ANNALES
POLONICI MATHEMATICI
LIX.3 (1994)

Analytic cell decomposition of sets definable in the structure $\mathbb{R}_{\exp}$

by Ta Lê Loi (Dalat and Kraków)

Abstract. We prove that every set definable in the structure $\mathbb{R}_{\exp}$ can be decomposed into finitely many connected analytic manifolds each of which is also definable in this structure.

Let $\mathcal{A}_n$ be the smallest ring of real-valued functions on $\mathbb{R}^n$ containing all polynomials and closed under exponentiation. We consider the smallest class $\mathcal{D}$ of subsets of Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, containing all analytic sets of the form

$$(*) \qquad \{x \in \mathbb{R}^n : f(x) = 0\}, \qquad \text{where } f \in \mathcal{A}_n \text{ and } n \in \mathbb{N},$$

and closed under taking: finite unions, finite intersections, complements and linear projections onto smaller dimensional Euclidean spaces. We adopt the name $\mathcal{D}$-*sets* for elements of $\mathcal{D}$.

In general, a $\mathcal{D}$-set is not subanalytic but the class $\mathcal{D}$ has some nice properties. As a direct consequence of Wilkie's Theorem [11], [12] of model completeness of the theory of the structure $\mathbb{R}_{\exp}$, each $\mathcal{D}$-set is the image of an analytic set of the form $(*)$ under a natural projection, thus by Khovanskiĭ's Theorem [4] it has only finitely many connected components. In particular, $\mathcal{D}$ is O-minimal (i.e. every $\mathcal{D}$-set of $\mathbb{R}$ is a finite union of intervals and points) so there are a Cell Decomposition Theorem and a Triangulation Theorem for this class (see [2], [6]).

In [3] L. van den Dries and C. Miller proved that each $\mathcal{D}$-set can be partitioned into finitely many connected analytic manifolds each of which is also a $\mathcal{D}$-set. In this paper we give another proof of this property (Theorem 2.8) avoiding making use of the O-minimality and the finite model completeness of the theory of the structure $\mathbb{R}_{\exp}$ as used in [3].

1991 *Mathematics Subject Classification*: 32B20, 32B25.
Key words and phrases: $\mathcal{D}$-sets, Wilkie's Theorem, semianalytic sets, analytic cell decomposition, Tarski's system.

I. Preliminaries

1.1. Definition. Let $\mathcal{R}_n$ denote the ring of real-valued functions on $\mathbb{R}^n$ generated over $\mathbb{R}$ by the coordinate functions $x_1, \ldots, x_n$ and their exponents $\exp(x_1), \ldots, \exp(x_n)$, i.e.

$$\mathcal{R}_n := \mathbb{R}[x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)].$$

A subset X of $\mathbb{R}^n$ is called $\mathcal{R}_n$-*analytic* iff it is the zero set of a function from $\mathcal{R}_n$.

A subset X of $\mathbb{R}^n$ is called $\mathcal{R}_n$-*semianalytic* iff

$$X = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : f_i(x) = 0,\ g_{ij}(x) > 0, j = 1, \ldots, q\}$$

where $f_i, g_{ij} \in \mathcal{R}_n$, $p, q \in \mathbb{N}$.

An $\mathcal{R}_n$-*analytic leaf* is a subset S of $\mathbb{R}^n$ of the form

$$S = \left\{ x \in \mathbb{R}^n : f_1(x) = \ldots = f_k(x) = 0,\ \delta(x) = \frac{D(f_1, \ldots, f_k)}{D(x_{i_1}, \ldots, x_{i_k})}(x) \neq 0 \right\}$$

where $f_1, \ldots, f_k \in \mathcal{R}_n$, $1 \leq i_1 < \ldots < i_k \leq n$, $k \in \mathbb{N}$.

An $\mathcal{R}_n$-*semianalytic leaf* is a subset of $\mathbb{R}^n$ which is the intersection of an $\mathcal{R}_n$-analytic leaf and an open set $\{x \in \mathbb{R}^n : g_1(x) > 0, \ldots, g_p(x) > 0\}$, $g_i \in \mathcal{R}_n$, $i = 1, \ldots, p$, $p \in \mathbb{N}$.

1.2. Remark. From the definition, $\mathcal{R}_n$ is a noetherian ring, closed under the operators $\partial/\partial x_i$ $(i = 1, \ldots, n)$, and every $\mathcal{R}_n$-semianalytic leaf is an analytic submanifold of $\mathbb{R}^n$.

