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Polynomial Hausdorff transformations

II. Regularity theorems and asymptotic properties of solutions of linear difference and differential equations*

by Z. Polniakowski (Poznań)

1. From the point of view of the theory of difference and differential equations the regularity and Mercerian theorems in the case of linear transformations of sequences and functions express the asymptotic properties of solutions of some linear equations.

In the case of polynomial Hausdorff transformations considered in I. we obtain Euler's linear difference or differential equations of finite order. In this paper we shall give the generalizations of those theorems in the case of difference equations for sequences and functions and in the case of differential equations. Some theorems of this kind are considered in the papers of Perron [5] and Späth [9] (1). As an application we give three Tauberian theorems (theorems 4A, 3B and 3C).

We shall use the notation introduced in I.

2. We consider first the case of sequences.

2.1.1. *Let* $W(a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(\nu)}) \neq 0$ *for* $n \geqslant \nu-1$ *and* $W(1, a_n^{(1)}, \ldots, a_n^{(\nu)}) \neq 0$ *for* $n \geqslant \nu$ $(\nu = 1, 2, \ldots, k)$, *where*

$$W(u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(\nu)}) = \begin{vmatrix} u_n^{(1)} & u_n^{(2)} & \ldots & u_n^{(\nu)} \\ u_{n-1}^{(1)} & u_{n-1}^{(2)} & \ldots & u_{n-1}^{(\nu)} \\ \ldots & \ldots & \ldots & \ldots \\ u_{n-\nu+1}^{(1)} & u_{n-\nu+1}^{(2)} & \ldots & u_{n-\nu+1}^{(\nu)} \end{vmatrix}.$$

If

$$w_n^{(1)} = \frac{1}{a_n^{(1)}}, \qquad w_n^{(\nu)} = \frac{W(1, a_n^{(1)}, \ldots, a_n^{(\nu-1)})}{W(a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(\nu)})}$$

$$\text{for} \quad \nu = 2, 3, \ldots, k \quad \text{and} \quad n \geqslant \nu-1,$$

* The first part of this paper appeared in Annales Polonici Mathematici 5 (1958), p. 1-24.

(1) See the references given at the end of I.

then $\Delta w_n^{(\nu)} \neq 0$ *for* $\nu = 1, 2, \ldots, k$ *and* $n \geqslant \nu-1$. *The relations*

$$(1)\qquad x_n^{(\nu)} + \frac{w_{n-1}^{(\nu)}}{\Delta w_{n-1}^{(\nu)}} \Delta x_{n-1}^{(\nu)} = x_n^{(\nu+1)}, \qquad \nu = 1, 2, \ldots, k \quad \text{and} \quad n \geqslant \nu$$

imply

$$x_n^{(\nu+1)} = \frac{W(x_n^{(1)}, a_n^{(1)}, \ldots, a_n^{(\nu)})}{W(1, a_n^{(1)}, \ldots, a_n^{(\nu)})} \quad \text{for} \quad \nu = 1, 2, \ldots, k \quad \text{and} \quad n \geqslant \nu.$$

In the proof we shall use the identity

$$(2)\qquad \begin{vmatrix} A_{nk} & A_{nl} \\ A_{mk} & A_{ml} \end{vmatrix} = A A_{nm;kl},$$

where $A = |a_{nk}|$ and $A_{nm;kl}$ denotes the algebraic complement of the subdeterminant $\begin{vmatrix} a_{nk} & a_{nl} \\ a_{mk} & a_{ml} \end{vmatrix}$.

We prove by induction. For $\nu = 1$ we obtain

$$\Delta w_{n-1}^{(1)} = -\frac{W(1, a_n^{(1)})}{a_{n-1}^{(1)} a_n^{(1)}} \neq 0,$$

$$x_n^{(2)} = x_n^{(1)} + \frac{w_{n-1}^{(1)}}{\Delta w_{n-1}^{(1)}} \Delta x_{n-1}^{(1)} = \frac{\Delta[w_{n-1}^{(1)} x_{n-1}^{(1)}]}{\Delta w_{n-1}^{(1)}}$$

$$= \frac{a_{n-1}^{(1)} x_n^{(1)} - a_n^{(1)} x_{n-1}^{(1)}}{\Delta a_{n-1}^{(1)}} = \frac{W(x_n^{(1)}, a_n^{(1)})}{W(1, a_n^{(1)})}, \qquad n \geqslant 1.$$

Next, if $A = W(x_n^{(1)}, a_n^{(1)}, \ldots, a_n^{(\nu)})$, $B = W(1, a_n^{(1)}, \ldots, a_n^{(\nu)})$ for $n \geqslant \nu$, then

$$w_{n-1}^{(\nu)} = (-1)^\nu \frac{B_{1,\nu+1}}{B_{11}}, \qquad w_n^{(\nu)} = (-1)^\nu \frac{B_{\nu+1,\nu+1}}{B_{\nu+1,1}}.$$

Hence by (2) we obtain for $n \geqslant \nu$

$$\Delta w_{n-1}^{(\nu)} = (-1)^{\nu+1} \frac{1}{B_{11} B_{\nu+1,1}} \begin{vmatrix} B_{11} & B_{1,\nu+1} \\ B_{\nu+1,1} & B_{\nu+1,\nu+1} \end{vmatrix} = (-1)^{\nu+1} \frac{B B_{1,\nu+1;1,\nu+1}}{B_{11} B_{\nu+1,1}} \neq 0.$$

Suppose now that $x_n^{(\nu)} = \dfrac{W(x_n^{(1)}, a_n^{(1)}, \ldots, a_n^{(\nu-1)})}{W(1, a_n^{(1)}, \ldots, a_n^{(\nu-1)})}$ for $n \geqslant \nu-1 \geqslant 1$; then

$$x_{n-1}^{(\nu)} = \frac{A_{1,\nu+1}}{B_{1,\nu+1}}, \qquad x_n^{(\nu)} = \frac{A_{\nu+1,\nu+1}}{B_{\nu+1,\nu+1}}$$

and

$$x_n^{(\nu+1)} = \frac{\Delta[w_{n-1}^{(\nu)} x_{n-1}^{(\nu)}]}{\Delta w_{n-1}^{(\nu)}} = \frac{B_{11} B_{\nu+1,1}}{B B_{1,\nu+1;1,\nu+1}} \left(\frac{A_{\nu+1,\nu+1}}{B_{\nu+1,1}} - \frac{A_{1,\nu+1}}{B_{11}} \right)$$

$$= \frac{1}{B B_{1,\nu+1;1,\nu+1}} \begin{vmatrix} A_{11} & A_{1,\nu+1} \\ A_{\nu+1,1} & A_{\nu+1,\nu+1} \end{vmatrix}$$

$$= \frac{A A_{1,\nu+1;1,\nu+1}}{B B_{1,\nu+1;1,\nu+1}} = \frac{W(x_n^{(1)}, a_n^{(1)}, \ldots, a_n^{(\nu)})}{W(1, a_n^{(1)}, \ldots, a_n^{(\nu)})}, \qquad n \geqslant \nu,$$

because $A_{m1} = B_{m1}$ for $m = 1, 2, \ldots, \nu+1$ and $A_{1,\nu+1;1,\nu+1} = B_{1,\nu+1;1,\nu+1}$.

We remark that theorem A, I, may also be formulated in the case of bounded sequences as follows:

THEOREM A_1. *If conditions* 1) *and* 2) *of theorem* A, I, *are satisfied, then the inequality* $\overline{\lim\limits_n} |s_n| \leqslant M$ *implies* $\overline{\lim\limits_n} |a_n/b_n| \leqslant MK$.

2.1.2. *Suppose that conditions* 1) *and* 2) *of* 2.1.1, I, *are satisfied and furthermore*

3) $\overline{\lim\limits_n} |s_n| \leqslant M$.

Then there exists a solution $\{\bar{x}_n\}$ *of the difference equation* (2), I, *satisfying the inequality* $\overline{\lim\limits_n} |\bar{x}_n| \leqslant MK$. *In case* 1a) *every solution of* (2), I, *has this property.*

After theorem A_1 the proof is like that of 2.1.1, I. We observe that if $b_n \neq 0$ for $n \geqslant N$, the inequalities in conditions 1a) and 1b) in theorems A and A_1 may be replaced by $\overline{\lim\limits_n} \dfrac{1}{b_n} \sum\limits_{\nu=0}^{n-1} |\Delta b_\nu| \leqslant K$ and $\overline{\lim\limits_n} \dfrac{1}{b_n} \sum\limits_{\nu=n}^{\infty} |\Delta b_\nu| \leqslant K$ respectively.

THEOREM 1A. *Let* $L(x_n) = x_n + \lambda_1(n) \Delta x_{n-1} + \ldots + \lambda_k(n) \Delta^k x_{n-k}$. *Suppose that the homogeneous difference equation* $L(x_n) = 0$ *has the solutions* $\{a_n^{(\nu)}\}$, $\nu = 1, 2, \ldots, k$, *linearly independent, for which*

1) $W(a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(\nu)}) \neq 0$ *for* $n \geqslant \nu-1$,

2a) $|w_n^{(\nu)}| \to \infty$ *and* $\sum\limits_{j=0}^{n-1} |\Delta w_j^{(\nu)}| \leqslant K |w_n^{(\nu)}|$

or

2b) $w_n^{(\nu)} \to 0$ *and* $\sum\limits_{j=n}^{\infty} |\Delta w_j^{(\nu)}| \leqslant K |w_n^{(\nu)}|$,

3) $w_n^{(\nu)} \neq 0$ *for* $n \geqslant \nu-1$.

Furthermore suppose that

4) $\overline{\lim_n}|s_n| \leqslant M$.

