ANNALES
POLONICI MATHEMATICI
LIX.3 (1994)

Proper intersection multiplicity and regular separation of analytic sets

by Ewa Cygan and Piotr Tworzewski (Kraków)

Abstract. We consider complex analytic sets with proper intersection. We find their regular separation exponent using basic notions of intersection multiplicity theory.

1. Separation. This part of the paper is the straightforward generalization of the corresponding section in [8] (cf. [4], IV.7). Let M be an m-dimensional normed complex vector space and X,Y closed sets in an open subset G of M. For $p > 0$, we say that X and Y are *p-separated at* $a \in G$ if $a \in X \cap Y$ and

$$\varrho(z, X) + \varrho(z, Y) \geq c\varrho(z, X \cap Y)^p,$$

in a neighbourhood of the point a, for some $c > 0$. ($\varrho(\cdot, Z)$ denotes the distance function to the set $Z \subset M$).

Let us start with the following obvious lemma (cf. [4], [8]).

Lemma 1.1 *Let $H_1 \subset G$ and H_2 be open subsets of normed, finite-dimensional complex vector spaces and let $f : H_1 \to H_2$ be a biholomorphism. Then closed subsets X and Y of G are p-separated at a point $a \in H_1$ if and only if $f(X \cap H_1)$ and $f(Y \cap H_1)$ are p-separated at $f(a)$.*

By the above lemma we can consider p-separation for closed subsets of complex manifolds. Let us mention that in this paper all manifolds are assumed to be second-countable.

Namely, we say that closed subsets X and Y of an m-dimensional complex manifold M are *p-separated at* $a \in M$ if for some (and hence for every) chart $\varphi : \Omega \to G \subset \mathbb{C}^m$ such that $a \in \Omega$, the sets $\varphi(X \cap \Omega)$ and $\varphi(Y \cap \Omega)$, closed in G, are p-separated at $\varphi(a)$.

1991 *Mathematics Subject Classification*: Primary 32B10, 14C17.
Key words and phrases: proper intersection, multiplicity, exponent of regular separation.
Supported by KBN Grant 2 1077 91 01.

LEMMA 1.2 (cf. [3], part 18). *Let G be an open subset of a normed finite-dimensional complex vector space. Suppose that $a \in X \cap Y$ is an accumulation point of $X \setminus Y$ and let $p > 0$. Then X and Y are p-separated at a if and only if there exist a neighbourhood U of a and $c > 0$ such that*

$$\varrho(x, Y) \geq c\varrho(x, X \cap Y)^p \quad \text{for } x \in X \cap U.$$

Proof. We must show that the above condition implies that X and Y are p-separated at a. We can assume that $c \in (0,1)$ and U is contained in the ball $B(a,1)$. Since a is an accumulation point of $X \setminus Y$, we have $p \geq 1$.

Take $r > 0$ such that $B(a, 4r) \subset U$. If $z \in B(a,r)$ then there exist $x \in X \cap B(a, 2r)$ and $y \in Y \cap B(a, 2r)$ such that $\varrho(z, X) = |z - x|$ and $\varrho(z, Y) = |z - y|$. Therefore

$$l := \varrho(z, X) + \varrho(z, Y) \geq |x - y| \geq \varrho(x, Y) \geq c\varrho(x, X \cap Y)^p.$$

Let w be a point of $(X \cap Y) \cap B(a, 4r)$ for which $\varrho(x, X \cap Y) = |x - w|$. Then $l \geq c|x - w|^p$. Moreover, $l \geq \varrho(z, X) = |z - x| \geq c|z - x|^p$. Combining these inequalities we deduce that

$$l \geq \frac{c}{2}(|x - w|^p + |z - x|^p) \geq \frac{c}{2^p}|z - w|^p \geq \frac{c}{2^p}\varrho(z, X \cap Y)^p,$$

which completes the proof.

LEMMA 1.3. *Let M be a complex manifold. If $a \in M$ and $p > 0$ then the following conditions are equivalent*:

(1) *X and Y are p-separated at a*,

(2) *$X \times Y$ and Δ_M are p-separated at (a,a)*,

where $\Delta_M = \{(x, x) \in M^2 : x \in M\}$ is the diagonal in M^2.

