Nous obtenons

(18) $$|\lambda| < \frac{1}{b(I+D)},$$

(18′) $$|\lambda| < \frac{1}{2D(m+n)}$$

et en réduisant au commun dénominateur nous aurons, d'après l'inégalité (18′)

(18″) $$|\lambda|\{2D(m+n)[a(I+D)-\pi b]|\lambda|a(I+D)+\pi b+2D(m+n)\}<1.$$

Désignons par p_1 un nombre positif arbitraire vérifiant l'inégalité

$$p_1 < \min\left[\frac{1}{b(I+D)}, \frac{1}{2D(m+n)}\right].$$

Alors il en résulte l'inégalité suivante:

(19) $$|\lambda| < \frac{1}{2D(m+n)[a(I+D)-\pi b]p_1+a(I+D)+\pi b+2D(m+n)} = \frac{1}{p_2}.$$

En vertu des inégalités (18), (18′) et (19), si les valeurs absolues du paramètre λ satisfont à l'inégalité (16), nous obtenons la thèse du théorème 5.

Travaux cités

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[3] — *Badanie równań całkowych mocno-osobliwych metodą punktu niezmienniczego*, Biuletyn W. A. T. 18(1955), p. 3-85.

Reçu par la Rédaction le 10. 7. 1957

ANNALES
POLONICI MATHEMATICI
VI (1959)

On summability of double sequences (II)

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In paper [2](1) we have proved strong consistency theorems for the regular summability of double sequences. In the present paper analogous theorems are proved for the restricted summability of double sequences. After preliminary definitions we prove, in section 2, some theorems of Toeplitz type for restricted summability; some instances of these results were proved in an alternative version by C. N. Moore ([5], p. 92). The main result, the consistency theorem, is then proved in section 4. The method is based, as in [2], on the application of two norm spaces. The results may be extended without any alteration to the summability of sequences of multiplicity greater than two.

1. Definitions and preliminaries. By a convergent sequence we shall always mean sequences convergent in Pringsheim's sense; the limit of such a sequence $x = \{x_{ik}\}$ will be denoted by $\lim\limits_{i,k\to\infty} x_{ik}$. To recall the notion of restricted convergence let us denote by S_n the set of the pairs (i, k) of indices such that $n^{-1} \leqslant (i+1)(k+1)^{-1} \leqslant n$; the sequence $x = \{x_{ik}\}$ is called *restrictedly convergent* to $x_{..}$ (Moore [6], p. 567) if, given any $\varepsilon > 0$ and n, there is a N such that $i, k > N$ together with $(i, k) \in S_n$ implies $|x_{ik} - x_{..}| < \varepsilon$; $x_{..}$ is then called the *limit in the restricted sense* of the sequence x and will be denoted by $[\lim\limits_{i,k\to\infty}] x_{ik}$.

Let $A = (a_{ik\mu\nu})$ be a four dimensional matrix; given a sequence $x = \{x_{ik}\}$, let us consider the transforms

$$A_{ik}(x) = \sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu} x_{\mu\nu}.$$

If these series converge (in Pringsheim's sense) for every i, k and there exists $\lim\limits_{i,k\to\infty} A_{ik}(x) = A_{..}(x)$, then the sequence x is called *A-summable* to $A_{..}(x)$; the quantity $A_{..}(x)$ will be written interchangeably $A\text{-}\lim\limits_{i,k\to\infty} x_{ik}$.

(1) Let us correct the following misprints of this paper: p. 176, line 6 is to be read $||x_0 - x'||^* < \varepsilon$, $||x_0 - x''||^* < \varepsilon$, p. 176, line 18 is to be read $z_{ik} + \varepsilon_{ik} x_{ik}$.

If only $[\lim_{i,k\to\infty}]A_{ik}(x) = [A]..(x)$ exists, the sequence x is called *restrictedly A-summable* to $[A]..(x)$; $[A]..(x)$ will be denoted also by $A\text{-}[\lim_{i,k\to\infty}]x_{ik}$.

The matrix A, as giving rise to the sequence $\{A_{ik}(x)\}$ of transforms, is called the *method of summability A*.