1.3. Proposition. *Every $\mathcal{R}_n$-semianalytic set has only finitely many connected components.*

Proof. First of all note that

$$\begin{aligned} &g > 0 && \text{iff} \quad \exists v\,(v^2 g - 1 = 0), \\ &f = 0 \text{ and } g = 0 && \text{iff} \quad f^2 + g^2 = 0, \text{ and} \\ &f = 0 \text{ or } g = 0 && \text{iff} \quad fg = 0. \end{aligned}$$

After introducing some new variables an $\mathcal{R}_n$-semianalytic set is a projection of an $\mathcal{R}_{n+m}$-analytic set. The proposition follows from Khovanskiĭ's result [4] or [5, Ch. I, §1.2]. ∎

1.4. Proposition (Tougeron). *Every $\mathcal{R}_n$-analytic set can be represented as a disjoint union of finitely many analytic manifolds S_i, each S_i being a connected component of an $\mathcal{R}_n$-analytic leaf $\widetilde{S}_i$. Consequently, every $\mathcal{R}_n$-semianalytic set can be represented as a union of finitely many analytic manifolds, each of which is a connected component of an $\mathcal{R}_n$-semianalytic leaf.*

P r o o f. The proposition follows from Remark 1.2, Proposition 1.3 and [9, Prop. 1.3] (see also [10]). ∎

The following proposition is analogous to Lemmas A and B in [1].

1.5. PROPOSITION. *Let X be a subset of $\mathbb{R}^n \times \mathbb{R}^m$ and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be the natural projection. If X can be represented as a union of finitely many S_i, where each S_i is a connected component of an $\mathcal{R}_{n+m}$-semianalytic leaf, then there are finitely many subsets B_j in X, each B_j being a connected component of an $\mathcal{R}_{n+m}$-semianalytic leaf such that*:

(i) $\pi(X) = \pi(\bigcup_j B_j)$.

(ii) *For each j, $\pi|_{B_j} : B_j \to \mathbb{R}^n$ is an immersion.*

P r o o f. Induction on $d = \dim X$. If $d = 0$ there is nothing to prove. Suppose $d > 0, X = \bigcup S_i$, where each S_i is a connected component of an $\mathcal{R}_{n+m}$-semianalytic leaf $\widetilde{S}_i$. By the inductive hypothesis the proposition is true for $\bigcup_{i:\dim S_i<d} S_i$, so we can suppose $X = \bigcup S_i$ with $\dim \widetilde{S}_i = d$ for all i. Fix i, write $S = S_i$ and

$$\widetilde{S} = \widetilde{S}_i = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : f_1(x, y) = \ldots = f_{n+m-d}(x, y) = 0, \\ \delta(x, y) \neq 0, g_1(x, y) > 0, \ldots , g_p(x, y) > 0\}$$

where $f_i, g_j \in \mathcal{R}_{n+m}$; $i = 1, \ldots , n + m - d$, $j = 1, \ldots , p$; δ is a jacobian of $(f_1, \ldots , f_{n+m-d})$.

If $n - \alpha = \max \operatorname{rank} \pi|_S$, then there exists a jacobian

$$\delta_1 = \frac{D(f_1, \ldots , f_{n+m-d})}{D(x_{i_1}, \ldots , x_{i_\alpha}, y_{j_1}, \ldots , y_{j_\beta})}, \quad \alpha + \beta = n + m - d, \text{ such that } \delta_1|_S \neq 0.$$

Therefore $\dim S \cap \{\delta_1 = 0\} < d$ and by Proposition 1.4, $S \cap \{\delta_1 = 0\}$ is as in the assumption. Hence, by the inductive hypothesis, it is sufficient to consider

$$\begin{aligned} S' &= S \cap \{\delta_1 \neq 0\} \\ &= \{f_1 = \ldots = f_{n+m-d} = 0,\ \delta_1 \neq 0,\ \delta \neq 0,\ g_1 > 0, \ldots , g_p > 0\}. \end{aligned}$$

Note that S' is a union of finitely many connected components of $\widetilde{S} \cap \{\delta_1 \neq 0\}$, $\pi|_{S'}$ has constant rank $n - \alpha$ and $\dim S' = d$.

For each $x \in \pi(S')$ the fibre $\pi^{-1}(x) \cap S'$ is a submanifold of $\mathbb{R}^{n+m}$ of codimension $d + \alpha - n$.

C a s e 1: $d+\alpha-n=0$, i.e. $\operatorname{rank} \pi|_{S'} = \dim S'$. Then $\pi|_{S'}$ is an immersion. Take the connected components of S' as B_j's. In this case the number of B_j's is finite by Proposition 1.3.

