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# On the density of the equilibrium distributions of plane sets

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**I.** Let $D$ be a domain (bounded or not) in the $z$-plane and the set $F$ be its boundary. We assume that $F$ is of positive capacity(¹). A point $\zeta \in D$ being fixed, we consider

$$\inf_{\mu} \iint \log|z-a|^{-1} d\mu(a)\, d\mu(z) \quad \text{if} \quad \zeta = \infty,$$

or

$$\inf_{\mu} \iint (\log|z-a|^{-1} - 2\log|\zeta-a|^{-1})\, d\mu(a)\, d\mu(z) \quad \text{if} \quad \zeta \neq \infty.$$

$\mu$ varies over the class of all non-negative Radon measures whose supports are contained in $F$ and whose total mass is equal to 1. It is proved in [2] that by the above assumptions there exists a unique measure which realises the considered infimum. This measure is called the *equilibrium distribution of the set $F$ with respect to the point* $\zeta$(²) and we denote it by $\mu_\zeta$. Then the Green function(³) $g(z, \zeta)$ of the domain $D$ is expressed by

$$g(z, \infty) = \int \log|z-a|^{-1} d\mu_\infty(a) - \gamma_\infty \tag{1}$$

or

$$g(z, \zeta) = \log|z-\zeta|^{-1} - \int \log|z-a|^{-1} d\mu_\zeta(a) - \gamma_\zeta \tag{1'}$$

with respect to $\zeta = \infty$ or $\zeta \neq \infty$. The quantity $\gamma_\zeta$ does not depend on $z$ and is expressed by the formules

$$\gamma_\zeta = 0 \quad \text{if} \quad \infty \notin D, \qquad \gamma_\zeta = g(\zeta, \infty) \quad \text{if} \quad \infty \in D.$$

The details and elegant proofs are to be found in [5].

(¹) For the definition cf. [5] and [3].

(²) Cf. [2]. O. Frostman uses the term *masse du balayage* instead of *equilibrium distribution*.

(³) We mean here and throughout this paper the generalised (in Wiener's sense) Green function.

Let $e$ be a Borel subset of $F$. Denote by $\chi_e$ its characteristic function. Then

$$\mu_\zeta(e) = \int \chi_e(a)\, d\mu_\zeta(a)$$

is a harmonic function of the point $\zeta$ in $D$. It is the solution of the generalised Dirichlet problem for the domain $D$ with the boundary values $\chi_e(z)$. This function is identical with the *harmonic measure* of the set $e$ with respect to the point $\zeta$. The details are to be found in [5] and [3].

The object of this paper is the investigation of the density of the measure $\mu_\zeta$ with respect to the linear Lebesgue measure in the case when $F$ or its part is an analytic or rectifiable arc. In the proofs we shall use some properties of the harmonic measure and of the conformal mapping.

**II.** Consider first the case when $D$ is a simply connected domain bounded by a Jordan curve $F$. Assume also that $\zeta \neq \infty$. Then the function $w(z, \zeta)$ which maps conformly $D$ into the interior of the unit circle of the $w$-plane in such a way that the point $z = \zeta$ corresponds to $w = 0$, is expressed by the formula

$$w(z, \zeta) = (z-\zeta)\exp\left\{\int \log(z-a)^{-1} d\mu_\zeta(a) + \gamma_\zeta - i\alpha\right\}, \tag{2}$$

$\alpha$ being an arbitrary real constant. This formula follows immediately by (1′) and by the known relations between the Green and mapping function, viz.

$$g(z, \zeta) = \log|w(z, \zeta)|^{-1}.$$

Now we assume that a part of the boundary $F$ is an analytic arc $L$. Then the mapping function $w$ may be continued analytically onto $L$. The real part $g(z, \zeta)$ and the imaginary part $-h(z, \zeta)$ of the function $-\log w(z, \zeta)$ satisfy the Cauchy-Riemann relation

$$\frac{\partial g(z, \zeta)}{\partial_z s} = \frac{\partial h(z, \zeta)}{\partial_z n},$$

where $s$ and $n$ denote (by the convenient orientation of the arc $L$) the normal and the arc element on $L$.

