C}$ is irrational, gives the following:

(8) $$P_i = (l+m_i)Q_i+Q_{i-1}, \qquad l(m_iQ_i+Q_{i-1})+CQ_i = m_iP_i+P_{i-1}.$$

We now have

$$\frac{P_{2i+1}}{Q_{2i+1}} = f\left(a_{i+1}-a+\frac{P_i}{Q_i}\right) = \frac{(m_iP_i+P_{i-1})Q_i-lP_iQ_i+P_i^2}{2(P_i-lQ_i)Q_i}.$$

Again, eliminating $m_i$ with the help of (8), we have

(9) $$\frac{P_{2i+1}}{Q_{2i+1}} = \frac{P_i^2+(C-l^2)Q_i^2}{2Q_i(P_i-lQ_i)}.$$

Now $\sqrt{C}+l$, being a root of $x^2+Ax+B=0$, gives

(10) $$B = l^2-C, \quad A = -2l,$$

and putting $x_n = P_i/Q_i$, we have by (7)

$$x_{n+1} = \frac{P_i^2-BQ_i^2}{Q_i(2P_i+AQ_i)},$$

which reduces to (9) on using (10), which proves the result (I′).



It is easy to show in an analogous way the following theorem for formula (5):

(Ia) *If in (5) we take for $x_0$ the $(p-2)$-th convergent and for $x_1$ the $(p-1)$-th convergent for $\sqrt{C}+l$ having a continued fraction given by (6), then $x_2$ is the $(2p-2)$-th convergent.*

**4. Proof of (II′).** Take $a$ to be given by

(11) $$a = a(a_1, a_2, \ldots, a_{i-1}, b, a_{i-1}, \ldots, a_1, d),$$

so that the period is symmetric except for the last term, and put

$$f(x) = a+\frac{1|}{|a_1}+\ldots+\frac{1|}{|a_{i-1}}+\frac{1}{b+x} = \frac{P_i+P_{i-1}x}{Q_i+Q_{i-1}x},$$
$$g(x) = \frac{1|}{|a_{i-1}}+\frac{1|}{|a_{i-2}}+\ldots+\frac{1|}{|a_1+x} = \frac{Q_{i-2}+(P_{i-2}-aQ_{i-2})x}{Q_{i-1}+(P_{i-1}-aQ_{i-1})x}.$$

Then the number $a$ given by the continued fraction satisfies the equation

(12) $$x = f[g(1/(x+d-a))].$$

Simplifying and putting

(13) $$P_iQ_{i-1}+P_{i-1}Q_{i-2} = a_i, \quad P_{i-1}(P_i+P_{i-2}) = \gamma_i,$$
$$P_{i-1}Q_i+Q_{i-1}P_{i-2} = \beta_i, \quad Q_{i-1}(Q_i+Q_{i-2}) = \delta_i$$

we get from (12),

(14) $$x^2+x[\delta_i(d-2a)+\beta_i-a_i] = (d-2a)a_i+\gamma_i.$$

It is easy to see that $\beta_i = a_i$. Newton's formula now gives

(15) $$x_1 = \frac{\delta_ix_0^2+(d-2a)a_i+\gamma_i}{2\delta_ix_0+\delta_i(d-2a)} = \frac{\delta_iP_{i-1}^2+Q_{i-1}^2[(d-2a)a_i+\gamma_i]}{\delta_iQ_{i-1}[2P_{i-1}+(d-2a)Q_{i-1}]}.$$

Also

$$P_{2i-1}/Q_{2i-1} = f[g(0)] = a_i/\delta_i.$$

It then remains to show that the right side of (15) $= a_i/\delta_i$. Simplifying the right side of (15) with the help of (13), we must show that

$$Q_{i-1}(P_i+P_{i-2})+P_{i-1}(Q_i+Q_{i-2})-2(P_iQ_{i-1}+P_{i-1}Q_{i-2}) = 0,$$

and this is easily verified if we recall that $a_i = \beta_i$. This proves (II′).