C a s e 2: $d + \alpha - n > 0$, i.e. $\operatorname{rank} \pi|_{S'} < \dim S'$. Define

$$\theta_{S'} := \frac{1}{1 + |x|^2 + |y|^2} \cdot \frac{\delta^2}{1 + \delta^2} \cdot \frac{\delta_1^2}{1 + \delta_1^2} \prod_{i=1}^{p} \frac{g_i^2}{1 + g_i^2}.$$

Then $\theta_{S'}$ is a quotient of functions in $\mathcal{R}_{n+m}$ and $\theta_{S'}(x,y) \to 0$ as $(x,y) \to \infty$ in S', or (x,y) tends to a point of $\overline{S'} \setminus S'$.

Define $S'' = \{(x,y) \in S' : \operatorname{grad}(\theta_{S'}|_{\pi^{-1}(x)\cap S'})(x,y) = 0\}$. Then

(a) S'' is as in the assumptions.
(b) $\dim S'' < \dim S'$.
(c) $\pi^{-1}(x) \cap S'' \neq \emptyset$, $\forall x \in \pi(S')$.

Indeed, to see (a) note that $S'' = \{(x,y) \in S' : d_y f_1 \wedge \ldots \wedge d_y f_{n+m-d} \wedge d_y \theta_{S'}(x,y) = 0\}$ (here $d_y f(x,y) := \sum_{i=1}^m \frac{\partial f}{\partial y_i}(x,y) dy_i$), and by the form of $\theta_{S'}$ and Proposition 1.4, (a) follows.

To prove (b), let T be a connected component of $\pi^{-1}(x) \cap S'$. Then T is not compact, because $\beta < m$ and the projection of $\pi^{-1}(x) \cap S'$ onto $\{y \in \mathbb{R}^m : y_{i_1} = \ldots = y_{i_\beta} = 0\}$ is open, and $\theta_{S'}|_T > 0$, $\theta_{S'}(x,y) \to 0$ as $(x,y) \in T, y \to \infty$ or (x,y) tends to a point of $\overline{T} \setminus T$. These imply $\theta_{S'}|_T$ is not constant, so $\dim S'' \cap T < \dim T$. Therefore $\dim S'' \cap \pi^{-1}(x) < \dim S' \cap \pi^{-1}(x), \forall x \in \pi(S')$. Hence $\dim S'' < \dim S'$.

Finally, $\theta_{S'}|_T$ has a positive maximum on T, i.e. $\exists\,(x,y) \in T(\operatorname{grad}\theta_{S'}|_T (x,y) = 0)$, and (c) follows.

As a result, we have $S'' \subset S'$, $\dim S'' < \dim S'$ and $\pi(S'') = \pi(S')$. By the inductive hypothesis, the proposition is proved. ∎

1.6. Corollary. *Let $F \in \mathcal{R}_{n+m}$ and $(x,y) = (x_1,\ldots,x_n,y_1,\ldots,y_m)$ be the coordinate functions of $\mathbb{R}^n \times \mathbb{R}^m$. Then there are $h_j = (h_{j1},\ldots,h_{jm})$, $h_{ji} \in \mathcal{R}_{n+m}$ and $g_{j1},\ldots,g_{jp} \in \mathcal{R}_{n+m}$, $j = 1,\ldots,l$, $i = 1,\ldots,m$, such that*

$$\{x : \exists y\,(F(x,y) = 0)\} = \bigcup_{j=1}^{l} \Big\{x : \exists y \Big(F(x,y) = 0, h_j(x,y) = 0, \frac{Dh_j}{Dy}(x,y) \neq 0,\ g_{js}(x,y) > 0, s = 1,\ldots,p\Big)\Big\}.$$

Proof (compare with [3, Lemma (5.13)]). By Propositions 1.4 and 1.5 there are finitely many subsets B_j of $F^{-1}(0)$ such that each B_j is a connected component of an $\mathcal{R}_{n+m}$-semianalytic leaf of the form

$$\Big\{(x,y) : f_1(x,y) = \ldots = f_k(x,y) = 0, \frac{D(f_1,\ldots,f_k)}{D(x_{i_1},\ldots,x_{i_\alpha},y_{j_1},\ldots,y_{j_\beta})}(x,y) \neq 0,\ g_1(x,y) > 0,\ldots,g_p(x,y) > 0\Big\},$$

and $\{x : \exists y\,(F(x,y) = 0)\} = \pi(F^{-1}(0)) = \pi(\bigcup_j B_j)$ and $\pi|_{B_j}$ are immersions. Moreover, each B_j can be taken to be of the form of Case 1 in the proof of Proposition 1.5, that is,

$$\alpha = n - d, \qquad \alpha + \beta = k = m + n - d.$$

Hence, $\beta = m \le k$, and for each j,

$$B_j \subset \Big\{ f_1 = \ldots = f_k = 0, \quad \frac{D(f_1,\ldots,f_k)}{D(x_{i_1},\ldots,x_{i_\alpha},y_1,\ldots,y_m)} \neq 0,\ g_1 > 0,\ldots,g_p > 0 \Big\}.$$