By $\boldsymbol{BC}$ and $[\boldsymbol{BC}]$ we shall denote the class of all bounded(²) convergent and all bounded restrictedly convergent sequences respectively; these are Banach spaces if addition and multiplication by scalars is defined in the usual way, the norm being $\|x\| = \sup_{i,k=0,1,\ldots} |x_{ik}|$. $\boldsymbol{BC}_0$ and $[\boldsymbol{BC}_0]$ will stand for the subspaces of $\boldsymbol{BC}$ and $[\boldsymbol{BC}]$ composed of the null-convergent and restrictedly null-convergent sequences respectively.

Let Z be any class of bounded convergent or bounded restrictedly convergent sequences; the method A is called *restrictedly conservative* (shortly *r-conservative*) for Z if it transforms every sequence of Z into a bounded restrictedly convergent sequence. The method A is called *restrictedly permanent* (briefly *r-permanent*) for Z if it is r-conservative for that class and the generalized limit $[A]..(x)$ is equal to the ordinary limit (or the limit in the restricted sense) for every $x \in Z$.

In the study of restricted summability the methods reducing the number of dimensions of the sequences seem to be helpful. Those are methods obtained by aid of three dimensional matrices $L = (l_{i\mu\nu})$, the transforms being defined as

$$L_i(x) = \sum_{\mu,\nu=0}^{\infty} l_{i\mu\nu} x_{\mu\nu}.$$

The definition of L-summability is then clear. Let us arrange all the transforms $A_{ik}(x)$ with $(i,k)\in S_n$ into a single sequence $L_1^{(n)}(x), L_2^{(n)}(x), \ldots$ and denote the matrix giving rise to this sequence by $L^{(n)} = (l_{i\mu\nu}^{(n)})$. It is obvious that the sequence $x = \{x_{ik}\}$ is restrictedly A-summable to $x..$ if and only if for each n it is $L^{(n)}$-summable to $x..$; this fact enables us to reduce certain arguments to the case of two-to-one dimensional transformations.

2. Toeplitzian theorems for restricted summability.

2.1 Proposition. *The method A transforms each bounded sequence of $\boldsymbol{BC}_0$ into a bounded sequence if and only if*

$$(\mathrm{a}_1) \qquad \sup_{i,k=0,1,\ldots} \sum_{\mu,\nu=0}^{\infty} |a_{ik\mu\nu}| < \infty.$$

(²) Unlike the notion of convergence which is meant in Pringsheim's sense, the notion of boundedness is the usual: the sequence $\{x_{ik}\}$ is bounded if $\sup\{|x_{ik}|: i,k = 0,1,\ldots\} < \infty$.

This is well known. The supremum in (a_1) will be denoted in the sequel by M.

The following lemma is also well known.

2.2 Lemma. *Let $(b_{n\nu})$ be a matrix; then the limit*

$$\lim_{n\to\infty} \sum_{\nu=0}^{\infty} b_{n\nu} z_\nu = B(z)$$

exists for each bounded sequence $z = \{z_\nu\}$ if and only if

(o_1) $\lim_{n\to\infty} b_{n\nu} = b_{.\nu}$ *exists for* $\nu = 0, 1, \ldots$,

(o_2) *the series* $\sum_{\nu=0}^{\infty} |b_{n\nu}|$ *are equiconvergent as* $n = 0, 1, 2, \ldots$(³).

If these conditions are satisfied, then

$$B(z) = \sum_{\nu=0}^{\infty} b_{.\nu} z_\nu.$$

2.3 Proposition. *The method A is r-conservative for $\boldsymbol{BC}_0$ if and only if (a_1) and the following conditions are satisfied:*

(b_1) *there exists*

$$[\lim_{i,k\to\infty}] a_{ik\mu\nu} = a_{..\mu\nu} \quad \text{for} \quad \mu,\nu = 0,1,\ldots,$$

(b_2) *for each m and n the series* $\sum_{\nu=0}^{\infty} |a_{ik\nu m}|$ *and* $\sum_{\nu=0}^{\infty} |a_{ikm\nu}|$ *are equiconvergent as* $(i,k)\in S_n$.