Then there exists a solution $\{\bar{x}_n\}$ *of the difference equation*

(3) $$L(x_n) = s_n,$$

such that $\overline{\lim_n}|\bar{x}_n| \leqslant MK^k$. *Under the hypothesis* $\lim_n|w_n^{(\nu)}| = \infty$ *for* $\nu = 1, 2, \ldots, k$ *every solution of* (3) *satisfies this inequality.*

Proof. We write system (1) in the form

$$\Delta x_{n-1}^{(\nu)} + \frac{\Delta w_{n-1}^{(\nu)}}{w_{n-1}^{(\nu)}}(x_n^{(\nu)} - x_n^{(\nu+1)}) = 0 \quad \text{for} \quad \nu = 1, 2, \ldots, k \quad \text{and} \quad n \geqslant \nu.$$

If $x_n^{(1)} = x_n$ then $x_n^{(k+1)} = L(x_n) = s_n$ by 2.1.1 and there follows $\overline{\lim_n} x_n^{(k+1)} \leqslant M$ by hypothesis. Using 2.1.2 we find from the above difference equations successively for $\nu = k, k-1, \ldots, 1$ that there exists a sequence $\{\bar{x}_n^{(\nu)}\}$ satisfying the difference equation with index ν and such that $\overline{\lim_n}|\bar{x}_n^{(\nu)}| \leqslant MK^{k-\nu+1}$. We assume now that $\bar{x}_n = \bar{x}_n^{(1)}$.

Let us remark that the sequence $\{a_n^{(\nu)}\}$ is a solution of the difference equation $L_\nu(x_n) = (-1)^{\nu-1}/w_n^{(\nu)}$, where

$$L_\nu(x_n) = \frac{W(x_n, a_n^{(1)}, \ldots, a_n^{(\nu-1)})}{W(1, a_n^{(1)}, \ldots, a_n^{(\nu-1)})}, \qquad \nu = 2, 3, \ldots, k \quad \text{and} \quad n \geqslant \nu - 1.$$

From the proved part of the theorem it follows that if $|w_n^{(\nu)}| \to \infty$ for $\nu = 1, 2, \ldots, j$ $(j \leqslant k)$, then for those ν we have $\lim_n a_n^{(\nu)} = 0$. Thus we obtain the second part of the theorem if we observe that the general solution of (3) is

$$x_n = \bar{x}_n + \sum_{\nu=1}^{k} c_\nu a_n^{(\nu)},$$

where c_ν are constants.

Let us notice that under the hypothesis $\lim_n s_n - s = 0$ instead of $\overline{\lim_n}|s_n| \leqslant M$ we may replace the relations in the assertion of theorem 1A by $\lim_n \bar{x}_n = s$ and $\lim_n x_n = s$ respectively.

Theorem 2A. *Suppose that* $W(z)$ *and* $W_1(z)$ *are polynomials of degrees* $k \geqslant 1$ *and* $l \geqslant 0$ $(l \leqslant k)$ *respectively, and* $\overline{\lim_n}|s_n| n^{-c} \leqslant M$ *with some* c *real. We consider the difference equation*

(4) $$L(x_n) = L^*(s_n), \qquad n \geqslant k,$$

where

$$L(x_n) = \sum_{\nu=0}^{k} \lambda_\nu \binom{n}{\nu} \Delta^\nu x_{n-\nu}, \quad L^*(x_n) = \sum_{\nu=0}^{l} \eta_\nu \binom{n}{\nu} \Delta^\nu x_{n-\nu}, \quad \lambda_\nu = (-1)^\nu \Delta^\nu W(0)$$

and $\eta_\nu = (-1)^\nu \Delta^\nu W_1(0)$. *By* r_ν, $\nu = 1, 2, \ldots, k$, *we denote the roots of the equation* $W(z) = 0$. *(This equation may be called the characteristic equation of* (4), *analogously to the case of Euler's differential equation.)*

a) *If* $\mathrm{re}\, r_\nu \neq c$ *for* $\nu = 1, 2, \ldots, k$, *then there exists a constant* K^* *which does not depend on the sequence* $\{s_n\}$ *and a solution* $\{\bar{x}_n\}$ *of the difference equation* (4) *such that* $\overline{\lim_n}|\bar{x}_n| n^{-c} \leqslant MK^*$.

b) *If* $\mathrm{re}\, r_\nu < c$ *for* $\nu = 1, 2, \ldots, k$, *then every solution* $\{x_n\}$ *of* (4) *satisfies the same inequality as* $\{\bar{x}_n\}$.

c) *If* $\mathrm{re}\, r_j \geqslant c$ *with some* $j \leqslant k$, *then for every* K *there exists a solution* $\{x_n^*\}$ *of* (4) *such that* $\overline{\lim_n}|x_n^*| n^{-c} > K$.

Proof. a) We assume first that $W(z) = (r-z)^k$ and $W_1(z) = 1$. In this case the difference equation $L(x_n) = s_n$ is equivalent to the system

(5) $$r x_n^{(\nu)} + n \Delta x_{n-1}^{(\nu)} = x_n^{(\nu+1)} \quad \text{for} \quad \nu = 1, 2, \ldots, k \quad \text{and} \quad n \geqslant \nu,$$

where $x_n^{(1)} = x_n$, $x_n^{(k+1)} = s_n$. (Compare 2.2.4, I.)

If neither r nor $r-c$ are positive integers, we substitute $x_n^{(\nu)} = \dfrac{\Gamma(n-r+c+1)}{\Gamma(n-r+1)} y_n^{(\nu)}$ for $\nu = 1, 2, \ldots, k+1$ and obtain the system

(6) $$y_n^{(\nu)} + \frac{n}{r-c} \Delta y_{n-1}^{(\nu)} = \frac{1}{r-c} \cdot \frac{n-r+c}{n-r} y_n^{(\nu+1)}, \quad \nu = 1, 2, \ldots, k \quad \text{and} \quad n \geqslant \nu.$$

Since $\dfrac{\Gamma(n-r+c+1)}{\Gamma(n-r+1)} \sim n^c$, we have $\overline{\lim_n}|y_n^{(k+1)}| \leqslant M$. Reasoning as in the proof of theorem 1A we infer, after 2.1.2 and 2.1.2, I, that in this particulary case there exists a sequence $\{\bar{y}_n\}$ satisfying (6) and such that $\overline{\lim_n}|\bar{y}_n| \leqslant MK^k$, where $K = 1/|\mathrm{re}\, r - c|$.

Taking $\bar{x}_n = \dfrac{\Gamma(n-r+c+1)}{\Gamma(n-r+1)} \bar{y}_n$, we see that the sequence $\{\bar{x}_n\}$ satisfies (4) and is $\overline{\lim_n}|\bar{x}_n| n^{-c} \leqslant MK^k$.

If r or $r-c$ are positive integers, we prove in a similar way the existence of $\bar{y}_n$ and $\bar{x}_n$ for $n \geqslant m$, where $m > r$, or $m > r-c$ respectively. The terms $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{m-1}$ we compute immediately from (4) for $L^*(s_n) = s_n$.

In the general case let

$$W_1(z) = l_0 W(z) + \sum_{\nu=1}^{k} l_\nu \frac{W(z)}{(r_\nu - z)^{p_\nu}} \ [2].$$

In virtue of the proved part of the theorem and of 2.2.3, I, the equation

$$\delta[(r_\nu - n)^{p_\nu}]\,\delta(x_n) = (s_n), \qquad 1 \leqslant \nu \leqslant k,$$

has a solution $\{x_n^{(\nu)}\}$ such that $\overline{\lim_n} |x_n^{(\nu)}|\, n^{-c} \leqslant MK_\nu^{p_\nu}$, where $K_\nu = 1/|\mathrm{re}\, r_\nu - c|$.

Taking $x_n^{(0)} = s_n$ and $\bar{x}_n = \sum_{\nu=0}^{k} l_\nu x_n^{(\nu)}$, we obtain

$$\delta[W(n)]\,\delta(\bar{x}_n) = \sum_{\nu=0}^{k} l_\nu\, \delta[W(n)]\,\delta(x_n^{(\nu)})$$

$$= \delta\left[l_0 W(n) + \sum_{\nu=0}^{k} l_\nu \frac{W(n)}{(r_\nu - n)^{p_\nu}}\right]\delta(s_n) = \delta[W_1(n)]\,\delta(s_n),$$

because the relations

$$W(n)\,\Delta^n x_0^{(\nu)} = \frac{W(n)}{(r_\nu - n)^{p_\nu}}\,\Delta^n s_0, \qquad \nu = 1, 2, \ldots, k,$$

hold for $n = 0, 1, 2, \ldots$ Hence $L(\bar{x}_n) = L^*(s_n)$ by 2.2.2, I. Moreover $\overline{\lim_n} |\bar{x}_n|\, n^{-c} \leqslant \sum_{\nu=0}^{k} |l_\nu|\, \overline{\lim_n} |x_n^{(\nu)}|\, n^{-c} \leqslant MK^*$, where $K^* = |l_0| + \sum_{\nu=1}^{k} |l_\nu|\, K_\nu^{p_\nu}$.

b) Let us notice that by 2.2.3, I, the homogeneous difference equation $L(x_n) = 0$ has solutions of the form

$$a_n^{(\nu)} = \begin{cases} \left[\dfrac{d^j}{dx^j}\dfrac{n!}{\Gamma(n-x+1)}\right]_{x=r_{k_1}} & \text{for} \quad \nu = j+1 = 1, 2, \ldots, k_1, \\ \left[\dfrac{d^j}{dx^j}\dfrac{n!}{\Gamma(n-x+1)}\right]_{x=r_{k_2}} & \text{for} \quad \nu = k_1+j+1 = k_1+1, \ldots, k_2, \\ \text{etc.,} & \end{cases}$$

if $r_1 = r_2 = \ldots = r_{k_1} \neq r_{k_1+1} = \ldots = r_{k_2} \ldots$ etc.

We have namely

$$L\left(\frac{d^j}{dx^j}\frac{n!}{\Gamma(n-x+1)}\bigg|_{x=r_\nu}\right) = \frac{d^j}{dx^j}\left[W(x)\,\frac{n!}{\Gamma(n-x+1)}\right]_{x=r_\nu} = 0 \quad \text{for} \quad j = 0, 1, \ldots, k_\nu - 1,$$

where k_ν is the degree of multiplicity of the root r_ν.

[2] This relation holds for every z if we assume that $W(z)/(r_\nu - z)^{p_\nu}$ denotes a polynomial of degree $k - p_\nu$.