Therefore,

$$B_j \subset \bigcup_{1\le i_1<\ldots<i_m\le k} \Big\{ f_{i_1} = \ldots = f_{i_m} = 0, \quad \frac{D(f_{i_1},\ldots,f_{i_m})}{D(y_1,\ldots,y_m)} \neq 0,\ g_1 > 0,\ldots,g_p > 0 \Big\}.$$

Hence the corollary is satisfied with the functions $h_J = (f_{i_1},\ldots,f_{i_m})$ and $g_1,\ldots,g_p$, $1 \le i_1 < \ldots < i_m \le k$ (where h_J, g_i, k depend on B_j). ∎

2. The class of $\mathcal{D}$-sets. Decomposition theorem. In this section we give another definition of the class of $\mathcal{D}$-sets defined at the beginning of this paper. We present here the proof of analytic cell decomposition of $\mathcal{D}$-sets (Theorem 2.8) based on Wilkie's Theorem on the Tarski property of this class (Theorem 2.3), Khovanskiĭ's result on the finiteness of the number of connected components (Lemma 2.8.2) and Proposition 1.5 above (compare with [3, Th. 8.8], where the proof is strongly based on model theory methods; see also [2]).

2.1. Definition. Let $\mathcal{D}_n$ denote the class of subsets of $\mathbb{R}^n$ each of which is the image of an $\mathcal{R}_{n+m}$-semianalytic set by the natural projection $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ for some $m \in \mathbb{N}$. Each set in $\mathcal{D}_n$ is called a *$\mathcal{D}_n$-set*. A *$\mathcal{D}$-set* is a $\mathcal{D}_n$-set for some $n \in \mathbb{N}$.

2.2. Proposition. (i) *For each $\mathcal{D}_n$-set S there are $m \in \mathbb{N}$ and $F \in \mathcal{R}_{n+m}$ such that $S = \pi(F^{-1}(0))$, where π is the natural projection of $\mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n$.*

(ii) *If $f_i, g_{ij} \in \mathcal{A}_n, i = 1,\ldots,p,\ j = 1,\ldots,q$, then the semianalytic set of the form*

$$\bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : f_i(x) = 0,\ g_{ij}(x) > 0,\ j = 1,\ldots,q\}$$

is a $\mathcal{D}_n$-set.

Proof. See [7, Prop. 1.2]. ∎

As a direct consequence of Wilkie's result on model completeness of the theory of the structure $\mathbb{R}_{\exp}$ (see [11], [12, Main Theorem]) we have the following theorem.

2.3. Theorem (Wilkie). $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ *is a Tarski system*, *i.e.*

• *If* $S, T \in \mathcal{D}_n$, *then* $S \cup T$, $S \cap T$ *and* $S \setminus T \in \mathcal{D}_n$.

• *If* $S \in \mathcal{D}_{n+1}$, *then* $\pi(S) \in \mathcal{D}_n$, *where* $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *is the natural projection.*

2.4. Proposition. *The closure*, *the interior and the boundary* (*in* $\mathbb{R}^n$) *of a* $\mathcal{D}_n$-*set are* $\mathcal{D}_n$-*sets.*

Proof. This follows from Theorem 2.3. ∎

2.5. Remark. (i) A $\mathcal{D}$-set, in general, is not subanalytic (e.g. $\{(x, y) : x > 0, y = \exp(-1/x)\}$ in $\mathbb{R}^2$).

(ii) By Propositions 1.4 and 1.5, the dimension of a $\mathcal{D}_n$-set S, defined by

$$\dim S := \max\{\dim \Gamma : \Gamma \text{ is an analytic submanifold of } \mathbb{R}^n \text{ contained in } S\},$$

equals $\max_j \dim B_j$, where B_j's are given in Proposition 1.5.

The following definition, inspired by Łojasiewicz's proof of Tarski's Theorem in [8], is introduced by L. van den Dries (see [3, §8]).