If these conditions are satisfied, then

$$[A]..(x) = \sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu} x_{\mu\nu}$$

for every $x \in \boldsymbol{BC}_0$.

Proof. Necessity. (b_1) is obviously necessary. Now let m and n be fixed, and let $\{z_n\}$ be any bounded sequence. Set

$$x_{\mu\nu} = \begin{cases} z_\nu & \text{for} \quad \mu = m,\ \nu = 0,1,\ldots, \\ 0 & \text{elsewhere.} \end{cases}$$

(³) The series $\sum_{n=0}^{\infty} a_{tn}$ are called *equiconvergent* as $t\in T$ if for every $\varepsilon > 0$ there is an N such that $|\sum_{n=p}^{q} a_{tn}| < \varepsilon$ for $q \geqslant p > N$, $t \in T$.

Then $x = \{x_{\mu\nu}\} \epsilon \boldsymbol{BC}_0$. The restricted A-summability of x implies by lemma 2.2 for every n the equiconvergence of the series $\sum_{\nu=0}^{\infty} |a_{ikm\nu}|$ if $(i, k) \epsilon S_n$. The equiconvergence of the series $\sum_{\mu=0}^{\infty} |a_{ik\mu m}|$ follows similarly.

To prove sufficiency let us observe first that $a_{..\mu\nu} = \lim_{i\to\infty} a_{ii\mu\nu}$ whence (a_1) implies

$$\sum_{\mu,\nu=0}^{\infty} |a_{..\mu\nu}| \leqslant M.$$

Now let D denote set of all elements $z = \{z_{ik}\}$ of $\boldsymbol{BC}_0$, for which $z_{ik} = 0$ for $i, k \geqslant p$ (p being, of course, dependent on z). Condition (b_2) implies for every fixed $z \epsilon D$ and every n the equiconvergence of the series

$$\sum_{\mu,\nu=0}^{\infty} |a_{ik\mu\nu} z_{\mu\nu}|$$

as $(i, k) \epsilon S_n$, which implies by (b_2) and lemma 2.2 the existence for each $z \epsilon D$ of the limit of $A_{ik}(z)$ as $i, k \to \infty$ through the set S_n; moreover, this limit is equal to $\sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu} z_{\mu\nu}$. Now, n being arbitrary, we deduce the existence of

$$[A]..(z) = \underset{i,k\to\infty}{[\lim]} A_{ik}(z) = \sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu} z_{\mu\nu}$$

for every $z \epsilon D$. The last formula defines an (obviously linear) functional on $\boldsymbol{BC}_0$, and since the set D is dense in $\boldsymbol{BC}_0$ we infer that $\underset{i,k\to\infty}{[\lim]} A_{ik}(x)$ exists on the whole of $\boldsymbol{BC}_0$ and is equal to $\sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu} z_{\mu\nu}$.

2.4. Proposition. *The method A is r-conservative for $\boldsymbol{BC}$ if and only if the conditions* (a_1), (b_1), (b_2), *and the following are satisfied*:

(c_1) *there exists*

$$\underset{i,k\to\infty}{[\lim]} \sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu} = s...$$

If these conditions are satisfied,

$$[A]..(x) = \sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu} x_{\mu\nu} + x..\Big(s.. - \sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu}\Big)$$

for each $x \epsilon \boldsymbol{BC}$.

Proof. The necessity of (c_1) follows from the fact that the sequence composed of ones is in $\boldsymbol{BC}$. Now let the conditions of our theorem be satisfied, $x = \{x_{ik}\} \epsilon \boldsymbol{BC}$ and set $z_{ik} = x_{ik} - x..$, $z = \{z_{ik}\}$. Then $z \epsilon \boldsymbol{BC}_0$ and

$$[A]..(z) = \sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu}(x_{\mu\nu} - x..).$$

On the other hand

$$[A]..(z) = \underset{i,k\to\infty}{[\lim]} \sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu}(x_{\mu\nu} - x..)$$

$$= \underset{i,k\to\infty}{[\lim]} \sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu} x_{\mu\nu} - x.. \underset{i,k\to\infty}{[\lim]} \sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu}$$

which gives

$$[A]..(x) = \sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu} x_{\mu\nu} + x..\Big(s.. - \sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu}\Big)$$

for every $x \epsilon \boldsymbol{BC}$.