It is proved in [5] that for every measurable set $e \subset L$ we have the equality

$$\mu_\zeta(e) = \int_e \frac{\partial g(z, \zeta)}{\partial_z n}\, d_z s. \tag{3}$$

In view of the definition of the functions $g$ and $h$ we obtain by (3) the expression for the difference of the argument of $w$ at the points of the boundary. We choose on $F$ two different points $\sigma$ and $\varrho$ and we introduce the notation $\Delta_\sigma^\varrho \arg w = \arg w(\varrho, \zeta) - \arg w(\sigma, \zeta)$. We have



$$|\Delta_\sigma^\varrho \arg w(z, \zeta)| = \left|\int_\sigma^\varrho d_z h(z, \zeta)\right| = \left|\int_\sigma^\varrho \frac{\partial g(z, \zeta)}{\partial n}\, d_z s\right| = 2\pi \mu_\zeta(l_{\sigma\varrho}), \tag{4}$$

where $l_{\sigma\varrho}$ denotes the arc leading from $\sigma$ to $\varrho$ in the established direction.

Now we fix $\sigma$. Denote by $|l_{\sigma\varrho}|$ the length of the arc $l_{\sigma\varrho}$. Then (4) implies directly the equality

$$\lim_{\varrho \to \sigma} \frac{\mu_\zeta(l_{\sigma\varrho})}{|l_{\sigma\varrho}|} = \left|\frac{d\arg w(z, \zeta)}{dz}\right| = \frac{dh(z, \zeta)}{d|z|}.$$

We have proved the following

THEOREM 1. *If the boundary of the domain $D$ is a Jordan curve $F$, and $F$ contains an analytic arc $L$ then the equilibrium distribution $\mu_\zeta$ of $F$ for every $\zeta \in D$ has the density with respect to the Lebesgue linear measure on $L$.*

**III.** Now we shall investigate the case when $F$ is a *rectifiable Jordan curve*. Consider the curves

$$I_\varepsilon = \{z\colon\, g(z, \zeta) = \varepsilon\} \qquad (0 < \varepsilon \leqslant 1)$$

and denote by $D^\varepsilon$ this part of $D$ which is bounded by $I_\varepsilon$ and contains the point $\zeta$. Choosing on $F$ an arbitrary point $\sigma$ we denote by $J_\sigma$ the curve

$$J_\sigma = \{z\colon\, h(z, \zeta) = h(\sigma, \zeta)\}.$$

On the curves $I_\varepsilon$ we define the measures $\mu_\zeta^\varepsilon$ in the following manner: We take into consideration the two points $\sigma$ and $\varrho \in F$ and the corresponding curves $J_\sigma$ and $J_\varrho$. We denote by $\sigma'$ and $\varrho'$ the points of coincidence of $I_\varepsilon$ with $J_\sigma$ and $J_\varrho$ respectively. Denote by $l_{\sigma\varrho}^\varepsilon$ the arc of $I_\varepsilon$ which leads from $\sigma'$ to $\varrho'$. Now we put

$$\mu_\zeta^\varepsilon(l_{\sigma\varrho}^\varepsilon) = \mu_\zeta(l_{\sigma\varrho}).$$

There is defined in this way an additive set function of the arc on $I_\varepsilon$. We extend this function in the usual manner to the (completely additive) measure.

LEMMA 1. *The measure $\mu_\zeta^\varepsilon$ is the equilibrium distribution of $I_\varepsilon$.*

Proof. The function $g(z, \zeta) - \varepsilon$ is the Green function of the domain $D^\varepsilon$ and $-h(z, \zeta)$ is its conjugate. We have by the construction

$$\frac{d\mu_\zeta^\varepsilon(l_{\sigma\varrho}^\varepsilon)}{d|l_{\sigma\varrho}^\varepsilon|} = \frac{dh(z, \zeta)}{d|z|} = \frac{dg(z, \zeta)}{d|z|}.$$

In view of theorem 1 and the unicity of the equilibrium distribution this equality gives our theorem.