2.6. Definition. (i) A map $f : S \to \mathbb{R}^m$ with $S \subset \mathbb{R}^n$ is called a $\mathcal{D}$-*map* if its graph belongs to $\mathcal{D}_{n+m}$. In this case it is called $\mathcal{D}$-*analytic* if there is an open neighborhood U of S in $\mathbb{R}^n$ with $U \in \mathcal{D}_n$ and an analytic $\mathcal{D}$-map $F : U \to \mathbb{R}^m$ such that $F|_S = f$.

(ii) $\mathcal{D}_n$-*analytic cells* in $\mathbb{R}^n$ are defined by induction on n:

$\mathcal{D}_1$-analytic cells are points $\{r\}$ or open intervals $(a, b), -\infty \le a < b \le \infty$.

If C is a $\mathcal{D}_n$-analytic cell and $f, g : C \to \mathbb{R}$ are $\mathcal{D}$-analytic such that $f < g$, then the sets

$$\begin{aligned}
(f, g) &:= \{(x, r) \in C \times \mathbb{R} : f(x) < r < g(x)\},\\
(-\infty, f) &:= \{(x, r) \in C \times \mathbb{R} : r < f(x)\},\\
(g, \infty) &:= \{(x, r) \in C \times \mathbb{R} : g(x) < r\},\\
\Gamma(f) &:= \operatorname{graph} f \quad \text{and} \quad C \times \mathbb{R}
\end{aligned}$$

are $\mathcal{D}_{n+1}$-analytic cells.

(iii) A $\mathcal{D}$-*analytic decomposition* of $\mathbb{R}^n$ is defined by induction on n:

A $\mathcal{D}$-analytic decomposition of $\mathbb{R}^1$ is a finite collection of intervals and points

$$\{(-\infty, a_1), \ldots, (a_k, \infty), \{a_1\}, \ldots, \{a_k\}\}, \qquad \text{where } a_1 < \ldots < a_k.$$

A $\mathcal{D}$-analytic decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into $\mathcal{D}_{n+1}$-analytic cells C such that the set of all the projections $\pi(C)$ is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ (here $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the natural projection).

We say that a decomposition *partitions* S if S is a union of some cells of the decomposition.

2.7. Remark. Obviously, every $\mathcal{D}_n$-analytic cell S is a $\mathcal{D}_n$-set. Moreover, it is a connected analytic submanifold of $\mathbb{R}^n$. In fact, there are $r \in \mathbb{N}$, $r \le n$ and a permutation σ of $\{1, \ldots, n\}$, $p(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(r)})$ such that $h = p|_S$ is a C^ω-diffeomorphism [1] from S onto an open cell in $\mathbb{R}^r$.

If, moreover, $f : S \to \mathbb{R}^m$, then f is $\mathcal{D}$-analytic iff $f \circ h^{-1}$ is a $\mathcal{D}$-map, analytic on $h(S)$. Indeed, define

$$U = \{x \in \mathbb{R}^n : p(x) \in p(S)\} \quad \text{and} \quad F(x) = f \circ h^{-1}(p(x)), \quad x \in U.$$

Then U, F satisfy the condition of Definition 2.6(i) for f, S.

2.8. Theorem (L. van den Dries & C. Miller).

(I_n) *For $S_1, \ldots, S_k \in \mathcal{D}_n$ there is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ partitioning $S_1, \ldots, S_k$.*

(II_n) *For every $\mathcal{D}$-function $f : S \to \mathbb{R}$ with $S \in \mathcal{D}_n$, there is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ partitioning S such that, for each cell $C \subset S$ in the decomposition, the restriction $f|_C : C \to \mathbb{R}$ is $\mathcal{D}$-analytic.*

To prove the theorem we need two lemmas.

2.8.1. Lemma. *Suppose A is an open $\mathcal{D}_n$-set and $f : A \to \mathbb{R}$ is a $\mathcal{D}$-map. Then*

$$R^0(f) := \{x \in A : f \text{ is continuous at } x\} \in \mathcal{D}_n,$$
$$A \setminus R^0(f) \in \mathcal{D}_n \quad \text{and} \quad \dim(A \setminus R^0(f)) < n.$$

Proof. By Proposition 2.4 the closure of the graph of f, $\overline{\Gamma(f)}$, is a $\mathcal{D}_{n+1}$-set, so

$$R^0(f) = \{x \in A : \exists \varepsilon, M > 0, \forall x' \in A, |x - x'| < \varepsilon \Rightarrow |f(x')| \le M \text{ and } \forall (x', y) \in \overline{\Gamma(f)}, |x - x'| < \varepsilon \Rightarrow (x', y) \in \Gamma(f)\}$$

and $A \setminus R^0(f)$ are $\mathcal{D}_n$-sets, by Theorem 2.3.