From 2.3 and 2.4 there follows directly

2.5. Proposition. *The method A is r-permanent for $\boldsymbol{BC}_0$ if and only if the conditions* (a_1), (b_1), (b_2) *are satisfied and* $a_{..\mu\nu} = 0$ *for* $\mu, \nu = 0, 1, \ldots$

2.6. Proposition. *The method A is r-permanent for $\boldsymbol{BC}$ if and only if the conditions* (a_1), (b_1), (b_2), (c_1) *are satisfied,* $a_{..\mu\nu} = 0$ *for* $\mu, \nu = 0, 1, \ldots$, *and* $s.. = 1$.

Now we shall prove analogous propositions involving the spaces $[\boldsymbol{BC}_0]$ and $[\boldsymbol{BC}]$. In the sequel T will denote the set of all sequences $\{\mu_n, \nu_n\}$ of pairs of indices such that either $\lim_{n\to\infty} (\mu_n+1)(\nu_n+1)^{-1} = 0$ or $\lim_{n\to\infty} (\mu_n+1)^{-1}(\nu_n+1) = 0$.

2.7. Proposition. *The method A is r-conservative for $[\boldsymbol{BC}_0]$ if and only if the conditions* (a_1), (b_1) *and the following are satisfied*:

(d_1) *for each m and $\{\mu_n, \nu_n\} \epsilon T$ the series $\sum_{\sigma=0}^{\infty} |a_{ik\mu_\sigma\nu_\sigma}|$ are equiconvergent as $(i, k) \epsilon S_m$.*

Proof. Necessity. Conditions (a_1) and (b_1) follow from the inclusion $\boldsymbol{BC}_0 \subset [\boldsymbol{BC}_0]$. Now let $\{\mu_n, \nu_n\} \epsilon T$ and let $z = \{z_n\}$ be an arbitrary bounded sequence. Write

$$x_{ik}(z) = \begin{cases} z_n & \text{for } (i, k) = (\mu_n, \nu_n), \quad n = 0, 1, \ldots, \\ 0 & \text{elsewhere}; \end{cases}$$

then $x(z)=\{x_{ik}(z)\}\in[\boldsymbol{BC}_0]$. Let m be fixed and write

$$C_{ik}(z)=A_{ik}(x(z))=\sum_{\sigma=0}^{\infty}a_{ik\mu_\sigma\nu_\sigma}z_\sigma;$$

arranging $C_{ik}(z)$ with $(i,k)\in S_m$ into a single sequence we obtain a convergent sequence, since $\underset{i,k\to\infty}{[\lim]}A_{ik}(x(z))$ exists. This gives in virtue of lemma 2.2 the condition (d_1).

Sufficiency. By (a_1) $A_{ik}(x)$ are linear functionals on $[\boldsymbol{BC}_0]$ with norms, $\|A_{ik}\|$, uniformly bounded. Let E denote the set of all elements $x=\{x_{ik}\}\in[\boldsymbol{BC}_0]$ such that $\{(i,k)\colon x_{ik}\neq 0\}\in T$. By (b_1) and (d_1) we infer from lemma 2.2 that $A_{ik}(x)$ tend to

$$C(x)=\sum_{\mu,\nu=0}^{\infty}a_{\cdot\cdot\mu\nu}x_{\mu\nu}$$

for every $x\in E$, as $i,k\to\infty$ through the set S_n, i. e.