Proof. Without loss of generality we can assume that M is an open subset of a normed complex vector space N with $\dim N \geq 1$, and take the norm $|(x, y)| = |x| + |y|$ in N^2. Observe that for $z \in M$,

$$\varrho((z, z), X \times Y) = \varrho(z, X) + \varrho(z, Y),$$

and

$$\varrho((z, z), (X \times Y) \cap \Delta_M) = 2\varrho(z, X \cap Y).$$

Now Lemma 1.2 completes the proof.

2. Special descriptions of analytic sets. Let us start with the following general lemma.

LEMMA 2.1. *Suppose that k, d are positive integers, $r = (k-1)d + 1$ and $L_1, \ldots, L_r$ are linear forms on $\mathbb{C}^k$ such that $L_{i_1}, \ldots, L_{i_k}$ are linearly* *independent for* $i_1, \ldots, i_k \in \{1, \ldots, r\}$ *with* $i_s \neq i_t$ *for* $s \neq t$. *Define*

$$\Lambda : (\mathbb{C}^k)^d \ni (v_1, \ldots, v_d) \to \Big(\prod_{i=1}^{d} L_1(v_i), \ldots, \prod_{i=1}^{d} L_r(v_i)\Big) \in \mathbb{C}^r.$$

Then there exists a positive constant $A > 0$ *such that* $|\Lambda(v_1, \ldots, v_d)| \geq A|v_1| \cdot \ldots \cdot |v_d|$ *for* $v_1, \ldots, v_d \in \mathbb{C}^k$.

Proof. It is easy to verify that $\Lambda(v_1, \ldots, v_d) = 0$ if and only if $\prod_{i=1}^{d} |v_i| = 0$. Since $\Lambda : (\mathbb{C}^k)^d \to \mathbb{C}^r$ is a d-linear mapping our lemma follows by a standard calculation.

Now, let D be an open connected subset in $\mathbb{C}^n$, Z a pure n-dimensional analytic subset of $D \times \mathbb{C}^k$ such that the natural projection $\pi|_Z : Z \ni (x, y) \to x \in D$ is proper. Then the mapping $\pi|_Z : Z \to D$ is a so-called branched covering. In particular, it has the following properties:

1) $\pi|_Z$ is surjective and open,

2) there exist a proper analytic subset S of D and a positive integer d such that the mapping $\pi|_{Z \setminus \pi^{-1}(S)} : Z \setminus \pi^{-1}(S) \to D \setminus S$ is locally biholomorphic, and

$$\#(\pi|_Z)^{-1}(x) = d \quad \text{if } x \in D \setminus S,$$
$$\#(\pi|_Z)^{-1}(x) < d \quad \text{if } x \in S.$$

The set $D \setminus S$ is called the *regular set* of $\pi|_Z$ and d its *multiplicity* (sheet number).

Each set Z as above has a special description, especially useful from the point of view of regular separation.

PROPOSITION 2.2. *There exists a holomorphic mapping* $F : D \times \mathbb{C}^k \to \mathbb{C}^r$, $r = (k-1)d + 1$, *such that*

1) $F^{-1}(0) = Z$,

2) $|F(z)| \geq \varrho(z, (\pi|_Z)^{-1}(\pi(z)))^d$ *for* $z \in D \times \mathbb{C}^k$.

Proof. Let $L_1, \ldots, L_r$ be linear forms on $\mathbb{C}^k$ satisfying the assumptions of Lemma 2.1. For every $s \in \{1, \ldots, r\}$ define

$$\Phi_{L_s} : D \times \mathbb{C}^k \ni (x, y) \to (x, L_s(y)) \in D \times \mathbb{C}, \qquad Z_{L_s} = \Phi_{L_s}(Z).$$

It is easy to show (cf. [9]) that the projection $\widetilde{\pi}|_{Z_{L_s}} : Z_{L_s} \ni (x, t) \to x \in D$ is proper. We can assume, by changing the forms if necessary, that the multiplicity of the branched covering $\widetilde{\pi}|_{Z_{L_s}}$ is equal to d for all $s \in \{1, \ldots, r\}$. There exist holomorphic functions $a_1^s, \ldots, a_d^s$ on D such that

$$Z_{L_s} = \{(x, t) \in D \times \mathbb{C} : P_s(x, t) = t^d + a_1^s(x)t^{d-1} + \ldots + a_d^s(x) = 0\}$$

for $s = 1, \ldots, r$.