Since $\Gamma(f) \in \mathcal{D}_{n+1}$, by Propositions 2.2 and 1.5, it follows that $\Gamma(f) = \pi(\bigcup_j B_j)$, where $\pi : \mathbb{R}^{n+1} \times \mathbb{R}^m \to \mathbb{R}^{n+1}$ is the natural projection, each B_j is a connected component of an $\mathcal{R}_{n+1+m}$-semianalytic leaf and $\pi|_{B_j}$ is an immersion.

Define $X = \bigcup_{j : \dim B_j = n} \pi(B_j)$ and $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $p(x, y) = x$, the projection on the first n coordinates. Then $p(X) \subset R^0(f)$. Indeed, for all $(x, y) \in X \subset \Gamma(f)$, there are $B = B_{j_0}$ and $z \in B$ such that $\pi(z) = (x, y)$ and $\dim B = n$. Since $\pi|_B$ is an immersion and p is 1-1 on $\pi(B)$ and $\dim B = n$, there is a neighborhood U of z in B such that $p \circ \pi(B \cap U)$ is a neighborhood

---

[1] "C^ω" stands for "analytic".

of x. So the germs at (x,y) of $\Gamma(f)$ and $\pi(B\cap U)$ are equal, i.e. f is continuous at x.

Therefore, $A\setminus R^0(f)\subset A\setminus p(X)\subset\bigcup_{j:\dim B_j<n}\pi(B_j)$, and this implies $\dim(A\setminus R^0(f))<n$. ∎

2.8.2. Lemma. *Let S be a $\mathcal{D}_{n+1}$-set. Suppose that for all x in $\mathbb{R}^n$, $S_x:=(\{x\}\times\mathbb{R})\cap S$ is finite. Then there is $N\in\mathbb{N}$ such that*

$$\operatorname{card} S_x\le N,\qquad \forall x\in\mathbb{R}^n.$$

Proof. By Proposition 2.2, $S=\pi(F^{-1}(0))$, where $F\in\mathcal{R}_{n+1+m}$ and $\pi:\mathbb{R}^{n+1}\times\mathbb{R}^m\to\mathbb{R}^{n+1}$ is the natural projection. By Khovanskiĭ's property (see [4] or [5, Ch. III, §3.14]) there is $N\in\mathbb{N}$ such that

$$\operatorname{nc}(F^{-1}(0)\cap(\{x\}\times\mathbb{R}\times\mathbb{R}^m))\le N,\qquad \forall x\in\mathbb{R}^n.$$

(Here nc denotes the number of connected components.) This implies

$$\operatorname{nc}(S_x)=\operatorname{nc}(\pi(F^{-1}(0))\cap\{x\}\times\mathbb{R})\le N,\qquad \forall x\in\mathbb{R}^n,$$

and from the assumption, $\operatorname{card} S_x\le N$, $\forall x\in\mathbb{R}^n$. ∎

2.8.3. Proof of Theorem 2.8. Induction on n.

Proof of (I_1). This follows from Propositions 2.2 and 1.3.

Proof of (II_1). Suppose $f:S\to\mathbb{R}$ is a $\mathcal{D}$-map and $S\in\mathcal{D}_1$. By (I_1) it suffices to prove (II_1) for $S=(a,b)$ and by Lemma 2.8.1 we can suppose that f is continuous on (a,b). By Proposition 2.2 there are $m\in\mathbb{N}$ and $F\in\mathcal{R}_{2+m}$ such that

$$\Gamma(f)=\{(x,y)\in S\times\mathbb{R}:\exists z\,(F(x,y,z)=0)\}.$$

From Corollary 1.6 there are $h_j=(h_{j1},\ldots,h_{j,m+1})$ with $h_{ji}\in\mathcal{R}_{2+m}$ and $g_{j1},\ldots,g_{jp}\in\mathcal{R}_{2+m}$, $i=1,\ldots,m+1$, $j=1,\ldots,l$, such that

$$\{x:\exists y,z\,(F(x,y,z)=0)\}=\bigcup_j\bigg\{x:\exists y,z\bigg(F(x,y,z)=h_j(x,y,z)=0,\\ \frac{Dh_j}{D(y,z)}(x,y,z)\ne0,\ g_{js}(x,y,z)>0,\ s=1,\ldots,p\bigg)\bigg\}.$$