It may be computed that the sequence $\{a_n^{(\nu)}\}$, $1 \leqslant \nu \leqslant k$, satisfies the difference equation $L_\nu(x_n) = c_\nu \dfrac{n!}{\Gamma(n-r_\nu+1)}$, where $L_\nu(x_n) = \dfrac{W(x_n, a_n^{(1)}, \ldots, a_n^{(\nu-1)})}{W(1, a_n^{(1)}, \ldots, a_n^{(\nu-1)})}$ for $\nu \geqslant 2$, $L_1(x_n) = x_n$, and $c_\nu \neq 0$ depends on $r_1, r_2, \ldots, r_\nu$ (see the proof of theorem 1A). In virtue of the proved part of the theorem and of the relation $n!/\Gamma(n-r+1) \sim n^r$, it follows successively that $\lim_n a_n^{(\nu)}\, n^{-c} = 0$, because the sequences $\{a_n^{(1)}\}, \{a_n^{(2)}\}, \ldots, \{a_n^{(\nu-1)}\}$ are (lineary independent) solutions of the homogeneous difference equation $L_\nu(x_n) = 0$. Since every solution $\{x_n\}$ of (4) may be written in the form

$$x_n = \bar{x}_n + \sum_{\nu=1}^{k} c_\nu a_n^{(\nu)},$$

where c_ν are constants, we infer that in the considered case $\overline{\lim_n} |x_n|\, n^{-} \leqslant MK^*$.

c) Suppose that, for example, $\mathrm{re}\, r_1 \geqslant c$ and that, for given $K > 0$, we have $\overline{\lim_n} |x_n'|\, n^{-c} \leqslant K$, where $\{x_n'\}$ is a solution of (4). It is easy to see that the sequence with terms $x_n^* = x_n' + (2K+1)\dfrac{n!}{\Gamma(n-r_1+1)}$ satisfies also (4) and we have $\overline{\lim_n} |x_n^*|\, n^{-c} \geqslant K+1$.

COROLLARY. *Suppose now that $s_n \sim sn^a$, with some a complex, instead of $\overline{\lim_n} |s_n|\, n^{-c} \leqslant M$ in theorem 2A. In the case $\mathrm{re}\, r_\nu \neq \mathrm{re}\, a$, $\nu = 1, 2, \ldots, k$, it follows from this theorem that there exists a sequence $\{\bar{x}_n\}$ satisfying (4) and such that $\bar{x}_n \sim s\dfrac{W_1(a)}{W(a)}\, n^a$. If $W(a) \neq 0$, then the condition $\mathrm{re}\, r_\nu < \mathrm{re}\, a$, $\nu = 1, 2, \ldots, k$, is necessary and sufficient for every solution $\{x_n\}$ of (4) to satisfy the above asymptotic relation.*

For the proof we observe that $\lim_n \left[s_n - s\dfrac{n!}{\Gamma(n-a+1)}\right] n^{-a} = 0$ by hypothesis. From theorem 2A it follows that in the first case the difference equation $L(x_n) = L^*\left(s_n - s\dfrac{n!}{\Gamma(n-a+1)}\right)$ has the solution $\{x_n'\}$ such that $\lim_n x_n' n^{-a} = 0$. Hence after 2.2.3, I, the sequence with terms $\bar{x}_n = x_n' + s\dfrac{W_1(a)}{W(a)} \cdot \dfrac{n!}{\Gamma(n-a+1)}$ satisfies (4) and we have $\lim_n \bar{x}_n n^{-a} = s\dfrac{W_1(a)}{W(a)}$.

The second case is proved by using the results of the proof of theorem 2A, case b), and applying the assertion c) of that theorem. It will be observed that the case $W(a)=0$ may be considered by the use of theorem 2A immediately.

From theorem 2A follows a Mercerian theorem of G. H. Hardy ([1], theorem 52).

Remark. Every sequence $\{x_n\}$ satisfying (4) may be considered as a transform of the sequence $\{s_n\}$. If $W(n)\neq 0$ then one of them is the Hausdorff transform $\{t_n^*\}$ of $\{s_n\}$ with $\mu_n = W_1(n)/W(n)$; the others differ from it by a linear combination of sequences $a_n^{(\nu)} = \frac{d^j}{dx^j}\frac{n!}{\Gamma(n-x+1)}\Big|_{x=r_\nu}$.

In the case $W_1(z)=1$ every solution $\{x_n\}$ of (4) is a Hausdorff transform of some sequence $\{s_n'\}$, where $s_n'=s_n$ for $n\geqslant k$. Namely, with the given solution $\{x_n^*\}$ of (4), if we take $L(x_n^*)=s_n'$ for $n=0,1,2,\ldots$, then $s_n'=s_n$ for $n=k,k+1,\ldots$ and $\Delta^n x_0^* = \frac{1}{W(n)}\Delta^n s_0'$ for $n=0,1,2,\ldots$ (see 2.2.2, I, for $t_n=s_n'$ and $s_n=x_n^*$).

In the case $\operatorname{re} r_\nu\neq 0$, $\nu=1,2,\ldots,k$, the method of summability represented by the difference equation (4) may be called *regular*, by which we mean that under the hypothesis $\lim_n s_n = s$ and $W(0)=W_1(0)$ there exists a solution $\{x_n^*\}$ of (4) tending to s.

From the above discussion we obtain almost immediately

THEOREM 3A. *The hypothesis*:

1) *$W(z)$ and $W_1(z)$ are defined as in theorem 2A and $W(n)\neq 0$ for $n=0,1,2,\ldots$,*

2) $(t_n)=\delta\left[\frac{W_1(n)}{W(n)}\right]\delta(s_n)$,

3) $s_n\sim sn^a$ $(|s|<\infty)$,

imply the relation $t_n\sim s\frac{W_1(a)}{W(a)}n^a$ *if and only if* $\operatorname{re} a>\max_\nu \operatorname{re} r_\nu$.

In the case $W_1(z)=1$ the hypothesis $\operatorname{re} a\neq \operatorname{re} r_\nu$, $\nu=1,2,\ldots,k$, implies $t_n'\sim s\frac{1}{W(a)}n^a$, where $(t_n')=\delta\left[\frac{W_1(n)}{W(n)}\right]\delta(s_n')$ and $s_n'=s_n$ for $n\geqslant k$.

We prove the case $\operatorname{re} a\leqslant \operatorname{re} r_\nu$ with some ν just as in the proof of theorem 1A, I.

2.2. In the sequel we shall use the following lemmas:

2.2.1. *Let $W(z)$ denote the polynomial of degree $p\geqslant 1$, whose zeros are r_ν; furthermore suppose that $W(n)\neq 0$ for $n=0,1,2,\ldots$ and that*

$(t_n)=\delta\left[\frac{1}{W(n)}\right]\delta\left(\frac{n!}{\Gamma(n-r+1)}\right)$. *Then the hypothesis $\operatorname{re} r>\max_{1<\nu<p}\operatorname{re} r_\nu$ implies $t_n\sim\frac{1}{W(r)}n^r$ and the hypothesis $\operatorname{re} r\leqslant \max_{1<\nu<p}\operatorname{re} r_\nu<c$ implies $\lim_n t_n n^{-c}=0$.*

For the proof it suffices to use 2.2.2, I, and apply theorem 2A (see corollary and theorem 3A).

2.2.2. *Suppose that $W(z)$ is the polynomial of degree $p\geqslant 1$. Then for every $\varphi\in\langle 0,2\pi)$ and every integer m there exists a root $r=r(\lambda)$ of the equation*

$$W(z)+\lambda=0,\quad \textit{where}\quad \arg\lambda/a=\varphi, \tag{7}$$

such that $r\sim\sqrt[p]{\left|\frac{\lambda}{a}\right|}\exp\left[\frac{(2m+1)\pi+\varphi}{p}i\right]$ as $|\lambda|\to\infty$. By a we denote the coefficient of z^p in the polynomial $W(z)$.

This follows from the well-known properties of the algebraic equations.

From 2.2.2 we immediately obtain

2.2.3. *Suppose that $\arg\lambda/a\neq 0 \pmod{\pi/2}$. Then for every real c there exists $A>0$ such that all the roots r_ν of equation (7) with $|\lambda|\geqslant A$ satisfy the inequality $\operatorname{re} r_\nu\neq c$.*

THEOREM 4A. *Suppose that*

1) *$w(z)$ is a polynomial of degree $k\geqslant 1$ and $w(n)\neq 0$ for $n=0,1,2,\ldots$,*

2) *$\operatorname{re} a>\max_\nu \operatorname{re}\varrho_\nu$, where ϱ_ν are zeros of $w(z)$,*

3) *$\overline{\lim_n}|n^{l-a}\Delta^l s_n|\leqslant M<\infty$ with some positive integer l,*

4) *$t_n\sim sn^a$, where $(t_n)=\delta[1/w(n)]\delta(s_n)$.*

Then $s_n\sim sw(a)n^a$.

Proof. Let $r_1,r_2,\ldots,r_{k+l}$ be the roots of the equation

$$w(z)\binom{z}{l}+\lambda=0 \tag{8}$$

and let a denote the coefficient of z^k in the polynomial $w(z)$. If we choose $\arg\lambda/a\neq 0\pmod{\pi/2}$, then by 2.2.2 and 2.2.3 we have for $|\lambda|\geqslant A>0$

1° $\operatorname{re} r_\nu\neq \operatorname{re} a$ for $\nu=1,2,\ldots,k+l$,

2° $\operatorname{re} r_n\neq \operatorname{re} r_m$ if $n\neq m$ and if r_n and r_m are not complex conjugate (in particular (8) has only simple roots).

Suppose first that $s=0$. We choose λ as above and take $p_n = (-1)^l\frac{1}{\lambda}\binom{n}{l}\Delta^l s_{n-l}+t_n$ for $n\geqslant l$; then $\overline{\lim_n}|p_n n^{-a}|\leqslant M/|\lambda| l!$. Using theorem 2A for $W(z) = w(z)\binom{z}{l} + \lambda$, $W_1(z) = \lambda w(z)$, we infer that there exists a solution $\{\bar{x}_n\}$ of the difference equation $L(x_n) = L^*(p_n)$ such that $\overline{\lim_n} |\bar{x}_n n^{-a}| \leqslant MK^*/|\lambda| l!$, where $K^* = \sum_{\nu=1}^{k+l} |l_\nu|/|\mathrm{re}(r_\nu - a)|$ and $l_\nu = W_1(r_\nu)/W'(r_\nu)$ (see the proof of theorem 2A).