The function $w_\varepsilon(z,\zeta)$ which maps conformly $D^\varepsilon$ onto a circle $|w|<1$ may be expressed as follows:

$$w(z,\zeta) = (z-\zeta)\exp\Big\{-\int \log(z-a)\,d\mu_\zeta^\varepsilon(a)+\gamma_\zeta+\varepsilon\Big\}.$$

The carriers of all the measures $\mu_\zeta^\varepsilon$ $(0<\varepsilon\leqslant 1)$ are contained in the common compact $K=\bar{D}-D^1$ and they have the common value of the total mass, which is equal to 1. We choose from $\{\mu_\zeta^\varepsilon\}$ a sequence $\{\mu_\zeta^{\varepsilon_n}\}$ which converges to some measure $\mu_\zeta^0$ by $\varepsilon_n\to 0$ (4). Then we have

$$(5)\quad w(z,\zeta) = \lim w_{\varepsilon_n}(z,\zeta) = \lim (z-\zeta)\exp\Big\{-\int\log(z-a)\,d\mu_\zeta^{\varepsilon_n}(a)+\gamma_\zeta+\varepsilon_n\Big\}$$
$$= (z-\zeta)\exp\Big\{-\int\log(z-a)\,d\mu_\zeta^0(a)+\gamma_\zeta\Big\}.$$

Hence, in view of the unicity of the equilibrium distribution $\mu_\zeta$, we obtain $\mu_\zeta^0=\mu_\zeta$. Further we have

$$(6)\qquad |\Delta_{\sigma'}^{\varrho'}\arg w_\varepsilon(z,\zeta)| = 2\pi\mu_\zeta^\varepsilon(l_{\sigma\varrho}^\varepsilon);$$

then, by (5) and (6) we obtain

$$|\Delta_\sigma^\varrho \arg w(z,\zeta)| = \lim_{\varepsilon_n\to 0} 2\pi\,(\mu_\zeta^{\varepsilon_n} l_{\sigma\varrho}^{\varepsilon_n}) = 2\pi\mu_\zeta(l_{\sigma\varrho}).$$

We have proved

LEMMA 2. *If $D$ is a simply connected domain and $F$ is a rectifiable Jordan curve, then we have*

$$|\Delta_\sigma^\varrho \arg w(z,\zeta)| = 2\mu_\zeta(l_{\sigma\varrho}).$$

Now we fix $\sigma\in F$. Then $\mu_\zeta(l_{\sigma\varrho})$ considered as a function of the point $\varrho$ is continuous and monotonic, i. e. if $l_{\sigma\varrho}\subset l_{\sigma\tau}$, then we have $\mu_\zeta(l_{\sigma\varrho})\leqslant\mu_\zeta(l_{\sigma\tau})$.

THEOREM 2. *The function $\mu_\zeta(l_{\sigma\varrho})$ is absolutely continuous with respect to the arc length $|l_{\sigma\varrho}|$.*

Proof. It is proved in [3] (p. 462) that the function $\arg w(z,\zeta)$ maps the sets of linear measure 0 on $F$ into the sets of linear measure 0 on the circle $|w|=1$. We make use of the Banach-Zarecki theorem (5), which states that if a function of one real variable is continuous and of bounded variation (in particular if it is monotonic and continuous on a compact) and maps the sets of Lebesgue measure 0 into the sets of Lebesgue measure 0, then this function is absolutely continuous. Our function satisfies the above conditions.

(4) We use here the principle of choosing for measures. For the details cf. [2].
(5) The proof of this theorem is to be found in [4], p. 219.



COROLLARY 1. *For almost all $z$ there exists the limit*

$$\lim_{\substack{\sigma\to z,\varrho\to z\\ z\in l_{\sigma\varrho}}}\frac{\mu(l_{\sigma\varrho})}{|l_{\sigma\varrho}|} = f_\zeta(z)$$

