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On Simpson's formula of cubature

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Simpson's formula for a function of 2 variables [1] is given by

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} F(x, y)\,dx\,dy = \frac{16F_{00}+4(F_{\pm 10}+F_{\pm 01})+F_{\pm 11}}{36}+R,$$

where $R = \frac{-1}{2880}(D_\xi^4+D_\eta^4+\frac{1}{2880}D_\xi^4 D_\eta^4)$. This formula requires 9 values of $F(x, y)$ placed in a square: F_{00} is the value at the centre of the square, $F_{\pm 11}$ is the sum of the values at the corners and $F_{\pm 10}+F_{\pm 01}$ is the sum of the remaining values. The aim of this paper is to investigate for Simpson's formula problems analogous to those proposed by Gołąb [2] in the case of quadrature.

Let $f(x, y)$ be a function of 2 variables and let

(A) $$f(x, y) = \sum_{\nu,\mu=p,q,r,s\ldots}^{\infty} a_{\nu\mu}x^\nu y^\mu$$

where $0 < p < q < r < s\ldots$, $a_{\nu\mu} \neq 0$ $(\nu, \mu = p, q, r, s)$. Let us write

(1) $$P(\lambda_0, \lambda_1, \lambda_2; h, k) = hk[\lambda_0 f(\tfrac{1}{2}h, \tfrac{1}{2}k)+\lambda_1\{f(\tfrac{1}{2}h, 0)+f(h, \tfrac{1}{2}k)+ +f(\tfrac{1}{2}h, k)+f(0, \tfrac{1}{2}k)\}+\lambda_2\{f(0, 0)+f(h, 0)+f(h, k)+f(0, k)\}].$$

We first propose the problem of determining $\lambda_0, \lambda_1, \lambda_2$ in such a way that

(2) $$R(h, k) = \int_0^h\int_0^k f(x, y)\,dx\,dy - P(\lambda_0, \lambda_1, \lambda_2; h, k)$$

is of the smallest order possible.

Making the coefficients of $h^p k^p$, $h^p k^q$, $h^q k^p$, $h^q k^q$ in (2) equal to zero, we have

(3) $$\frac{1}{2^{2p}}\lambda_0+\frac{2}{2^p}\lambda_1+\lambda_2 = \frac{1}{(p+1)(p+1)},$$

$$(4)\qquad \frac{1}{2^{p+q}}\lambda_0+\left(\frac{1}{2^p}+\frac{1}{2^q}\right)\lambda_1+\lambda_2=\frac{1}{(p+1)(q+1)},$$

$$(5)\qquad \frac{1}{2^{2q}}\lambda_0+\left(\frac{1}{2^q}+\frac{1}{2^q}\right)\lambda_1+\lambda_2=\frac{1}{(q+1)(q+1)}.$$

Equations (3), (4), and (5) are enough to determine $\lambda_0, \lambda_1, \lambda_2$ and give unique values for them, since the determinant

$$W=\begin{vmatrix}\frac{1}{2^{2p}} & \frac{2}{2^p} & 1\\ \frac{1}{2^{p+q}} & \frac{1}{2^p}+\frac{1}{2^q} & 1\\ \frac{1}{2^{2q}} & \frac{2}{2^q} & 1\end{vmatrix}=\left(\frac{1}{2^p}-\frac{1}{2^q}\right)^3>0.$$

In order to be able to increase the order of smallness of $R(h,k)$, we must have, besides (3), (4), and (5),

$$(6)\qquad \frac{1}{2^{r+p}}\lambda_0+\left(\frac{1}{2^r}+\frac{1}{2^p}\right)\lambda_1+\lambda_2=\frac{1}{(p+1)(r+1)},$$

$$(7)\qquad \frac{1}{2^{r+q}}\lambda_0+\left(\frac{1}{2^r}+\frac{1}{2^q}\right)\lambda_1+\lambda_2=\frac{1}{(r+1)(q+1)},$$

$$(8)\qquad \frac{1}{2^{2r}}\lambda_0+\frac{2}{2^r}\lambda_1+\lambda_2=\frac{1}{(r+1)^2}.$$

The condition of consistency then demands that we have

$$(9)\qquad A_{rp}\equiv\begin{vmatrix}\frac{1}{2^{2p}} & \frac{2}{2^p} & 1 & \frac{1}{(p+1)^2}\\ \frac{1}{2^{p+q}} & \frac{1}{2^p}+\frac{1}{2^q} & 1 & \frac{1}{(p+1)(q+1)}\\ \frac{1}{2^{2q}} & \frac{2}{2^q} & 1 & \frac{1}{(q+1)^2}\\ \frac{1}{2^{r+p}} & \frac{1}{2^r}+\frac{1}{2^p} & 1 & \frac{1}{(r+1)(p+1)}\end{vmatrix}=0$$

and $A_{rq}=0$, $A_{rr}=0$ where the suffixes indicate the elements of the last row. Thus A_{rq} is obtained from A_{rp} by replacing the last row by

$$\left(\frac{1}{2^{r+q}},\ \frac{1}{2^r}+\frac{1}{2^q},\ 1,\ \frac{1}{(r+1)(q+1)}\right).$$