This difference equation satisfies also the sequence $\{s_n\}$, because by (4), I, we have

$$\delta[\mu_n^{(1)}]\,\delta(p_n) = \delta[\mu_n^{(2)}]\,\delta(s_n),$$

where $\mu_n^{(1)} = W_1(n)$, $\mu_n^{(2)} = W(n)$. Whence

$$s_n = \bar{x}_n + \sum_{\nu=1}^{k+l} c_\nu \frac{n!}{\Gamma(n - r_\nu + 1)}, \qquad n = 0, 1, 2, \ldots,$$

with some complex $c_1, c_2, \ldots, c_{k+l}$.

We prove that the inequality $\mathrm{re}\, r_\nu > \mathrm{re}\, a$ implies $c_\nu = 0$. Suppose that, for example, $c_1 \neq 0$ and $r_1 = \sigma + i\tau$, $r_2 = \sigma - i\tau$, $\tau \neq 0$, $\max_{1 \leqslant \nu \leqslant k+l} \mathrm{re}\, r_\nu = \sigma > \mathrm{re}\, a$. Since

$$t_n = x_n^* + \sum_{\nu=1}^{k+l} c_\nu t_n^{(\nu)},$$

where $\{x_n^*\}$ and $\{t_n^{(\nu)}\}$ are the Hausdorff transforms of the sequences $\{\bar{x}_n\}$ and $\{n!/\Gamma(n - r_\nu + 1)\}$ respectively, corresponding to $\mu_n = 1/w(n)$, we obtain by theorem 2A

$$\overline{\lim_n} \left| \Big(t_n - \sum_{\nu=1}^{k+l} c_\nu t_n^{(\nu)}\Big) n^{-a} \right| = \overline{\lim_n} |x_n^* n^{-a}| \leqslant \frac{MK^*}{|\lambda| l!} K_1 .$$

Whence by 2.2.1

$$\overline{\lim_n} |t_n| n^{-\sigma} = \overline{\lim_n} |c_1 t_n^{(1)} + c_2 t_n^{(2)}| n^{-\sigma} = \overline{\lim_n} \left| \frac{c_1}{w(r_1)} n^{\tau i} + \frac{c_2}{w(r_2)} n^{-\tau i} \right| > 0,$$

because the sequence $\{a\cos(\tau \ln n) + b\sin(\tau \ln n)\}$ diverges for every real $\tau \neq 0$, a and b such that $|a| + |b| > 0$. (This follows from the fact that the set of the limit points of the sequence $\{\cos(\tau \ln n)\}$ is dense in the interval $\langle -1, 1\rangle$.)

If $r_1 = \sigma$ is real, we have similarly

$$\lim_n |t_n| n^{-\sigma} = \lim_n |c_1 t_n^{(1)}| n^{-\sigma} = |c_1 / w(r_1)| > 0 .$$

In both cases we obtain $\overline{\lim_n} |t_n n^{-a}| = \infty$ in contradiction to the hypothesis. This proves that $c_1 = 0$.

Since $n^{-a} n!/\Gamma(n - r_\nu + 1) \to 0$ if $\mathrm{re}\, r_\nu < \mathrm{re}\, a$, we finally obtain $\overline{\lim_n} |s_n n^{-a}| \leqslant MK^*/|\lambda| l!$. We observe that by 2.2.2 there follows

$$|r_\nu| \sim \left(\left| \frac{\lambda}{a} \right| l! \right)^{1/(k+l)}, \qquad |\mathrm{re}\, r_\nu| \sim \lambda_\nu |r_\nu| \quad \text{as} \quad |\lambda| \to \infty,$$

where $\lambda_\nu = \left| \cos\left(\frac{(2\nu+1)\pi + \varphi}{k+l} \right) \right|$ and $\varphi = \arg \frac{\lambda}{a}$. Next,

$$|l_\nu| = \left| \frac{W_1(r_\nu)}{W'(r_\nu)} \right| \sim \frac{l!\,|\lambda|}{k+l} |r_\nu|^{1-l} \quad \text{as} \quad |\lambda| \to \infty,$$

$$\left| \frac{l_\nu}{\mathrm{re}(r_\nu - a)} \right| \sim \frac{|a|}{(k+l)\lambda_\nu} \left(l! \left| \frac{\lambda}{a} \right| \right)^{k/(k+l)}, \qquad K^* = K^*(\lambda) \sim B|\lambda|^{k/(k+l)} \quad (|\lambda| \to \infty),$$

where $B = \frac{1}{k+l} |a|^{l/(k+l)} (l!)^{k/(k+l)} \sum_{\nu=1}^{k+l} \frac{1}{\lambda_\nu}$. Writing $\varphi(x) = MK^*(x)/|x| l!$ we have $\lim_{|x| \to \infty} \varphi(x) = 0$. Since $\overline{\lim_n} |s_n n^{-a}| \leqslant \varphi(\lambda)$ for arbitrarily large $|\lambda| \geqslant A$, we obtain $\lim_n s_n n^{-a} = 0$.

In the general case we observe that the sequence with terms $t_n^* = t_n - s\frac{n!}{\Gamma(n-a+1)}$ is the Hausdorff transform of the sequence $s_n^* = s_n - s\,w(a)\frac{n!}{\Gamma(n-a+1)}$ for $n \geqslant k$, corresponding to $\mu_n = \frac{1}{w(n)}$. It is easy to see that $\lim_n t_n^* n^{-a} = 0$ and that the sequence $\{n^{l-a} \Delta^l s_n^*\}$ is bounded. Whence $\lim_n s_n^* n^{-a} = \lim_n s_n n^{-a} - s\,w(a) = 0$ by the proved part of the theorem.

3. We shall now prove analogous theorems in the case of functional linear difference equations and differential equations.

3.1. We assume that the functions considered in this section are complex and defined for $x \geqslant a$.

3.1.1. *Let $U(\varphi_1, \varphi_2, \ldots, \varphi_\nu) \neq 0$ for $x \geqslant a + \nu - 1$ and $U(1, \varphi_1, \ldots, \varphi_\nu) \neq 0$ for $x \geqslant a + \nu$ $(\nu = 1, 2, \ldots, k)$, where*

$$U(f_1, f_2, \ldots, f_\nu) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_\nu(x) \\ f_1(x-1) & f_2(x-1) & \ldots & f_\nu(x-1) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(x-\nu+1) & f_2(x-\nu+1) & \ldots & f_\nu(x-\nu+1) \end{vmatrix}.$$

If

$$u_1(x) = \frac{1}{\varphi_1(x)},\quad u_\nu(x) = \frac{U(1,\varphi_1,\ldots,\varphi_{\nu-1})}{U(\varphi_1,\varphi_2,\ldots,\varphi_\nu)}\ \textit{for}\ \nu = 2,3,\ldots,k,\ x \geqslant a+\nu-1$$

then $\Delta u_\nu(x) \neq 0$ *for* $\nu = 1,2,\ldots,k$ *and* $x \geqslant a+\nu-1$. *The relations*

$$(9)\quad y_\nu(x) + \frac{u_\nu(x-1)}{\Delta u_\nu(x-1)}\Delta y_\nu(x-1) = y_{\nu+1}(x),\ \nu = 1,2,\ldots,k\ \textit{and}\ x \geqslant a+\nu$$

imply

$$y_{\nu+1}(x) = \frac{U(y_1,\varphi_1,\ldots,\varphi_\nu)}{U(1,\varphi_1,\ldots,\varphi_\nu)},\qquad \nu = 1,2,\ldots,k\quad \textit{and}\quad x \geqslant a+\nu.$$

For the proof we use 2.1.1 with $a_m^{(\nu)} = \varphi_\nu(x-n+m)$, $x_m^{(\nu)} = y_\nu(x-n+m)$, $m = 0,1,2,\ldots,n$ and $\nu = 1,2,\ldots,k$.

We shall use the following theorem:

THEOREM B_1. *If conditions* 1) *and* 2) *of theorem* B, I, *are satisfied then the inequality* $\overline{\lim}_{x\to\infty}|s(x)| \leqslant M$ *implies* $\overline{\lim}_{x\to\infty}|f(x)/g(x)| \leqslant MK$.

3.1.2. *Suppose that conditions* 1), 2) *and* 3) *of* 3.1.1, I, *are satisfied and furthermore*

4) $\overline{\lim}_{x\to\infty}|s(x)| \leqslant M$.

Then there exists a function $\bar{y}(x)$ *satisfying for* $x \geqslant a+1$ *the difference equation* (12), I, *and such that* $\overline{\lim}_{x\to\infty}|\bar{y}(x)| \leqslant MK$. *If* $g(x)$ *satisfies* 1a) *and* $y^*(x)$ *is a solution of* (12), I, *bounded in some interval* $\langle x_0, x_0+1\rangle$ (*with* $x_0 \geqslant a$) *in which* $g(x)$ *and* $\bar{y}(x)$ *are bounded, then* $\overline{\lim}_{x\to\infty}|y^*(x)| \leqslant MK$.

After theorem B_1 the proof is similar to that of 3.1.1, I.

THEOREM 1B. *Let*

$$L[y(x)] = y(x) + \lambda_1(x)\Delta y(x-1) + \ldots + \lambda_k(x)\Delta^k y(x-k)$$

and suppose that the homogeneous difference equation $L(y) = 0$ *has the solutions* $\varphi_\nu(x)$, $\nu = 1,2,\ldots,k$, *linearly independent, for which*

1) $U(\varphi_1,\varphi_2,\ldots,\varphi_\nu) \neq 0$, $x \geqslant a+\nu-1$,

2a) $|u_\nu(x)| \to \infty$ *as* $x\to\infty$ *and* $\sum_{j=1}^{[x]}|\Delta u_\nu(x-j)| \leqslant K|u_\nu(x)|$

or

2b) $u_\nu(x) \to 0$ *as* $x\to\infty$ *and* $\sum_{j=0}^{\infty}|\Delta u_\nu(x+j)| \leqslant K|u_\nu(x)|$,

3) $u_\nu(x) \neq 0$ *for* $x \geqslant a+\nu-1$.