$$\underset{i,k\to\infty}{[\lim]}A_{ik}(x)=C(x)$$

for every $x\in E$. The functional $C(x)$ with arbitrary $x\in[\boldsymbol{BC}_0]$ is obviously linear on $[\boldsymbol{BC}_0]$. To prove our theorem it is sufficient to show that the set E is dense in $[\boldsymbol{BC}_0]$. Let $x=\{x_{ik}\}\in[\boldsymbol{BC}_0]$ and let $\varepsilon>0$ be given; then for every n there is a N_n such that $|x_{ik}|<\varepsilon$ for $(i,k)\in S_n$, $i,k\geqslant N_n$; we may suppose freely that $N_1\leqslant N_2\leqslant\ldots$ Set

$$x_{ik}^0=\begin{cases}0 & \text{for}\quad (i,k)\in S_n,\quad i,k\geqslant N_n,\quad n=1,2,\ldots,\\ x_{ik} & \text{elsewhere};\end{cases}$$

obviously $x^0=\{x_{ik}^0\}\in E$ and $\|x-x^0\|<\varepsilon$.

Just as for the space $\boldsymbol{BC}$, one can prove

2.8. Proposition. *The method A is r-conservative for $[\boldsymbol{BC}]$ if and only if the conditions (a_1), (b_1), (d_1), and (c_1) are satisfied.*

If these conditions are satisfied,

$$[A]_{\cdot\cdot}(x)=\sum_{\mu,\nu=0}^{\infty}a_{\cdot\cdot\mu\nu}x_{\mu\nu}+x_{\cdot\cdot}\Big(s_{\cdot\cdot}-\sum_{\mu,\nu=0}^{\infty}a_{\cdot\cdot\mu\nu}\Big).$$

2.9. Proposition. *The method A is r-permanent for $[\boldsymbol{BC}]$ if and only if the conditions (a_1), (b_1), (d_1), (c_1) are satisfied with $a_{\cdot\cdot\mu\nu}=0$ for $\mu,\nu=0,1,\ldots$ and $s_{\cdot\cdot}=1$.*

The quantity

$$\chi(A)=s_{\cdot\cdot}-\sum_{\mu,\nu=0}^{\infty}a_{\cdot\cdot\mu\nu}$$

defined for methods r-conservative for $\boldsymbol{BC}$ is usually called the *characteristic* of A.

3. The space of restrictedly summable sequences. Let A be an r-conservative method for the space $\boldsymbol{BC}_0$; we shall denote by $[\boldsymbol{BA}_0]$ the space of all bounded sequences $x=\{x_{ik}\}$ restrictedly A-summable to zero. This is a linear space under the usual definition of addition and multiplication by scalars. Let us introduce two norms in $[\boldsymbol{BA}_0]$

$$\|x\|=\sup_{i,k=0,1,\ldots}|x_{ik}|,$$

$$\|x\|^*=\sum_{\mu,\nu=0}^{\infty}2^{-(\mu+\nu)}|x_{\mu\nu}|+\sum_{\sigma=1}^{\infty}2^{-\sigma}\sup_{(i,k)\in S_\sigma}|A_{ik}(x)|.$$

By 2.1 $\|x\|^*\leqslant(M+2)\|x\|$, i. e. the norm $\|\ \|$ is not weaker than $\|\ \|^*$. It is easily seen that the conditions (n_1) and (n_2) of [2] (see also [1]) are verified. Thus $\langle[\boldsymbol{BA}_0],\|\ \|,\|\ \|^*\rangle$ is a two-norm space in which the two-norm convergence γ [1] is defined.

Let us denote by F the set of those sequences $x=\{x_{ik}\}$ for which the set $\{(i,k)\colon x_{ik}\neq 0\}$ is finite. We shall prove that this set F is γ-dense in $[\boldsymbol{BA}_0]$. This follows directly from

3.1. Lemma. *Let the method A be r-conservative for $\boldsymbol{BC}_0$ and let $x=\{x_{ik}\}\in[\boldsymbol{BA}_0]$. Then for every $\varepsilon>0$ and n there is a positive integer p and a sequence $z=\{z_{ik}\}$ such that $|z_{ik}|\leqslant|x_{ik}|$ for $i,k=0,1,\ldots$ and*

$$z_{ik}=\begin{cases}x_{ik} & \textit{for}\quad i,k\leqslant n,\\ 0 & \textit{for}\quad \max(i,k)>n+p,\end{cases}$$