*and $f(z)$ is summable with respect to the arc length on $F$, i. e., if $z(s)$ is a natural parametrisation of $F$, then for each pair of points $u_1$ and $u_2$ $(0\leqslant u_i\leqslant |F|)$ we have*

$$\mu_\zeta(l_{z(u_1),z(u_2)}) = \int_{u_1}^{u_2} f(z(s))\,ds.$$

COROLLARY 2. *The Green function $g$ and the mapping function $w$ of the domain bounded by a rectifiable Jordan curve may be written in the form*

$$g(z,\zeta) = \log|z-\zeta|^{-1} - \int_F \log|z-a|^{-1} f_\zeta(a)\,d_a s-\gamma_\zeta,$$

$$w(z,\zeta) = (z-\zeta)\exp\Big\{\int_F \log|z-a|^{-1} f(a)\,d_a s+\gamma_\zeta - ia\Big\}.$$

Theorem 2 and both corollaries may be extended to the case when $\zeta=\infty$. Then there exists on $F$ a summable function $f$ such that almost everywhere on $F$ we have

$$f(z) = \lim_{\substack{\sigma\to z,\varrho\to z\\ z\in l_{\sigma\varrho}}}\frac{\mu_\infty(l_{\sigma\varrho})}{|l_{\sigma\varrho}|}.$$

This function may be used for the expression of $g(z,\infty)$ and $w(z,\infty)$ in terms of the Lebesgue integrals.

**IV.** We consider two domains $D_1$ and $D_2$ bounded by the Jordan curves $F_1$ and $F_2$ respectively. Let $\zeta_1$ and $\zeta_2$ be two points, of the domains $D_1$ and $D_2$ respectively. Let $U$ be a function which maps conformly $D_1$ onto $D_2$ so that $U(\zeta_1)=\zeta_2$. We shall prove

THEOREM 3. *The equilibrium distribution of the Jordan curves is invariant with respect to the conformal mapping, i. e. we have*

$$\mu_{\zeta_1}^1(e) = \mu_{\zeta_2}^2(U(e))\qquad (U(e)=\{z'\colon z'=U(z),\ z\in e\}).$$

Proof. It will be sufficient to prove the theorem in the case when $e$ is an arc, say $e=l_{\sigma\varrho}$. Denote by $w_i(z,\zeta_i)$ $(i=1,2)$ the function which maps conformly $D_i$ onto the unit circle $|w|<1$, so that $w_i(\zeta_i,\zeta_i)=0$. Then we have

$$U(z) = w_2^{-1}(w_1(z,\zeta_1),\zeta_2),$$

and hence

$$2\pi\mu_{\zeta_2}^2(U(e)) = \Delta_{U(\sigma)}^{U(\varrho)}\arg w_2(z,\zeta_2) = |\arg w_1(\varrho,\zeta_1)-\arg w_1(\sigma,\zeta_1)| = 2\pi\mu_{\zeta_1}^1(e).$$

V. In the following we shall investigate the variation of the equilibrium distribution when the corresponding domain varies. Let us consider two domains $D$ and $D^*$ with the boundaries $F$ and $F^*$ respectively. We leave out the previous assumptions regarding the simple connectedness of the domains and the rectifiability of the boundaries. We assume only that $F$ and $F^*$ are of positive capacity. Denote by $\mu_\zeta$ and $\mu_\zeta^*$ the corresponding equilibrium distributions.

THEOREM 4. *If $D$ is contained strongly in $D^*$, $\zeta \in D$ and the sets $E = F \frown F^*$ and $H = F - F^*$ are of positive capacity, then for every Borel set $e \subset E$ such that the set $\bar{e} \frown (\overline{E-e})$ is of capacity 0 and $e$ is of positive capacity, we have*

$$\mu_\zeta(e) > \mu_\zeta^*(e).$$

Proof. We fix the set $e$ satisfying the assumptions of the theorem and we denote its characteristic function by $\chi_e$. We have

$$\mu_\zeta(e) = \int \chi_e(a)\, d\mu_\zeta(a) \quad \text{and} \quad \mu_\zeta^*(e) = \int \chi_e(a)\, d\mu_\zeta^*(a).$$

Then $\mu_\zeta(e)$ and $\mu_\zeta^*(e)$ treated as the functions of $\zeta$ are in $D$ the solutions of the generalised Dirichlet problem for $D$ and $D^*$ respectively, with the boundary condition $\chi_e(z)$. Evidently we have

$$0 \leqslant \mu_\zeta(e) \leqslant 1 \quad \text{and} \quad 0 \leqslant \mu_\zeta^*(e) \leqslant 1.$$