**5.** A theorem analogous to (II′) for formula (5) does not seem to hold true, as the following example shows:

$$\sqrt{19} = 4(2, 1, 3, 1, 2, 8)$$

and the successive convergents are

$$4,\ \frac{9}{2},\ \frac{13}{3},\ \frac{48}{11},\ \frac{61}{14},\ \frac{170}{39},\ \frac{1421}{326},\ \frac{3012}{691},\ \frac{4433}{1017},\ \frac{16\,311}{3742},\ \frac{20\,744}{4759},\ \ldots$$

Taking $x_0' = \frac{9}{2}$, $x_1' = \frac{13}{3}$, we have $x_2' = \frac{231}{53}$, which lies between $\frac{170}{39}$ and $\frac{1421}{326}$. But on taking $x_0' = \frac{61}{14}$, $x_1' = \frac{170}{39}$, we get $x_2' = \frac{20\,744}{4759}$. In the first case we get what is called by Weber [2] a *Nebenbruch*, and it appears that in order to obtain similar relations between Newton's formula and formula (5) and the successive convergents one must take into consideration the *Nebenbrüche* also. This is clear on examining the example of $\sqrt{89}$ and $\sqrt{13}$ considered by Mikusiński. To this problem we propose to return later.

**References**

[1] J. Mikusiński, *Sur la méthode d'approximation de Newton*, Ann. Polon. Math. 1 (1954), p. 184-194.

[2] H. Weber, *Lehrbuch der Algebra I*, Braunschweig 1898, p. 404.

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# Interprétation géométrique des conditions d'intégrabilité d'un système d'équations aux différentielles totales

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Considérons un système d'équations aux différentielles totales

$$(1)\qquad dz^i = P^i(x, y, z^1, \ldots, z^n)\,dx + Q^i(x, y, z^1, \ldots, z^n)\,dy \qquad (i = 1, 2, \ldots, n),$$

où les fonctions $P^i(x, y, z^1, \ldots, z^n)$ et $Q^i(x, y, z^1, \ldots, z^n)$ sont de classe $C^1$ dans un domaine $\Omega$. Le système (1) est dit *complètement intégrable dans* $\Omega$, lorsqu'on a dans $\Omega$

$$(2)\qquad Q_x^i - P_y^i + \sum_{j=1}^{n} (Q_{z^j}^i P^j - P_{z^j}^i Q^j) \equiv 0 \qquad (i = 1, 2, \ldots, n).$$

Nous nous proposons de donner une interprétation géométrique des premiers membres des identités (2). L'idée de cette interprétation est la suivante.

Le système (1) définit en chaque point $(\xi, \eta, \zeta^1, \ldots, \zeta^n)$ du domaine $\Omega$ un plan à deux dimensions

$$(3)\qquad z^i - \zeta^i = P^i(\xi, \eta, \zeta^1, \ldots, \zeta^n)(x-\xi) + Q^i(\xi, \eta, \zeta^1, \ldots, \zeta^n)(y-\eta) \qquad (i = 1, 2, \ldots, n).$$

Soit $(x_0, y_0, z_0^1, \ldots, z_0^n)$ un point du domaine $\Omega$ et considérons la surface cylindrique à $n+1$ dimensions dont les équations paramétriques sont

$$(4)\qquad x = x_0 + r\cos\psi, \quad y = y_0 + r\sin\psi, \quad z^i = h^i \quad (i = 1, 2, \ldots, n),$$

où $r > 0$ est fixé et suffisamment petit et $\psi, h^1, \ldots, h^n$ sont des paramètres. Désignons par $\Sigma_r$ la partie de la surface (4) contenue dans $\Omega$ et soit

$$(5)\qquad \xi = x_0 + r\cos\varphi, \quad \eta = y_0 + r\sin\varphi, \quad \zeta^i = h^i \quad (i = 1, 2, \ldots, n)$$

un point quelconque appartenant à $\Sigma_r$. Le plan (3) passant par le point (5) coupe la surface (4) le long de la courbe dont l'équation paramétrique,