$$|A_{ik}(x)-A_{ik}(z)|<\varepsilon\quad \textit{for}\quad (i,k)\in S_n.$$

Proof. Let us arrange the transforms $A_{ik}(x)$ with $(i,k)\in S_n$ into a single sequence $B_1(x),B_2(x),\ldots$; denoting

$$B_n(x)=\sum_{\mu,\nu=0}^{\infty}b_{n\mu\nu}x_{\mu\nu}$$

let us write

$$B_{ns}(x)=\sum_{\mu,\nu=0}^{s}b_{n\mu\nu}x_{\mu\nu},$$

$$u_s=\{B_{ns}(x)\}_{n=0,1,\ldots},\qquad u_0=\{B_n(x)\}_{n=0,1,\ldots}.$$

Then $u_n,u_0\in\boldsymbol{C}_0$, the space of (single) null-convergent sequences. Obviously $\|u_s\|\leqslant\|x\|\cdot M$

$$\lim_{s\to\infty}B_{ns}(x)=B_n(x),$$

whence u_s converge weakly to u_0 in the space C_0. By a theorem of Mazur ([3], p. 81) there exist non-negative $\lambda_n, \ldots, \lambda_{n+p}$ such that $\lambda_n + \ldots + \lambda_{n+p} = 1$ and

$$\|\lambda_n u_n + \ldots + \lambda_{n+p} u_{n+p} - u_0\| < \varepsilon.$$

Setting $x_s = \{x_{ik}^s\}$

$$x_{ik}^{(s)} = \begin{cases} x_{ik} & \text{for} \quad i, k \leqslant s, \\ 0 & \text{elsewhere} \end{cases}$$

we see that

$$\lambda_n u_n + \ldots + \lambda_{n+p} u_{n+p} = \{A_{ik}(\lambda_n x_n + \ldots + \lambda_{n+p} x_{n+p})\} \qquad ((i, k) \in S_n).$$

Hence writing $z = \lambda_n x_n + \ldots + \lambda_{n+p} x_{n+p}$ we obtain the desired conditions.

Let S denote the ball $\{x\colon \|x\| \leqslant 1\}$; we shall prove that the following property is satisfied:

(n_3) *For every $x_0 \in S$ and $\varepsilon > 0$ there is a $\delta > 0$ such that every $x \in S$ such that $\|x\|^* < \delta$ may be written in the form $x = x' - x''$ where $\|x_0 - x'\|^* < \varepsilon$ and $\|x_0 - x''\|^* < \varepsilon$.*

3.2. Theorem. *The space $\langle [AC_0], \|\ \|, \|\ \|^* \rangle$ has the property* (n_3).

Proof. Let us denote by P_n the set of the pairs (i, k) of indices for which $\max(i, k) > n$. Choose $\eta = \varepsilon/(4M+7)$ and n so large that

$$2 \sum_{(\mu,\nu) \in P_n} 2^{-(\mu+\nu)} + 2^{-n+1} M < \eta.$$

Let $x_0 = \{x_{ik}^0\}$; by lemma 3.1 there exists an element $z = \{z_{ik}\} \in F$ such that

$$z_{ik} = \begin{cases} x_{ik}^0 & \text{for} \quad i, k \leqslant n, \\ 0 & \text{for} \quad (i, k) \in P_{n+p}, \end{cases}$$

$$|A_{ik}(x_0) - A_{ik}(z)| < \varepsilon \quad \text{for} \quad (i, k) \in S_n.$$

Thus $z \in S$, and $S_1 \subset S_2 \subset \ldots$ implies

$$\|z - x_0\|^* = \sum_{(\mu,\nu) \in P_n} 2^{-(\mu+\nu)} |x_{ik}^0 - z_{ik}| + \sum_{\sigma=1}^{\infty} 2^{-\sigma} \sup_{(i,k) \in S_\sigma} |A_{ik}(x_0) - A_{ik}(z)|$$

$$\leqslant 2 \sum_{(\mu,\nu) \in P_n} 2^{-(\mu+\nu)} + (1 + 2^{-1} + \ldots + 2^{-n}) \sup |A_{ik}(x_0) - A_{ik}(z)| +$$