The equalities on the left or the right side hold if and only if $e$ or $E-e$ is of capacity 0. The first of these cases implies $\mu_\zeta(e) = \mu_\zeta^*(e) = 0$, the second is impossible in view of the assumption that $F-F^*$ is of positive capacity. Consider the function

$$\varphi(\zeta) = \mu_\zeta^*(e) - \mu_\zeta(e);$$

$\varphi(\zeta)$ is harmonic in $D$. We shall denote its boundary values, at those points, where they exist, also by $\varphi(\zeta)$. We have

$$\varphi(z) = 0 \quad \text{if} \quad z \in e - (\overline{E-e}),$$
$$\varphi(z) = 0 \quad \text{if} \quad z \in (F \frown F^*) - \bar{e},$$
$$\varphi(z) > 0 \quad \text{if} \quad z \in F - F^*,$$
$$-1 \leqslant \varliminf_{\zeta \to z} \varphi(\zeta) \leqslant \varlimsup_{\zeta \to z} \varphi(\zeta) \leqslant 1 \quad \text{if} \quad z \in \bar{e} \frown (\overline{E-e}).$$

Denote by $\psi(\zeta)$ the solution of the generalised Dirichlet problem for the domain $D$ with the boundary condition

$$\psi(z) = \begin{cases} 0 & \text{on} \quad F^* \frown F, \\ \varphi(\zeta) & \text{on} \quad F - F^*. \end{cases}$$



We shall show that $\varphi \equiv \psi$ in $D$.

Let $r(\zeta)$ be the Evans function of the set $\bar{e} \frown (\overline{E-e})$, i. e. the function which has the following properties:

1. $r(\zeta)$ is of the shape

$$r(\zeta) = \int \log|\zeta - a|^{-1} d\varrho(a),$$

where $\varrho$ is a convenable Radon measure of the carrier contained in $\bar{e} \frown (\overline{E-e})$ and the total mass is equal to 1.

2. $r(\zeta)$ has the limit value equal to $+\infty$ at every point of the set $\bar{e} \frown (\overline{E-e})$. Outside this set it is evidently harmonic(6).

We consider in $D$ the function

$$\varphi(\zeta) - \psi(\zeta) + \varepsilon r(\zeta) + \varepsilon \log \delta \qquad (\varepsilon > 0,\ \varepsilon = \text{const})$$

where $\delta = \sup_{z, a \in D} |z - a|$. It is easy to see that the boundary values of this function are non-negative on $F$. Then this function is in $D$ non-negative. In view of $\varepsilon$ being arbitrary we obtain the inequality

$$\varphi(\zeta) - \psi(\zeta) \geqslant 0.$$

By a similar treatment we obtain the converse inequality

$$\psi(\zeta) - \varphi(\zeta) \geqslant 0.$$

Hence $\varphi(\zeta) \equiv \psi(\zeta)$.

Since $\psi(\zeta) > 0$ on $F - F^*$ then $\psi(\zeta) > 0$ in $D$, and

$$\psi(\zeta) = \varphi(\zeta) = \mu_\zeta(e) - \mu_\zeta(e) > 0.$$

COROLLARY 3. *If $E$ is a rectifiable arc, then for every arc $l \in E$ we have*

$$\mu_\zeta^*(l) < \mu_\zeta(l).$$

*Denoting by $f_\zeta$ and $f_\zeta^*$ the densities of the measures $\mu_\zeta$ and $\mu_\zeta^*$ respectively we have*

$$f_\zeta^*(z) < f_\zeta(z)$$

*almost everywhere on $E$. If $E$ is an analytical arc then this inequality holds in the whole arc.*

We shall prove a theorem which is stronger than the one contained in the above corollary, viz.

THEOREM 5. *If $D \subset D^*$, $F$ and $F^*$ have a common point $z_0$ and in some neighbourhood of $z_0$, $F$ and $F^*$ are the analytical arcs, then we have the inequality*

$$f_\zeta(z_0) < f_\zeta^*(z_0).$$

(6) Cf. Evans' paper [1].