Furthermore suppose that

4) *there exists* $X \geqslant a$ *such that in every interval* $\langle a_1, b_1\rangle$, *where* $X \leqslant a_1 < b_1 < \infty$, *we have the inequalities* $|\Delta u_\nu(x)| \geqslant \varepsilon$ ($\nu = 1,2,\ldots,k-1$) *and* $|u_\nu(x)| \leqslant A$ ($\nu = 1,2,\ldots,k$) *with some* $\varepsilon > 0$ *and* $A < \infty$ (*depending on* a_1 *and* b_1).

5) $\overline{\lim}_{x\to\infty}|f(x)| \leqslant M$.

Then there exists a solution $\bar{y}(x)$ *of the difference equation*

$$(10)\qquad L(y) = f(x)\quad \textit{for}\quad x \geqslant a+k,$$

such that $\overline{\lim}_{x\to\infty}|\bar{y}(x)| \leqslant MK^k$. *In case* 2a) *each solution* $y^*(x)$ *of* (10), *bounded in every finite interval* $\langle x_0, x_1\rangle$ *with large* $x_0, x_1 \geqslant \max(a, X)$, *satisfies this inequality.*

For the proof we write system (9) in the form

$$\Delta y_\nu(x-1) + \frac{\Delta u_\nu(x-1)}{u_\nu(x-1)}[y_\nu(x) - y_{\nu+1}(x)] = 0,\quad \nu = 1,2,\ldots,k,\quad x \geqslant a+\nu,$$

and put $y_1(x) = y(x)$, $y_{k+1}(x) = f(x)$. Just as in the proof of theorem 1A we show, using 3.1.2, that there exists a function $\bar{y}_\nu(x)$ satisfying the above difference equation with index ν and such that $\overline{\lim}_{x\to\infty}|\bar{y}_\nu(x)| \leqslant MK^{k-\nu+1}$ ($\nu = 1,2,\ldots,k$). We set $\bar{y}(x) = \bar{y}_1(x)$. Let us remark that in the proof of this part of the theorem we have not used condition 4).

Suppose now that condition 2a) is fulfilled and that $y^*(x)$ is a solution of (10), bounded in the interval $\langle x_0, x_1\rangle$ with $\max(a, X) \leqslant x_0 < x_1 - k < \infty$. By hypothesis it follows that in this case the functions $y_\nu(x)$, $\nu = 1,2,\ldots,k$, where $y_1(x) = y^*(x)$, are bounded in the intervals $\langle x_0+\nu-1, x_1\rangle$. Using 3.1.2 successively to the equations in question for $\nu = k, k-1,\ldots,1$, we obtain $\overline{\lim}_{x\to\infty}|y_1(x)| = \overline{\lim}_{x\to\infty}|y^*(x)| \leqslant MK^k$.

3.1.3. *Let* $W_j(z) = \sum_{\nu=0}^{k_j}\lambda_\nu^{(j)}\binom{z}{\nu}$, $L_j(y) = \sum_{\nu=0}^{k_j}\lambda_\nu^{(j)}\binom{x}{\nu}\Delta^\nu y(x-\nu)$ *for* $j = 0,1,2$; $W_j(z)$ *is then the characteristic polynomial of the difference equation* $L_j(y) = 0$.

If $L_0(y) = L_1(y) + L_2(y)$, *then* $W_0(z) = W_1(z) + W_2(z)$, *if* $L_0(y) = L_2[L_1(y)]$, *then* $W_0(z) = W_1(z)W_2(z)$.

The additive property is obvious. We observe that for the functions $y(x)$ representable by their Newton-Gregory series both relations follow almost immediately from (16), I. In the general case it suffices to prove the latter for $W_2(z) = a+bz$, $b \neq 0$. We obtain

$$L_0(y) = L_2[L_1(y)] = aL_1(y) + bx\sum_{\nu=0}^{k_1}\lambda_\nu^{(1)}\Delta\left[\binom{x-1}{\nu}\Delta^\nu y(x-1-\nu)\right]$$

$$= aL_1(y) + bx\lambda_0^{(1)}\Delta y(x-1) + bx\sum_{\nu=1}^{k_1}\lambda_\nu^{(1)}\left[\binom{x-1}{\nu}\Delta^{\nu+1}y(x-1-\nu) + \binom{x-1}{\nu-1}\Delta^\nu y(x-\nu)\right].$$

Computing the coefficient of $\binom{x}{\nu}\Delta^\nu y(x-\nu)$ we obtain

$$\lambda_\nu^{(0)} = a\lambda_\nu^{(1)} + b\nu(\lambda_{\nu-1}^{(1)}+\lambda_\nu^{(1)}) \quad \text{for} \quad \nu = 0,1,\ldots,k+1,$$

where we set $\lambda_{-1}^{(1)} = \lambda_{k+1}^{(1)} = 0$. Since $\lambda_\nu^{(j)} = \Delta^\nu W_j(0)$, we have

$$\Delta^\nu W_0(0) = W_2(0)\Delta^\nu W_1(0) + \nu\Delta W_2(0)\Delta^\nu W_1(1) = \Delta^\nu W_1(0) W_2(0)$$

for $\nu = 0,1,\ldots,k+1$, and there follows $W_1(z)W_2(z) = W_0(z)$.

3.1.4. *If*

1) $\lim_n a_n^{(j\nu)} = a^{(j\nu)}$, *where* $|a^{(j\nu)}| < \infty$, *for* $j, \nu = 1,2,\ldots,k$,

2) *the matrices* $(a^{(j\nu)})$ *are not singular* $(j,\nu = 1,\ldots,m$ *for* $m = 1,\ldots,k)$,

3) $|\sum_{\nu=1}^{k} a_n^{(j\nu)} c_n^{(\nu)}| \leqslant M$ *for* $j = 1,2,\ldots,k$ *and* $n = 0,1,2,\ldots$,

then the sequences $\{c_n^{(\nu)}\}$ *are bounded* $(\nu = 1,2,\ldots,k)$.

We prove the lemma by induction. For $k = 1$ the assertion is obvious. We assume that it is true for $k-1 \geqslant 1$ and prove indirectly that the sequence $\{c_n^{(k)}\}$ is bounded. If $\lim_m |c_m^{(k)}| = \infty$, where $\{c_m^{(k)}\}$ is a subsequence of $\{c_n^{(k)}\}$, then we obtain

$$\lim_m \sum_{\nu=1}^{k-1} a_m^{(j\nu)} \frac{c_m^{(\nu)}}{c_m^{(k)}} = -a^{(jk)}, \quad j = 1,2,\ldots,k.$$

Whence in virtue of the hypothesis the sequences $\{c_m^{(\nu)}/c_m^{(k)}\}$, $\nu = 1,2,\ldots,k-1$, are bounded. Let $\lim c_{m_p}^{(\nu)}/c_{m_p}^{(k)} = s_\nu$ with some sequence of indices $\{m_p\}$ tending to ∞, and $\nu = 1,2,\ldots,k-1$. We find that the values $x_\nu = s_\nu$ for $\nu = 1,2,\ldots,k-1$ and $x_k = 1$ satisfy the system of equations

$$\sum_{\nu=1}^{k} a^{(j\nu)} x_\nu = 0 \quad \text{for} \quad j = 1,2,\ldots,k$$

in contradiction to 2). Then $\overline{\lim_n} |\sum_{\nu=1}^{k-1} a_n^{(j\nu)} c_n^{(\nu)}| \leqslant M + \overline{\lim_n} |a_n^{(jk)} c_n^{(k)}| \leqslant M_1$ for $j = 1,2,\ldots,k$, and the boundedness of $\{c_n^{(1)}\}, \{c_n^{(2)}\}, \ldots, \{c_n^{(k-1)}\}$ follows now from the hypothesis.

3.1.5. *Suppose that*

1) *the functions* $c_\nu(x)$, $\nu = 1,2,\ldots,k$ *are periodic, with a period* $\omega = 1$,

2) *the functions* $f_\nu(x)$, $\nu = 1,2,\ldots,k$ *are continuous for* $x \in \langle x_0, x_0+k\rangle$,

3) $U(f_1, f_2, \ldots, f_\nu) \neq 0$ *for* $x \in \langle x_0+\nu-1, x_0+\nu\rangle$, $\nu = 1,2,\ldots,k$,

4) $|\sum_{\nu=1}^{k} c_\nu(x) f_\nu(x)| \leqslant M < \infty$ *for* $x \in \langle x_0, x_0+k\rangle$.

Then the functions $c_\nu(x)$ *are bounded* $(\nu = 1,2,\ldots,k)$.

Let $\{x_n\}$ be any convergent sequence from the interval $\langle x_0, x_0+1\rangle$. Using 3.1.4 for $c_n^{(\nu)} = c_\nu(x_n)$, $a_n^{(j\nu)} = f_\nu(x_n+j-1)$, we obtain $a^{(j\nu)} = f_\nu(\bar{x}+j-1)$, where $\bar{x} = \lim_n x_n$, and $\overline{\lim_n} |c_\nu(x_n)| < \infty$ for $\nu = 1,2,\ldots,k$. From this we immediately obtain the desired result.