For each $j=1,\ldots,l$ define

$$A_j=\bigg\{x\in S:\exists z\bigg(h_j(x,f(x),z)=0,\\ \frac{Dh_j}{D(y,z)}(x,f(x),z)\ne0,\ g_{js}(x,f(x),z)>0,\ s=1,\ldots,p\bigg)\bigg\}.$$

Then $A_j\in\mathcal{D}_1$ and $S=\bigcup_j A_j$. By (I_1) there is a decomposition of $\mathbb{R}$ partitioning $A_1,\ldots,A_l$. On each interval of the decomposition contained in A_j, f is continuous and satisfies the conditions of the implicit function theorem, so f is analytic on this interval, and (II_1) follows.

Now suppose that $(\mathrm{I}_1), \ldots, (\mathrm{I}_n), (\mathrm{II}_1), \ldots, (\mathrm{II}_n)$ hold.

Proof of (I_{n+1}). Suppose $S_1, \ldots, S_k \in \mathcal{D}_{n+1}$. Set $Y = \bigcup_{\alpha=1}^k \mathrm{bd}_n(S_\alpha)$, where

$$\mathrm{bd}_n(S) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : (x,y) \text{ is a boundary point in } \{x\} \times \mathbb{R} \text{ of } S_x = S \cap (\{x\} \times \mathbb{R})\}.$$

Then $Y \in \mathcal{D}_{n+1}$, by Theorem 2.3, and by Propositions 2.2, 1.3 and Lemma 2.8.2 there is $N \in \mathbb{N}$ such that

$$\operatorname{card} Y_x \le N, \quad \forall x \in \mathbb{R}^n.$$

For each $i = 1, \ldots, N$, $B_i := \{x \in \mathbb{R}^n : \operatorname{card} Y_x = i\}$ is a $\mathcal{D}_n$-set, by Theorem 2.3. There are functions $f_{i1}, \ldots, f_{ii}$ on B_i such that $-\infty := f_{i0} < f_{i1} < \ldots < f_{ii} < f_{i,i+1} := \infty$ and

$$Y_x = \{f_{i1}(x), \ldots, f_{ii}(x)\} \quad \text{for } x \in B_i.$$

Note that f_{ij},where $j = 1, \ldots, i$, are $\mathcal{D}$-maps, because

$$\Gamma(f_{ij}) = \{(x,y) : x \in B_i, \exists (x,y_1), \ldots, (x,y_i) \in Y, y_1 < \ldots < y_j = y < \ldots < y_i\}.$$

For any $\alpha = 1, \ldots, k$ define

$$C_{\alpha ij} = \{x \in B_i : (x, f_{ij}(x)) \in (S_\alpha)_x\},$$
$$D_{\alpha ij} = \{x \in B_i : \{x\} \times (f_{ij}(x), f_{i,j+1}(x)) \subset (S_\alpha)_x\}.$$

Then $C_{\alpha ij}$ and $D_{\alpha ij}$ are $\mathcal{D}$-sets.

From (I_n), (II_n) there is a $\mathcal{D}$-analytic decomposition, say $\mathcal{P}$, of $\mathbb{R}^n$ partitioning all $C_{\alpha ij}$ and $D_{\alpha ij}$ such that for each $C \in \mathcal{P}$, if $C \subset B_i$ then $f_{ij}|_C$ is $\mathcal{D}$-analytic. The collection

$$\bigcup_{i=1}^N \bigcup_{\substack{C \in \mathcal{P} \\ C \subset B_i}} \{(f_{ij}|_C, f_{i,j+1}|_C), \Gamma(f_{il}|_C) : j = 0, \ldots, i,\ l = 1, \ldots, i\} \cup \{C \times \mathbb{R} : C \in \mathcal{P},\ C \cap B_i = \emptyset,\ \forall i = 1, \ldots, N\}$$

is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^{n+1}$ partitioning $S_1, \ldots, S_k$.

Proof of (II_{n+1}). Suppose $S \subset \mathbb{R}^{n+1}$ and $f : S \to \mathbb{R}$ is a $\mathcal{D}$-function. By (I_{n+1}) we can suppose that S is a $\mathcal{D}_{n+1}$-analytic cell and it suffices to find a decomposition of S into $\mathcal{D}$-analytic cells such that the restriction of f to each cell is $\mathcal{D}$-analytic.