Define a holomorphic mapping

$$F_1 : D \times \mathbb{C}^k \ni (x, y) \to (P_1(x, L_1(y)), \ldots, P_r(x, L_r(y))) \in \mathbb{C}^r.$$

It follows immediately that $F_1^{-1}(0) = Z$. To prove the second condition we fix $(x, y) \in D \times \mathbb{C}^k$ such that x is a regular point of the branched covering $\pi|_Z : Z \to D$. If $(\pi|_Z)^{-1}(x) = \{(x, y_i) : i = 1, \ldots, d\}$ then $P_s(x, t) = (t - L_s(y_1)) \cdot \ldots \cdot (t - L_s(y_d))$, $s = 1, \ldots, r$, and so

$$F_1(x, y) = \Big(\prod_{i=1}^{d} L_1(y - y_i), \ldots, \prod_{i=1}^{d} L_r(y - y_i)\Big).$$

Now, Lemma 2.1 implies

$$|F_1(x, y)| \geq A|y - y_1| \cdot \ldots \cdot |y - y_d| \geq A\varrho(y, \{y_1, \ldots, y_d\})^d.$$

By continuity we can extend this inequality to all points $z = (x, y) \in D \times \mathbb{C}^k$. It is clear that $F = A^{-1}F_1$ is the required mapping and the proof is complete.

3. Proper intersections. For the convenience of the reader we recall some basic facts on proper intersections of analytic sets.

Let X and Y be pure dimensional analytic subsets of a complex manifold M of dimension m. We say that X and Y *meet properly* on M if

$$\dim(X \cap Y) = \dim X + \dim Y - m\,.$$

Then we have the *intersection product* $X \cdot Y$ of X and Y which is an analytic cycle on M defined by the formula

$$X \cdot Y = \sum_C i(X \cdot Y, C)\, C,$$

where the summation extends over all analytic components C of $X \cap Y$ and $i(X \cdot Y, C)$ denotes the intersection multiplicity along the component C in the sense of Draper ([2], Def. 4.5; cf. [10]). Such multiplicities are positive integers.

Now, let M be a complex manifold and let Z be a pure n-dimensional analytic subset of M. For $a \in M$ we denote by $\deg_a Z$ the degree of Z at a (see [2], p. 194). This degree is equal to the so-called Lelong number of Z at a.

In this paper we will consider a natural extension of this definition to n-dimensional analytic cycles. Namely, if $A = \sum_C \alpha_C C$ is an n-dimensional analytic cycle on M then the sum

$$\deg_a A = \sum_C \alpha_C \deg_a C$$

is well defined and we call it the *degree* of the cycle A at the point a.

4. Main results. Let us begin with a general useful fact for branched coverings.

THEOREM 4.1. *Let D be an open connected subset of $\mathbb{C}^n$ and let Z be a pure n-dimensional analytic subset of $D \times \mathbb{C}^k$ such that $\pi|_Z : Z \to D$ is proper with multiplicity d. Suppose that E is closed in D and $V = E \times \mathbb{C}^k$. Then Z and V are d-separated at every $a \in Z \cap V$.*

Proof. Fix $a \in Z \cap V$ and $r > 0$ such that $\overline{B(a, 2r)} \subset D \times \mathbb{C}^k$. By the mean value theorem there exists $C > 0$ such that $|F(z') - F(z'')| \le C|z' - z''|$ for $z', z'' \in B(a, 2r)$, where F is the function from Proposition 2.2.