THEOREM 2B. *Let* $W(z)$ *and* $W_1(z)$ *be polynomials defined in the theorem* 2A *and suppose that* $\overline{\lim_{x\to\infty}} |f(x)| x^{-c} \leqslant M$ *with given* c *real. We consider the difference equation*

(11) $$L(y) = L^*(f) \quad \textit{for} \quad x \geqslant a,$$

where $L(y) = \sum_{\nu=0}^{k} \lambda_\nu \binom{x}{\nu} \Delta^\nu y(x-\nu)$, $L^*(y) = \sum_{\nu=0}^{l} \eta_\nu \binom{x}{\nu} \Delta^\nu y(x-\nu)$, $\lambda_\nu = \Delta^\nu W(0)$, $\eta_\nu = \Delta^\nu W_1(0)$.

a) *If* $\mathrm{re}\, r_\nu \neq c$ *for* $\nu = 1,2,\ldots,k$ (r_ν *are defined in theorem* 2A) *then* (11) *has a solution* $\bar{y}(x)$ *such that* $\overline{\lim_{x\to\infty}} |\bar{y}(x)| x^{-c} \leqslant MK^*$, *where* K^* *does not depend on* $f(x)$.

b) *If* $\mathrm{re}\, r_\nu < c$, $\nu = 1,2,\ldots,k$, *then every solution* $\hat{y}(x)$ *of* (11) *bounded in the interval* $\langle x_0, x_0+k\rangle$ *with some* $x_0 \geqslant \max(0, a, X)$, *satisfies the same inequality as* $\bar{y}(x)$. (*The value* X *is defined in the proof.*)

c) *If* $\mathrm{re}\, r_j \geqslant c$ *with some* $j \leqslant k$, *then for every* K *there exists a solution* $y^*(x)$ *of* (11) *such that* $\overline{\lim_{x\to\infty}} |y^*(x)| x^{-c} > K$.

The proof is in principle similar to that of theorem 2A. In case a) we observe that under the hypothesis $W(z) = (r-z)^k$ and $W_1(z) = 1$ difference equation (11) is by 3.2.1, I, and 3.1.3 equivalent to the system of difference equations

$$ry_\nu(x) - x\Delta y_\nu(x-1) = y_{\nu+1}(x), \quad \nu = 1,2,\ldots,k \quad \text{and} \quad x \geqslant a+\nu,$$

where $y_1(x) = y(x)$, $y_{k+1}(x) = f(x)$. Making the substitution

$$y_\nu(x) = \frac{\Gamma(x-r+c+1)}{\Gamma(x-r+1)} z_\nu(x) \quad \nu = 1,2,\ldots,k+1, \quad \text{for} \quad x \geqslant A,$$

where $A = \max(a, r, r-c)$ if r is a real number, and $A = a$ otherwise, we obtain the system of difference equations

$$z_\nu(x) - \frac{x}{r-c} \Delta z_\nu(x-1) = \frac{1}{r-c} \cdot \frac{x-r+c}{x-r} z_{\nu+1}(x), \quad \nu = 1,2,\ldots,k \quad \text{and} \quad x \geqslant A+\nu.$$

Using 3.1.2. and 3.1.2, I, we prove, just as in the proof of theorem 2A, that in the considered case there exists a solution $\bar{y}(x)$ of (11) satisfying the inequality $\overline{\lim_{x\to\infty}} |\bar{y}(x)| x^{-c} \leqslant M \frac{1}{|\mathrm{re}\, r - c|^k}$.

In the general case let $L_\nu(y) = \sum_{j=0}^{p_\nu} \lambda_j^{(\nu)} \binom{x}{j} \Delta^j y(x-j)$, where $\lambda_j^{(\nu)} = \Delta^j W^{(\nu)}(0)$, $W^{(\nu)}(z) = (r_\nu - z)^{p_\nu}$ and let $L_\nu^*(y)$ be defined by the equalities $L_\nu^*(L_\nu(y)) = L(y)$, $\nu = 1, 2, \ldots, k$. From the proved part of the theorem it follows that the difference equation $L_\nu(y) = f(x)$ for $1 \leqslant \nu \leqslant k$ has a solution $\bar{y}_\nu(x)$ such that $\overline{\lim_{x\to\infty}} |\bar{y}_\nu(x)| x^{-c} \leqslant MK_\nu^{p_\nu}$, where $K_\nu = 1/|\operatorname{re} r_\nu - c|$.

If $\bar{y}(x) = l_0 f(x) + \sum_{\nu=1}^{k} l_\nu \bar{y}_\nu(x)$, then $\overline{\lim_{x\to\infty}} |\bar{y}(x)| x^{-c} \leqslant MK^*$, where $K^* = |l_0| + \sum_{\nu=1}^{k} |l_\nu| K_\nu^{p_\nu}$ and

$$L(\bar{y}) = l_0 L(f) + \sum_{\nu=1}^{k} l_\nu L(\bar{y}_\nu) = l_0 L(f) + \sum_{\nu=1}^{k} l_\nu L_\nu^*(f) = L^*(f),$$

because $W(z)/(r_\nu - z)^{p_\nu}$ is the characteristic polynomial of the difference equation $L_\nu^*(y) = 0$ by 3.1.3.

b) If $\hat{y}(x)$ satisfies (11) then

$$\hat{y}(x) = \bar{y}(x) + \sum_{\nu=1}^{k} c_\nu(x) \varphi_\nu(x),$$

where $c_\nu(x)$ are periodic functions with a period $\omega = 1$ and

$$\varphi_\nu(x) = \frac{d^j}{dr^j} \frac{\Gamma(x+1)}{\Gamma(x-r+1)}\bigg|_{r=r_\nu}, \qquad 0 \leqslant j \leqslant k_\nu - 1.$$

We choose X so that $W(x) \neq 0$ and that $\bar{y}(x) x^{-c}$ is bounded for $x \geqslant X$. Supposing that $\hat{y}(x)$ is bounded in $\langle x_0, x_0 + k\rangle$ for some $x_0 \geqslant \max(0, a, X)$ we see that hypothesis of 3.1.5 is fulfilled with $f_\nu(x) = \varphi_\nu(x)$ since, as may be computed, we have $U(\varphi_1, \ldots, \varphi_\nu) = \lambda_\nu \frac{\Gamma(x+1)}{\prod_{j=1}^{\nu} \Gamma(x - r_j + 1)}$, where λ_ν depends on $r_1, r_2, \ldots, r_\nu$ $(1 \leqslant \nu \leqslant k)$. Whence $c_\nu(x)$ are bounded by 3.1.5.

Assuming $\operatorname{re} r_\nu < c$, $\nu = 1, 2, \ldots, k$, we have $\lim_{x\to\infty} \varphi_\nu(x) x^{-c} = 0$ and there follows $\overline{\lim_{x\to\infty}} |\hat{y}(x)| x^{-c} \leqslant MK^*$.

c) The proof is similar to that of theorem 2A, case c).

Just as in the case of sequences we may state the following corollary to theorem 2B: Suppose that $f(x) \sim sx^a$ as $x \to \infty$ instead of $\overline{\lim_{x\to\infty}} |f(x)| x^{-c} \leqslant M$, with complex a. In the case $\operatorname{re} r_\nu \neq \operatorname{re} a$, $\nu = 1, 2, \ldots, k$, there exists a solution $y(x)$ of (11) such that $y(x) \sim s \frac{W_1(a)}{W(a)} x^a$ as $x \to \infty$. If $W(a) \neq 0$, then the condition $\operatorname{re} r_\nu < \operatorname{re} a$, $\nu = 1, 2, \ldots, k$, is necessary and sufficient for every solution $\hat{y}(x)$ of (11) such that $\hat{y}(x)$ is bounded in some interval $\langle x_0, x_0 + k\rangle$ for large x_0 to satisfy the above asymptotic relation.

3.1.6. *Suppose that* $f(x) = \Gamma(x+1)/\Gamma(x-r+1)$ *for* $x \geqslant x_0 \geqslant 0$, $f(x) = 0$ *for* $x < x_0$ *and that* $c_k(x)$ *is defined by* (14), I. *Then the hypothesis* $\operatorname{re} r > -1$ *implies* $c_k(x) \sim x^r \Big/ \binom{k+r}{k}$ *as* $x \to \infty$ *and the hypothesis* $\operatorname{re} r \leqslant -1 < c$ *implies* $\lim_{x\to\infty} c_k(x) x^{-c} = 0$.

For the proof it suffices to apply theorem 2B since $L(c_k(x)) = f(x)$ for $W(z) = \binom{z+k}{k}$, the transform $c_k(x)$ is bounded in every finite interval $\langle x_0, x_1\rangle$ with $-1 < x_0 < x_1$ and $f(x) \sim x^r$ as $x \to \infty$.

THEOREM 3B. *Suppose that*

1) $\operatorname{re} a > -1$,

2) $f(x) = 0$ *for* $x < 0$, *the transform* $c_k(x)$ *is defined by* (14), I, *and* $c_k(x) \sim sx^a$ *as* $x \to \infty$,

3) $\overline{\lim_{x\to\infty}} |x^{l-a} \Delta^l f(x)| \leqslant M$.

Then $f(x) \sim s\binom{k+a}{k} x^a$ *as* $x \to \infty$.

Proof. Suppose first that $s = 0$ and $L(y)$ and $L^*(y)$ are defined as in theorem 2B with $W(z) = \binom{z+k}{k}\binom{z}{l} + \lambda$, $W_1(z) = \lambda \binom{z+k}{k}$. Just as in the proof of theorem 4A we show, by the use of theorem 2B, that for $|\lambda| \geqslant \lambda_0$ there exists a function $\bar{y}(x)$ satisfying the difference equation $L(y) = L^*(p(x))$ $(x > k-1)$, where $p(x) = \frac{1}{\lambda}\binom{x}{l} \Delta^l f(x-l) + c_k(x)$ for $x > -1$, and such that $\overline{\lim_{x\to\infty}} |\bar{y}(x) x^{-a}| \leqslant \frac{MK^*}{|\lambda| l!}$.

Using 3.1.3 and observing that $L^*(c_k(x)) = \lambda f(x)$ for $x > k-1$, we easily see that the difference equation $L(y) = L^*(p)$ is satisfied also by the function $f(x)$; hence for $x > k-1$ we have

$$f(x) = \bar{y}(x) + \sum_{\nu=1}^{k+l} d_\nu(x) \frac{\Gamma(x+1)}{\Gamma(x - r_\nu + 1)},$$

where $d_\nu(x)$ are periodic with a period $\omega = 1$ and r_ν are zeros of the polynomial $W(z)$.

We choose $X > k-1$ so that $W(x) \neq 0$ and that the functions $c_k(x)x^{-a}$ and $\bar{y}(x)x^{-a}$ are bounded for $x \geqslant X$. Hence the function $f(x) = \frac{1}{\lambda} L^*\big(c_k(x)\big)$ is bounded in every interval $\langle x_0, x_1 \rangle$ with $X+k \leqslant x_0 < x_1 < \infty$ and in virtue of 3.1.5 we infer, just as in the proof of theorem 2B, that every $d_\nu(x)$ is bounded.