$$+ \sum_{\sigma=n+1}^{\infty} 2^{-\sigma} 2M < 2\eta.$$

Now let $x \in S$, $\|x\|^* < \delta < 2^{-(n+p)}\eta$, $x = \{x_{ik}\}$. Then $|x_{ik}| \leqslant 2^{i+k} \|x\|^* < \eta < 1$ as $i, k \leqslant n+p$. Obviously $\min(|x_{ik}^0 + x_{ik}|, |x_{ik}^0 - x_{ik}|) \leqslant 1$, whence there exist $\varepsilon_{ik} = \pm 1$ such that $|x_{ik}^0 - \varepsilon_{ik} x_{ik}| \leqslant 1$. Let us set

$$x'_{ik} = \begin{cases} z_{ik} + \varepsilon_{ik} x_{ik} & \text{if} \quad i, k \leqslant n+p, \quad \varepsilon_{ik} = 1, \\ z_{ik} & \text{if} \quad i, k \leqslant n+p, \quad \varepsilon_{ik} = -1, \\ x_{ik} & \text{elsewhere,} \end{cases}$$

$$x''_{ik} = \begin{cases} z_{ik} & \text{if} \quad i, k \leqslant n+p, \quad \varepsilon_{ik} = 1, \\ z_{ik} + \varepsilon_{ik} x_{ik} & \text{if} \quad i, k \leqslant n+p, \quad \varepsilon_{ik} = -1, \\ 0 & \text{elsewhere,} \end{cases}$$

$$x' = \{x'_{ik}\}, \quad x'' = \{x''_{ik}\}.$$

Then $x', x'' \in S$, $x = x' - x''$ and for $(i, k) \in S_n$

$$|A_{ik}(x') - A_{ik}(z)| \leqslant \sum_{\mu,\nu \leqslant n+p} |a_{ik\mu\nu} \varepsilon_{\mu\nu} x_{\mu\nu}| + \Big| \sum_{(\mu,\nu) \in P_{n+p}} a_{ik\mu\nu} x_{\mu\nu} \Big|$$

$$\leqslant M\eta + \Big| \sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu} x_{\mu\nu} \Big| + \sum_{\mu,\nu \leqslant n+p} |a_{ik\mu\nu} x_{\mu\nu}|$$

$$\leqslant 2M\eta + |A_{ik}(x)| \leqslant 2M\eta + \sup_{(i,k) \in S_n} |A_{ik}(x)| < 2(M+1)\eta,$$

since $|A_{ik}(x)| < 2^n \|x\|^* < \eta$ for $(i, k) \in S_n$. Thus

$$\sum_{\sigma=0}^{\infty} 2^{-\sigma} \sup_{(i,k) \in S_\sigma} |A_{ik}(x' - z)| \leqslant (1 + 2^{-1} + \ldots + 2^{-n}) \sup_{(i,k) \in S_n} |A_{ik}(x') - A_{ik}(z)| +$$

$$+ \sum_{\sigma=n+1}^{\infty} 2^{-\sigma} \sup_{(i,k) \in S_\sigma} |A_{ik}(x') - A_{ik}(z)| \leqslant 4(M+1)\eta + 2 \cdot 2^{-n} M \leqslant (4M+2)\eta.$$

Also

$$\sum_{\mu,\nu=0}^{\infty} 2^{-(\mu+\nu)} |x'_{\mu\nu} - z_{\mu\nu}| \leqslant \sum_{\mu,\nu \leqslant n+p} 2^{-(\mu+\nu)} |x_{\mu\nu}| + \sum_{(\mu,\nu) \in P_{n+p}} 2^{-(\mu+\nu)} |x_{\mu\nu}| \leqslant 3\eta,$$

whence

$$\|x' - z\|^* \leqslant (4M+5)\eta, \qquad \|x_0 - x'\|^* \leqslant \|x_0 - z\|^* + \|z - x'\|^* < (4M+7)\eta.$$

Similarly $\|x_0 - x''\|^* < (2M+4)\eta$.