For $z \in B(a, r) \cap V$ there is $w \in Z \cap B(a, 2r)$ such that $\varrho(z, Z) = |z - w|$. Then

$$\begin{aligned}\varrho(z, Z) = |z - w| &\ge C^{-1}|F(z) - F(w)| \\ &= C^{-1}|F(z)| \ge C^{-1}\varrho(z, (\pi|_Z)^{-1}(\pi(z)))^d.\end{aligned}$$

But $(\pi|_Z)^{-1}(\pi(z)) \subset Z \cap V$ and so

$$\varrho(z, Z) \ge C^{-1}\varrho(z, Z \cap V)^d \quad \text{for } z \in B(a, r) \cap V.$$

Now, Lemma 1.2 implies that Z and V are d-separated at a.

We can now prove our main result.

THEOREM 4.2. *Let M be a complex manifold and let X, Y be pure dimensional analytic subsets of M. Suppose that X and Y meet properly on M, $a \in X \cap Y$ and $p = \deg_a(X \cdot Y)$. Then X and Y are p-separated at a.*

Proof. Consider the intersection of $X \times Y$ and Δ_M in $M \times M$. Write $n = \dim(X \times Y) = \dim X + \dim Y$, $k = 2m - n$ and suppose that G, H, W are open unit balls in $\mathbb{C}^{n-m}$, $\mathbb{C}^m$, $\mathbb{C}^k$ respectively. Define $D = G \times H$.

By ([2], Prop. 4.6, Cor. 5.2) there exists a chart $\varphi : \Omega \to D \times W$ defined on an open neighbourhood Ω of $b = (a, a)$ in $M \times M$ such that:

(1) $Z = \varphi(\Omega \cap (X \times Y))$ is an analytic subset of $D \times \mathbb{C}^k$ of pure dimension n such that the natural projection $\pi|_Z : Z \to D$ is proper,

(2) $(\pi|_Z)^{-1}(0) = \{0\} = \varphi(b)$,

(3) $\varphi(\Omega \cap \Delta_M) = G \times \{0\} \times W$,

(4) $\deg_0(Z \cdot (0 \times W)) = p = \deg_a(X \cdot Y)$.

Observe that the multiplicity of the branched covering $\pi|_Z : Z \to D$ is p and so, by Theorem 4.1, the sets Z and $(G \times \{0\}) \times \mathbb{C}^k$ are p-separated at 0. Then $X \times Y$ and Δ_M are p-separated at $b = (a, a)$. Now Lemma 1.3 shows that X and Y are p-separated at a and the proof is complete.

References

[1] R. Achilles, P. Tworzewski and T. Winiarski, *On improper isolated intersection in complex analytic geometry*, Ann. Polon. Math. 51 (1990), 21–36.

[2] R. Draper, *Intersection theory in analytic geometry*, Math. Ann. 180 (1969), 175–204.

[3] S. Łojasiewicz, *Ensembles semi-analytiques*, I.H.E.S. Bures-sur-Yvette, 1965.

[4] —, *Introduction to Complex Analytic Geometry*, Birkhäuser, Basel, 1991.

[5] —, *Sur la séparation régulière*, Univ. Studi Bologna, Sem. Geom. 1985, 119–121.

[6] A. Płoski, *Multiplicity and the Łojasiewicz exponent*, Polish Academy of Sciences, Warsaw, preprint 359.

[7] —, *Une évaluation pour les sous-ensembles analytiques complexes*, Bull. Polish Acad. Sci. Math. 31 (1983), 259–262.

[8] P. Tworzewski, *Isolated intersection multiplicity and regular separation of analytic sets*, Ann. Polon. Math. 58 (1993), 213–219.

[9] P. Tworzewski and T. Winiarski, *Analytic sets with proper projections*, J. Reine Angew. Math. 337 (1982), 68–76.

[10] —, —, *Cycles of zeroes of holomorphic mappings*, Bull. Polish Acad. Sci. Math. 37 (1989), 95–101.

[11] T. Winiarski, *Continuity of total number of intersection*, Ann. Polon. Math. 47 (1986), 155–178.

INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
REYMONTA 4
30–059 KRAKÓW, POLAND
E-mail: CYGAN@IM.UJ.EDU.PL
TWORZEWS@IM.UJ.EDU.PL

Reçu par la Rédaction le 10.11.1993