We prove now that $d_\nu(x) = 0$ if $\operatorname{re} r_\nu > \operatorname{re} a$. Suppose that, for example, $d_1(\xi) \neq 0$ with some $\xi \in \langle 0, 1 \rangle$ and that r_1, r_2 are defined as in the proof of theorem 4A. We have

$$c_k(x) = y^*(x) + \sum_{\nu=1}^{k+l} d_\nu(x) t_\nu(x), \qquad x > -1,$$

where $y^*(x)$ and $t_\nu(x)$ are the transforms of $\varphi(x)$ and $\varphi_\nu(x)$ respectively, defined by (14), I. We take here

$$\varphi(x) = \begin{cases} \bar{y}(x) & \text{for} \quad x > k-1, \\ f(x) & \text{for} \quad x \leqslant k-1, \end{cases}$$

$$\varphi_\nu(x) = \begin{cases} \dfrac{\Gamma(x+1)}{\Gamma(x-r_\nu+1)} & \text{for} \quad x > k-1, \\ 0 & \text{for} \quad x \leqslant k-1. \end{cases}$$

Hence, as in the proof of theorem 4A, we obtain by 3.1.6

$$\overline{\lim_n} \frac{|c_k(\xi+n)|}{(\xi+n)^\sigma} = \overline{\lim_n} \frac{|d_1(\xi)t_1(\xi+n)+d_2(\xi)t_2(\xi+n)|}{(\xi+n)^\sigma}$$
$$= \overline{\lim_n} \left| \frac{d_1(\xi)}{\binom{r_1+k}{k}} (\xi+n)^{r_1^i} + \frac{d_2(\xi)}{\binom{r_2+k}{k}} (\xi+n)^{-r_1^i} \right| > 0$$

and $\overline{\lim_{x\to\infty}} |c_k(x)x^{-a}| = \infty$ in contradiction to the hypothesis.

Similarly we prove the case of $r_1 = \sigma$ real.

This concludes the proof in the case of $s = 0$. In the general case we observe that if $c_k^*(x)$ is the transform of the function $f^*(x) = f(x) - s\binom{k+a}{k}\dfrac{\Gamma(x+1)}{\Gamma(x-a+1)}$ for $x \geqslant 0$ and $f^*(x) = 0$ for $x < 0$, then $\lim_{x\to\infty} c_k^*(x)x^{-a} = \lim_{x\to\infty} c_k(x)x^{-a} - s = 0$ by 3.1.6 and by the hypothesis. Since the function $x^{l-a}\Delta^l f^*(x)$ is bounded for large x, we infer from the proved part of theorem that $\lim_{x\to\infty} f^*(x)x^{-a} = \lim_{x\to\infty} f(x)x^{-a} - s\binom{k+a}{k} = 0$.

3.2. We assume that the functions considered in this section are complex and defined for $x \geqslant a$ and that they have for those x the k-th continuous derivative.

3.2.1. *Suppose that* $W(\varphi_1, \varphi_2, \ldots, \varphi_\nu) \neq 0$ and $W(1, \varphi_1, \ldots, \varphi_\nu) \neq 0$ *for* $x \geqslant a$ *and* $\nu = 1, 2, \ldots, k$. $W(f_1, f_2, \ldots, f_n)$ *denotes here Wronskian of the functions* $f_1(x), f_2(x), \ldots, f_n(x)$.

If $w_1(x) = \dfrac{1}{\varphi_1(x)}$, $w_\nu(x) = \dfrac{W(1, \varphi_1, \ldots, \varphi_{\nu-1})}{W(\varphi_1, \varphi_2, \ldots, \varphi_\nu)}$ *for* $\nu = 2, 3, \ldots, k$, *then* $w_\nu'(x) \neq 0$ *for* $x \geqslant a$ *and* $\nu = 1, 2, \ldots, k$; *taking*

$$(12) \qquad y_\nu(x) + \frac{w_\nu(x)}{w_\nu'(x)} y_\nu'(x) = y_{\nu+1}(x) \quad \textit{for} \quad \nu = 1, 2, \ldots, k \quad \textit{and} \quad x \geqslant a,$$

we obtain

$$y_{\nu+1} = \frac{W(y_1, \varphi_1, \ldots, \varphi_\nu)}{W(1, \varphi_1, \ldots, \varphi_\nu)}, \qquad \nu = 1, 2, \ldots, k.$$

For the proof we show that if $F(x) = \dfrac{W(y, \varphi_1, \ldots, \varphi_n)}{W(z, \varphi_1, \ldots, \varphi_n)}$, then

$$(13) \qquad F'(x) = -\frac{W(y, z, \varphi_1, \ldots, \varphi_n) W(\varphi_1, \ldots, \varphi_n)}{W^2(z, \varphi_1, \ldots, \varphi_n)}.$$

Let $A = W(y, z, \varphi_1, \ldots, \varphi_n)$; then by (2) we have

$$F'(x)W^2(z, \varphi_1, \ldots, \varphi_n) = W'(y, \varphi_1, \ldots, \varphi_n)W(z, \varphi_1, \ldots, \varphi_n) -$$
$$- W(y, \varphi_1, \ldots, \varphi_n)W'(z, \varphi_1, \ldots, \varphi_n)$$
$$= A_{n+1,2}A_{n+2,1} - A_{n+2,2}A_{n+1,1} = -AA_{n+1,n+2;1,2}.$$

We prove 3.2.1 by induction. For $\nu = 1$ is $w_1'(x) = -\varphi_1'(x)/\varphi_1^2(x) \neq 0$ and

$$y_2(x) = \frac{1}{w_1'(x)} \cdot \frac{d}{dx}(y_1 w_1) = \frac{W(y, \varphi_1)}{W(1, \varphi_1)}.$$

Let $y_\nu(x) = \dfrac{W(y_1, \varphi_1, \ldots, \varphi_{\nu-1})}{W(1, \varphi_1, \ldots, \varphi_{\nu-1})}$; then by (13) for $n = \nu-1$, $z = \varphi_\nu$ is

$$\frac{d}{dx}(y_\nu w_\nu) = (-1)^{\nu-1} \frac{d}{dx} \frac{W_1(y_1, \varphi_1, \ldots, \varphi_{\nu-1})}{W(\varphi_\nu, \varphi_1, \ldots, \varphi_{\nu-1})}$$
$$= -\frac{W(y_1, \varphi_1, \ldots, \varphi_\nu) W(\varphi_1, \ldots, \varphi_{\nu-1})}{W^2(\varphi_1, \ldots, \varphi_\nu)};$$

taking $y_\nu(x) = 1$ in the above formula we obtain

$$w_\nu'(x) = -\frac{W(1, \varphi_1, \ldots, \varphi_\nu) W(\varphi_1, \ldots, \varphi_{\nu-1})}{W^2(\varphi_1, \ldots, \varphi_\nu)} \neq 0.$$

Hence $y_{\nu+1}(x) = \dfrac{1}{w_\nu'(x)} \cdot \dfrac{d}{dx}(y_\nu w_\nu) = \dfrac{W(y_1, \varphi_1, \ldots, \varphi_\nu)}{W(1, \varphi_1, \ldots, \varphi_\nu)}$.

In the sequel we shall use the following theorem:

THEOREM C_1. *If conditions* 1), 2), 3) *of theorem* C, I, *are satisfied, then the inequality* $\overline{\lim_{x\to x_0}}|s(x)| \leqslant M$ *implies* $\overline{\lim_{x\to x_0}}|f(x)/g(x)| \leqslant MK$.

3.2.2. *Suppose that conditions* 1), 2), 3), 4) *of* 3.3.1, I, *are satisfied and furthermore*

5) $\overline{\lim_{x\to x_0}}|s(x)| \leqslant M$.

Then there exists a function $\bar{y}(x)$ *satisfying* (17), I, *and such that* $\overline{\lim_{x\to x_0}}|\bar{y}(x)| \leqslant MK$. *If* $g(x)$ *satisfies* 1a), *then every solution* $y(x)$ *of* (17), I, *satisfies the same inequality as* $\bar{y}(x)$.

The proof is like that of 3.3.1, I. We observe that if $g(x) \neq 0$ in I, the inequalities in conditions 1a) and 1b) in theorems C and C_1 may be replaced by $\overline{\lim_{x\to x_0}}\frac{1}{|g(x)|}\Big|\int_y^x |g'(t)|\,dt\Big| \leqslant K$ and $\overline{\lim_{x\to x_0}}\frac{1}{|g(x)|}\Big|\int_x^{x_0} |g'(t)|\,dt\Big| \leqslant K$ respectively.

THEOREM 1C. *Let* $L(y) = y + \sum_{\nu=1}^{k}\lambda_\nu(x)y^{(\nu)}$ *and let* $I \in (a, b)$ *denote some neighbourhood of the point* x_0. *Suppose that the homogeneous differential equation* $L(y) = 0$ *has in* (a, b) *the solutions* $\varphi_\nu(x)$, $\nu = 1, 2, \ldots, k$, *linearly independent, for which*

1a) $|w_\nu(x)| \to \infty$ *as* $x \to x_0$ *and* $\Big|\int_y^x |w_\nu'(t)|\,dt\Big| \leqslant K|w_\nu(x)|$ *for* $x, y \in I$,

or

1b) $w_\nu(x) \to 0$ *as* $x \to x_0$ *and* $\Big|\int_x^{x_0} |w_\nu'(t)|\,dt\Big| \leqslant K|w_\nu(x)|$ *for* $x \in I$,

2) $w_\nu(x) \neq 0$ *for* $x \in (a, b)$,

3) $w_\nu'(x)f(x)$ *are continuous in* (a, b).

If $\overline{\lim_{x\to x_0}}|f(x)| \leqslant M$, *then there exists in* (a, b) *a solution* $\bar{y}(x)$ *of the differential equation*

$$L(y) = f(x), \tag{14}$$

such that $\overline{\lim_{x\to x_0}}|\bar{y}(x)| \leqslant MK^k$. *If* $\lim_{x\to x_0}|w_\nu(x)| = \infty$ *for* $\nu = 1, 2, \ldots, k$, *then every solution of* (14) *satisfies the same inequality as* $\bar{y}(x)$.