Now (9) gives on simplification

$$\frac{1}{(p+1)}\left(\frac{1}{2^{2p}}-\frac{1}{2^{2q}}\right)\left[\frac{1}{2^p}\left(\frac{1}{r+1}-\frac{1}{q+1}\right)+\right.$$
$$\left.+\frac{1}{2^q}\left(\frac{1}{p+1}-\frac{1}{r+1}\right)+\frac{1}{2^r}\left(\frac{1}{q+1}-\frac{1}{p+1}\right)\right]=0$$

and since $q>p$, we have

$$(10)\qquad \frac{1}{2^p}\left(\frac{1}{r+1}-\frac{1}{q+1}\right)+\frac{1}{2^q}\left(\frac{1}{p+1}-\frac{1}{r+1}\right)+\frac{1}{2^r}\left(\frac{1}{q+1}-\frac{1}{p+1}\right)=0.$$

The above condition is the same as that treated by Gołąb, who has shown that the only values p, q, r can take when they satisfy (10) are $p=1$, $q=2$, $r=3$.

Similarly $A_{rq}=0$ also leads to the same equation (10). From $A_{rr}=0$ we get

$$\begin{vmatrix}\frac{1}{2^{2p}} & \frac{2}{2^p} & 1 & \frac{1}{(p+1)^2}\\ \frac{1}{2^{p+q}} & \left(\frac{1}{2^p}+\frac{1}{2^q}\right) & 1 & \frac{1}{(p+1)(q+1)}\\ \frac{1}{2^{2q}} & \frac{2}{2^q} & 1 & \frac{1}{(q+1)^2}\\ \frac{1}{2^{2r}} & \frac{2}{2^r} & 1 & \frac{1}{(r+1)^2}\end{vmatrix}=0,$$

i. e.

$$\begin{vmatrix}\frac{1}{2^p}\left(\frac{1}{2^p}-\frac{1}{2^q}\right) & \left(\frac{1}{2^p}-\frac{1}{2^q}\right) & \frac{1}{p+1}\left(\frac{1}{p+1}-\frac{1}{q+1}\right)\\ \left(\frac{1}{2^{2p}}-\frac{1}{2^{2q}}\right) & 2\left(\frac{1}{2^p}-\frac{1}{2^q}\right) & \left(\frac{1}{(p+1)^2}-\frac{1}{(q+1)^2}\right)\\ \left(\frac{1}{2^{2p}}-\frac{1}{2^{2r}}\right) & 2\left(\frac{1}{2^p}-\frac{1}{2^r}\right) & \left(\frac{1}{(p+1)^2}-\frac{1}{(r+1)^2}\right)\end{vmatrix}=0.$$

It is easy to see that the non-zero factor $(1/2^p-1/2^q)$ can be taken out from this determinant of the third order, which then reduces to

$$\left(\frac{1}{2^p}-\frac{1}{2^q}\right)^2\left(\frac{1}{(p+1)^2}-\frac{1}{(r+1)^2}\right)-\left(\frac{1}{2^p}-\frac{1}{2^r}\right)^2\left(\frac{1}{(p+1)^2}-\frac{1}{(q+1)^2}\right)+$$
$$+\frac{2}{p+1}\left(\frac{1}{p+1}-\frac{1}{q+1}\right)\left(\frac{1}{2^p}-\frac{1}{2^r}\right)\left(\frac{1}{2^q}-\frac{1}{2^r}\right)=0.$$

Making a further simplification we easily have

$$\frac{2}{(p+1)(q+1)}\left(\frac{1}{2^{p+r}}+\frac{1}{2^{q+r}}-\frac{1}{2^{p+q}}-\frac{1}{2^{2r}}\right)+\left[\frac{1}{p+1}\left(\frac{1}{2^q}-\frac{1}{2^r}\right)\right]^2+$$
$$+\left[\frac{1}{q+1}\left(\frac{1}{2^p}-\frac{1}{2^r}\right)\right]^2-\left[\frac{1}{r+1}\left(\frac{1}{2^p}-\frac{1}{2^q}\right)\right]^2=0.$$

In other words, we have

$$\left[\frac{1}{p+1}\left(\frac{1}{2^q}-\frac{1}{2^r}\right)-\frac{1}{q+1}\left(\frac{1}{2^p}-\frac{1}{2^r}\right)+\frac{1}{r+1}\left(\frac{1}{2^p}-\frac{1}{2^q}\right)\right]\times$$
$$\times\left[\frac{1}{p+1}\left(\frac{1}{2^q}-\frac{1}{2^r}\right)-\frac{1}{q+1}\left(\frac{1}{2^p}-\frac{1}{2^r}\right)-\frac{1}{r+1}\left(\frac{1}{2^p}-\frac{1}{2^q}\right)\right]=0.$$