4. Consistency theorems. Property (n_3) together with (n_1) and (n_2) imply for the space $\langle [BA_0], \|\ \|, \|\ \|^* \rangle$ that the limit of any sequence of γ-linear functionals is also γ-linear, the method being of course supposed r-permanent for BC_0. Let B be another method of such a type and let each sequence of $[\boldsymbol{BA}_0]$ be restrictedly B-summable. Then $B..(x) = \lim_{n\to\infty} B_{nn}(x)$ for $x \in [\boldsymbol{BA}_0]$; the functionals $B_{nn}(x)$ are γ-linear since they are limits of γ-linear functionals (the partial sums of the involved series); thus $B..(x)$ is γ-linear on $[\boldsymbol{BA}_0]$. For $x \in \boldsymbol{BC}_0$ we have $B..(x) = 0$, and since the set $\boldsymbol{BC}_0$ is γ-dense in $[\boldsymbol{BA}_0]$, we have $B..(x) = 0$ on $[\boldsymbol{BA}_0]$. Thus we have proved

4.1. Theorem. *Let the methods A and B be r-permanent for $\boldsymbol{BC}_0$ and let every bounded sequence $x = \{x_{ik}\}$ restrictedly A-summable to zero be restrictedly B-summable. Then B-$[\lim_{i,k\to\infty} x_{ik}] = 0$.*

The methods A and B are called *r-consistent* for the class Z of sequences if each sequence of Z is restrictedly summable by both methods to the same value. It is easily seen that if the methods A and B are r-consistent for bounded sequences, then the constants defined by (b_1) and (c_1) coincide for both methods, whence $\chi(A) = \chi(B)$.

Using the device of [4], p. 140 (concerning our case see also [2], p. 180), one can prove

4.2. Theorem. *Let the methods A and B be r-consistent for bounded convergent sequences and let $\chi(A) \neq 0$. If each bounded restrictedly A-summable sequence x is restrictedly B-summable, then $[A]..(x) = [B]..(x)$.*

References

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[5] C. N. Moore, *Summable series and convergence factors*, American Mathematical Society Colloquium Publications 22 (1938), p. 1-105.

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Reçu par la Rédaction le 10. 7. 1957

ANNALES
POLONICI MATHEMATICI
VI (1959)

Sur l'allure asymptotique des solutions de l'équation différentielle $u''+a(t)u'+b(t)u = 0$

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§ 1. Dans la monographie sur la théorie de la stabilité des équations différentielles M. R. Bellman consacre un petit paragraphe à l'étude de l'allure asymptotique des intégrales de l'équation différentielle linéaire du second ordre

$$u''+a(t)u'+u = 0, \tag{1.1}$$

dans l'hypothèse que $a(t)$ est une fonction continue tendant vers l'infini lorsque t croît indéfiniment ([1], Ch. V, Sect. 20).

L'auteur remarque que la comparaison de l'équation (1.1) avec une équation de la même forme, mais dont le coefficient $a(t)$ est constant, pourrait nous mener à supposer que toute intégrale de l'équation (1.1) doit tendre vers zéro lorsque $t\to+\infty$, mais qu'en réalité il n'en doit pas être ainsi. Aussi est-il plus raisonnable, remarque-t-il, de comparer l'équation (1.1) avec les équations

$$u''+a(t)u' = 0 \quad \text{et} \quad a(t)u'+u = 0 \tag{1.2}$$

ce qui permet de mieux comprendre le rôle que doit jouer, dans le problème de l'allure asymptotique des solutions de l'équation (1.1), l'intégrale

$$\int_0^\infty \frac{ds}{a(s)}. \tag{1.3}$$

En effet, si elle est finie, les intégrales non banales de la deuxième des équations (1.2) tendent, pour $t\to+\infty$, vers des limites finies, différentes de zéro et, par suite, on peut espérer qu'il existe au moins une intégrale de l'équation (1.1) qui jouit de la même propriété. Et inversement, si l'intégrale (1.3) est égale à ∞, on peut espérer que toute intégrale de l'équation (1.1) tende vers zéro lorsque $t\to+\infty$.