For the proof we write (12) in the form

$$y_\nu'(x) + \frac{w_\nu'(x)}{w_\nu(x)}[y_\nu(x) - y_{\nu+1}(x)] = 0, \qquad \nu = 1, 2, \ldots, k,$$

and reason as in the proofs of theorems 1A and 1B, using 3.2.2.

3.2.3. *Let* $W_j(z) = \sum_{\nu=0}^{k_j}\lambda_\nu^{(j)}\binom{z}{\nu}$, $L_j(y) = \sum_{\nu=0}^{k_j}\lambda_\nu^{(j)}\frac{x^\nu}{\nu!}y^{(\nu)}(x)$ *for* $j = 0, 1, 2$. *If* $L_0(y) = L_1(y) + L_2(y)$, *then* $W_0(z) = W_1(z)W_2(z)$, *if* $L_0(y) = L_2(L_1(y))$, *then* $W_0(z) = W_1(z)W_2(z)$.

In the case of functions representable by their Maclaurin expansions the above relations follow from the identity of Guderman (see 3.4.2, I). We prove the general case similarly to 3.1.3.

THEOREM 2C. *Let* $W(z)$, $W_1(z)$ *be polynomials defined in theorem* 2A *and let* r_ν *be zeros of the first polynomial. We consider the differential equation*

$$L(y) = L^*(f), \qquad x \geqslant a, \tag{15}$$

where $L(y) = \sum_{\nu=0}^{k}\lambda_\nu\frac{x^\nu}{\nu!}y^{(\nu)}$, $L^*(y) = \sum_{\nu=0}^{l}\eta_\nu\frac{x^\nu}{\nu!}y^{(\nu)}$, $\lambda_\nu = \Delta^\nu W(0)$, $\eta_\nu = \Delta^\nu W_1(0)$ *and suppose that* $f^{(l)}(x)$ *is continuous for* $x \geqslant a$ *and that* $\overline{\lim_{x\to\infty}}|f(x)|x^{-c} \leqslant M$.

a) *If* $\operatorname{re} r_\nu \neq c$ *for* $\nu = 1, 2, \ldots, k$, *then there exists a constant* K^* *which does not depend on* $f(x)$ *and a solution* $\bar{y}(x)$ *of* (15) *such that* $\overline{\lim_{x\to\infty}}|\bar{y}(x)|x^{-c} \leqslant MK^*$.

b) *If* $\operatorname{re} r_\nu < c$ *for* $\nu = 1, 2, \ldots, k$, *then every solution* $y(x)$ *of* (15) *satisfies the same inequality as* $\bar{y}(x)$.

c) *If* $\operatorname{re} r_j \geqslant c$ *with some* $j \leqslant k$, *then for every* K *there exists a solution* $y^*(x)$ *of* (15) *such that* $\overline{\lim_{x\to\infty}}|y^*(x)|x^{-c} > K$.

For the proof we observe that

$$L\big(x^a z(x)\big) = x^a \sum_{\nu=0}^{k}\Delta^\nu W(a)\frac{x^\nu}{\nu!}z^{(\nu)}(x) \tag{16}$$

for any complex a. Hence we may suppose that $c = 0$.

In case a) and under the hypothesis $W(z) = (r-z)^k$, $W_1(z) = 1$ we may apply theorem 1C with $x_0 = \infty$ and $f(x)r^k$ instead of $f(x)$, since in that case we have $w_\nu(x) = x^{-r}$ for $\nu = 1, 2, \ldots, k$. (See also Perron [5].) We observe that for $\varphi(x) = x^{-r}$ is

$$\lim_{x\to\infty}\frac{1}{\varphi(x)}\int_y^x |\varphi'(t)|\,dt = \left|\frac{r}{\operatorname{re} r}\right| \quad \text{if} \quad \operatorname{re} r < 0,$$

$$\lim_{x\to\infty}\frac{1}{\varphi(x)}\int_x^\infty |\varphi'(t)|\,dt = \left|\frac{r}{\operatorname{re} r}\right| \quad \text{if} \quad \operatorname{re} r > 0.$$

We conclude the proof similarly to that of theorems 2A and 2B by the use of the function x^r instead of $\Gamma(x+1)/\Gamma(x-r+1)$ and of 3.2.3.

COROLLARY. *Suppose that $f(x) \sim sx^a$ as $x \to \infty$ instead of $\overline{\lim_{x\to\infty}} |f(x)| x^{-c} \leqslant M$ in theorem 2C. In the case $\operatorname{re} r_\nu \neq \operatorname{re} a$, $\nu = 1, 2, \ldots, k$, it follows that there exists a function $\bar{y}(x)$ satisfying (15) and such that $\bar{y}(x) \sim s \frac{W_1(a)}{W(a)} x^a$ as $x \to \infty$. If $W(a) \neq 0$, then the condition $\operatorname{re} r_\nu < \operatorname{re} a$, $\nu = 1, 2, \ldots, k$, is necessary and sufficient for every solution $y(x)$ of (15) to satisfy the above asymptotic relation.*

For the proof we observe that in the case $\operatorname{re} r_\nu \neq \operatorname{re} a$, $\nu = 1, 2, \ldots, k$, the differential equation $L(y) = L^*(f - sx^a)$ has, by theorem 2C, a solution $\hat{y}(x)$ such that $\lim_{x\to\infty} \hat{y}(x) x^{-a} = 0$. It is easy to see that the function $\bar{y}(x) = \hat{y}(x) + s \frac{W_1(a)}{W(a)} x^a$ satisfies also (15) and we have $\lim_{x\to\infty} \bar{y}(x) x^{-a} = sW_1(a)/W(a)$. We conclude the proof just as in the case of sequences.

Let us remark that the Cesàro transform

$$C_k(x) = \frac{k}{x^k} \int_0^x (x-t)^{k-1} f(t)\,dt, \qquad x > 0,$$

satisfies the differential equation $L(y) = f(x)$ with $W(z) = \binom{z+k}{k}$. Hence it follows from the above corrollary that the relation $f(x) \sim sx^a$ implies $C_k(x) \sim sx^a \Big/ \binom{k+a}{k}$ as $x \to \infty$, if $\operatorname{re} a > -1$.

THEOREM 3C. *Suppose that*

1) *$f(x)$ has for $x \geqslant 0$ the continuous derivative of order $k+l$,*
2) *$\operatorname{re} a > -1$,*
3) *$\overline{\lim_{x\to\infty}} |x^{l-a} f^{(l)}(x)| \leqslant M$,*
4) *$C_k(x) = \frac{k}{x^k} \int_0^x (x-t)^{k-1} f(t)\,dt \sim sx^a$ as $x \to \infty$.*

Then $f(x) \sim s \binom{k+a}{k} x^a$ as $x \to \infty$.

Proof. Suppose first that $s = 0$ and that $L(y)$ and $L^*(y)$ are defined as in theorem 2C with $W(z) = \binom{z+k}{k}\binom{z}{l} + \lambda$, $W_1(z) = \lambda \binom{z+k}{k}$. Just as in the proofs of theorems 3A and 3B we show that there exists a function $\bar{y}(x)$ satisfying for $x > 0$ the differential equation $L(y) = L^*(p(x))$, where $p(x) = \frac{1}{\lambda} \cdot \frac{x^l}{l!} f^{(l)}(x) + C_k(x)$, and such that $\overline{\lim_{x\to\infty}} |\bar{y}(x) x^{-a}| \leqslant \frac{MK^*}{|\lambda| l!}$.

Since $L^*(C_k(x)) = \lambda f(x)$, we obtain $L(f) = L^*(p(x))$ by 3.2.3, whence

$$f(x) = \bar{y}(x) + \sum_{\nu=1}^{k+l} c_\nu x^{r_\nu} \quad \text{for} \quad x > 0,$$

where r_ν are zeros of the polynomial $W(z)$.

We prove now that the inequality $\operatorname{re} r_\nu > \operatorname{re} a$ implies $c_\nu = 0$. Suppose that, for example, $c_1 \neq 0$ and that r_1, r_2 are defined as in the proof of theorem 3A. Then

$$C_k(x) = y^*(x) + \sum_{\nu=1}^{k_1} c_\nu \frac{x^{r_\nu}}{\binom{k+r_\nu}{k}},$$

where $y^*(x)$ is the (C, k) transform of the function $\bar{y}(x) - \sum_{\nu=k_1+1}^{k+l} c_\nu x^{r_\nu}$ and r_ν are so ordered that $\operatorname{re} r_\nu > -1$ for $1 \leqslant \nu \leqslant k_1$ and $\operatorname{re} r_\nu \leqslant -1$ otherwise. (If $k_1 = k+l$, the latter sum is equal to zero.)

Since $\overline{\lim_{x\to\infty}} |y^*(x) x^{-a}| \leqslant \frac{MK^*}{|\lambda| l!} K_1$ by theorem 2C, we have

$$\overline{\lim_{x\to\infty}} |C_k(x)| x^{-\sigma} = \overline{\lim_{x\to\infty}} \left| C_1 \frac{x^{\tau i}}{\binom{k+r_1}{k}} + C_2 \frac{x^{-\tau i}}{\binom{k+r_2}{k}} \right| > 0,$$

and just as in the proof of theorem 3A we infer that $\lim_{x\to\infty} f(x) x^{-a} = 0$.

In the general case it is easy to see that the function $C_k^*(x) = C_k(x) - sx^a$ is the (C, k) transform of the function $f^*(x) = f(x) - s\binom{k+a}{k} x^a$ (see (16)) and $\lim_{x\to\infty} C_k^*(x) x^{-a} = 0$. Observing that $x^{l-a} \frac{d^l x^a}{dx^l} = a(a-1)\ldots(a-l+1)$ we find by the proved part of theorem that $\lim_{x\to\infty} f^*(x) x^{-a} = \lim_{x\to\infty} f(x) x^{-a} - s\binom{k+a}{k} = 0$.

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