We can show that the second factor cannot be zero. For on putting $p+1 = a$, $q-p=\beta$, and $r-q=\gamma$, we can write the second expression thus:

$$\frac{2}{a(a+\beta)(a+\beta+\gamma)}\cdot\frac{\beta(a+\beta+\gamma)(2^\gamma-1)-a(2a+2\beta+\gamma)2^\gamma(2^\beta-1)}{2^{a+\beta+\gamma}}$$

which is not zero since $2^\beta-1\geqslant\beta$. This can easily be seen from the fact that

$$\begin{aligned}\beta(a+\beta+\gamma)(2^\gamma-1)-a(2a+2\beta+\gamma)2^\gamma(2^\beta-1)\\ \leqslant\beta(a+\beta+\gamma)(2^\gamma-1)-a(2a+2\beta+\gamma)2^\gamma\beta\\ =-2^\gamma\beta[(2a-1)(a+\beta)+\gamma(a-1)]-\beta(a+\beta+\gamma)<0\end{aligned}$$

since $a\geqslant 2$, $\gamma\geqslant 1$. Thus $A_{rp}=A_{rq}=A_{rr}=0$ all lead to equation (10) and thus we have shown that in order to be able to increase the order of $R(h,k)$ further we must have $p=1$, $q=2$, $r=3$ and in this case

$$\lambda_0=\tfrac{16}{36},\quad \lambda_1=\tfrac{4}{36},\quad \lambda_2=\tfrac{1}{36}.$$

We shall call these values the Simpsonian values. We may thus ask ourselves the question: For what values of p, q are the values of $\lambda_0,\lambda_1,\lambda_2$ Simpsonian?

In order to reply to this question, we shall require to solve the equations

$$(11)\qquad \frac{16}{36}\cdot\frac{1}{2^{2p}}+\frac{4}{36}\cdot\frac{2}{2^p}+\frac{1}{36}=\frac{1}{(p+1)^2},$$

$$(12)\qquad \frac{16}{36}\cdot\frac{1}{2^{p+q}}+\frac{4}{36}\left(\frac{1}{2^p}+\frac{1}{2^q}\right)+\frac{1}{36}=\frac{1}{(p+1)(q+1)},$$

$$(13)\qquad \frac{16}{36}\cdot\frac{1}{2^{2q}}+\frac{4}{36}\cdot\frac{2}{2^q}+\frac{1}{36}=\frac{1}{(q+1)^2}.$$

Equations (11) and (13) mean that we may solve the diophantine equation

$$\frac{16}{36}\cdot\frac{1}{2^{2x}}+\frac{4}{36}\cdot\frac{2}{2^x}+\frac{1}{36}=\frac{1}{(x+1)^2},$$

which can be rewritten as

$$(14)\qquad 2^{2x}=\frac{8(2^x+2)(x+1)^2}{(5-x)(7+x)},$$

whence we see at once that $x<5$. On verification we see that $x=1,2,3$ are the only solutions of (14). Since (11), (12), (13) must be simultaneously satisfied and since the pairs

$$(15)\qquad p=1,\ q=2;\quad p=1,\ q=3;\quad p=2,\ q=3$$

satisfy (12), we see that the values of $\lambda_0,\lambda_1,\lambda_2$ are Simpsonian only when p,q belong to the set (15).

We have thus proved the following result:

Let $f(x,y)$ be a regular function having the development (A). *If we suppose that $R(h,k)$ is infinitely small of the greatest possible order, then the coefficients $\lambda_0,\lambda_1,\lambda_2$ have the form*

$$\lambda_0=\frac{\left(\frac{1}{p+1}-\frac{1}{q+1}\right)^2}{\left(\frac{1}{2^p}-\frac{1}{2^q}\right)^2},\qquad \lambda_1=\frac{\left(\frac{1}{p+1}-\frac{1}{q+1}\right)\left(\frac{1}{q+1}\cdot\frac{1}{2^p}-\frac{1}{p+1}\cdot\frac{1}{2^q}\right)}{\left(\frac{1}{2^p}-\frac{1}{2^q}\right)^2}$$

and $\lambda_2=\dfrac{1}{(p+1)^2}-\dfrac{1}{2^{2p}}\lambda_0-\dfrac{2}{2^p}\lambda_1$. *Thus the values of $\lambda_0,\lambda_1,\lambda_2$ depend on p and q and not on the coefficients $a_{\mu\nu}$.*

Further if the order of smallness of $R(h,k)$ is equal to $O(h^{s+1}k^{s+1})$ then $p=1, q=2, r=3$.

References

[1] J. F. Steffenson, *Interpolation*, New York 1950, p. 227.

[2] S. Gołąb, *Contribution à la formule Simpsonnienne de quadrature*, Ann. Polon. Math. 1 (1956), p. 166-175.

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