Proof. The case when $F = F^*$ in some neighbourhood of $z_0$ will be omitted, because it has been considered.

We shall use the notations

$$K_\varrho = \{z\colon |z-z_0| < \varrho\} \qquad (\varrho > 0),$$

$$D_\varrho = (D^* - \overline{K}_\varrho) + D, \qquad F_\varrho = \overline{D}_\varrho - D.$$

$\mu_\zeta^\varrho$ denotes the equilibrium distribution of $F_\varrho$ with respect to the point $\zeta$. The set $F \frown F_\varrho = F \frown K_\varrho$ is for all sufficiently small $\varrho$ an analytic arc. Theorem 4 easily implies the inequality

$$(7) \qquad \mu_\zeta(K_\varrho) < \mu_\zeta^\varrho(K_\varrho) < \mu_\zeta^*(K_\varrho).$$

Then for the density $f_\zeta^\varrho$ of the measure $\mu_\zeta^\varrho$ we have the inequality

$$f_\zeta(z_0) < f_\zeta^\varrho(z_0).$$

In order to prove our theorem it will be sufficient to show that $f_\zeta^\varrho(z_0) \leqslant f_\zeta^*(z_0)$. This follows directly from the equalities

$$f_\zeta^\varrho(z_0) = \lim_{\varrho\to 0} \frac{\mu_\zeta^\varrho(K_\varrho)}{2\varrho}, \qquad f_\zeta^*(z_0) = \lim_{\varrho\to 0} \frac{\mu_\zeta^*(K_\varrho)}{2\varrho}$$

and the inequality (7).

The assumptions of theorem 5 remaining valid, we consider the special case when $D$ and $D^*$ are simply connected. The functions which map conformly $D$ and $D^*$ into the unit circle so that $\zeta$ corresponds to 0 will be denoted by $w(z, \zeta)$ and $w^*(z, \zeta)$ respectively. We have the expression

$$w(z, \zeta) = \exp\{-g(z, \zeta) + ih(z, \zeta)\}$$

and an analogical formula for $w^*$. This easily implies the expression for the absolute value of the derivative of $w$, viz.

$$\left|\frac{dw(z, \zeta)}{dz}\bigg|_{z_0}\right| = \frac{dg(z, \zeta)}{dn}\bigg|_{z_0} = f_\zeta(z_0)$$

and an analogical formula for $w^*$. Hence, in view of theorem 5, we obtain

COROLLARY 4 (The result of Lindelöf). *If $D$ and $D^*$ are simply connected domains, then by the assumptions of theorem 5 we have*

$$\left|\frac{dw(z, \zeta)}{dz}\bigg|_{z_0}\right| < \left|\frac{dw^*(z, \zeta)}{dz}\bigg|_{z_0}\right|.$$

Using the above results we shall generalise theorem 2.

Let $D$ be a $p$-connected domain, $1 \leqslant p \leqslant \infty$. Its boundary $F$ be of positive capacity. Assume that one of the components of $F$ is a Jordan curve $K$ or a sum of a Jordan curve $K$ and some continua. Then there holds



THEOREM 6. *If a part $L$ of $K$ is a rectifiable arc, then for every $\zeta \in D$ the equilibrium distribution $\mu_\zeta$ of $F$ is on $L$ absolutely continuous with respect to the linear Lebesgue measure.*

Proof. The curve $K$ intersects the plane into two domains $G$ and $G'$. One of them, say $G$, is contained in the complement of $D$. We join the extremities of the arc $L$ by an arc $L'$ which is comprised in $G$, and we do it in such a way that both $L$ and $L'$ form a Jordan curve. Denote by $D^*$ that domain bounded by $L \cup L'$ which contains $D$. Let $\mu_\zeta^*$ be the equilibrium distribution of $L \cup L'$ with respect to the point $\zeta$. According to theorem 4 for every measurable set $e \subset L$ we have the inequality

$$(8) \qquad \mu_\zeta(e) < \mu_\zeta^*(e).$$

Since $\mu_\zeta^*$ is on $L$ absolutely continuous with respect to the linear Lebesgue measure, in view of (8) $\mu_\zeta$ is also absolutely continuous.

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