Case 1: $\dim S < n+1$. By Remark 2.7 there are $r = \dim S$ $(< n+1)$ and $h : S \to \mathbb{R}^r$ of the form $h(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(r)})$ such that h is a C^ω-diffeomorphism from S onto the $\mathcal{D}_r$-analytic cell $h(S)$. Note that $f \circ h^{-1} : h(S) \to \mathbb{R}$ is a $\mathcal{D}$-function.

By (II_r) there is a decomposition of $h(S)$ into cells B such that, for each B, $f\circ h^{-1}|_B$ is $\mathcal{D}$-analytic. This implies S is decomposed into the cells $h^{-1}(B)\cap S$ on each of which f is $\mathcal{D}$-analytic (make use of Remark 2.7).

Case 2: $\dim S = n+1$. Then S is open. By Lemma 2.8.1, (I_{n+1}) and Case 1 we can assume that f is continuous on S. Similarly to the proof of (II_1), there is $F\in\mathcal{R}_{n+2+m}$ such that

$$\Gamma(f)=\{(x,y)\in\mathbb{R}^{n+1}\times\mathbb{R} : \exists z\,(F(x,y,z)=0)\}.$$

By Corollary 1.6,

$$\{x:\exists y,z\,(F(x,y,z)=0)\}=\bigcup_{j=1}^{l}\bigg\{x:\exists y,z\bigg(F(x,y,z)=h_j(x,y,z)=0,\\ \frac{Dh_j}{D(y,z)}(x,y,z)\neq 0,\ g_{js}(x,y,z)>0,\ s=1,\dots,p\bigg)\bigg\},$$

where $h_j=(h_{j1},\dots,h_{j,m+1})$, $h_{ij},g_{j1},\dots,g_{jp}\in\mathcal{R}_{n+2+m}$, $j=1,\dots,l$.

For $j=1,\dots,l$ define

$$A_j=\bigg\{x\in S:\exists z\bigg(h_j(x,f(x),z)=0,\\ \frac{Dh_j}{D(y,z)}(x,f(x),z)\neq 0,\ g_{js}(x,f(x),z)>0,\ s=1,\dots,p\bigg)\bigg\}.$$

Then $S=\bigcup_j A_j$ and $A_j\in\mathcal{D}_{n+1}$. By (I_{n+1}) there is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^{n+1}$ partitioning $A_1,\dots,A_l$. For each cell of the decomposition with dimension $<n+1$, we apply Case 1. For each cell C of dimension $n+1$ with $C\subset A_i$ we can apply the implicit function theorem to the continuous function $f|_C$. This finishes the proof. ∎

2.9. Corollary. *The class of $\mathcal{D}$-sets has the Łojasiewicz property*:

(Ł) *Every $\mathcal{D}$-set has only finitely many connected components and each component is also a $\mathcal{D}$-set.*

2.10. Corollary (C^ω-stratification of $\mathcal{D}$-sets). *Let $S_1,\dots,S_k$ be $\mathcal{D}$-sets. Then there is a C^ω-stratification of $\mathbb{R}^n$ compatible with $S_1,\dots,S_k$. Precisely, there is a finite family $\{\Gamma^d_\alpha\}$ of subsets of $\mathbb{R}^n$ such that*:

(S1) *Γ^d_α are disjoint, $\mathbb{R}^n=\bigcup_{\alpha,d}\Gamma^d_\alpha$ and $S_i=\bigcup\{\Gamma^d_\alpha:\Gamma^d_\alpha\cap S_i\neq\emptyset\}$, $i=1,\dots,k$.*

(S2) *Each Γ^d_α is a $\mathcal{D}_n$-analytic cell of dimension d.*

(S3) *$\overline{\Gamma^d_\alpha}\setminus\Gamma^d_\alpha$ is a union of some cells Γ^e_β with $e<d$.*

Proof. The following lemma is proved in [2, Ch. 7, Th. 1.8].

2.10.1. Lemma. $\dim(\overline{C}\setminus C)<\dim C$ *for every nonempty $\mathcal{D}$-set $C\subset\mathbb{R}^n$.*

Now, applying Theorem 2.8 iteratively, we construct families $\mathcal{F}^d$ by decreasing induction on d: Let $\mathcal{P}^n$ be a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ partitioning $S_1, \ldots, S_k$. Define $\mathcal{F}^n = \{C \in \mathcal{P}^n : \dim C = n\}$. Suppose that $\mathcal{F}^n, \ldots, \mathcal{F}^{d+1}$ are constructed ($d \geq 0$). Let $\mathcal{P}^d$ be a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ partitioning $S_1, \ldots, S_k, \overline{C} \setminus C$ where $C \in \mathcal{F}^{d+1} \cup \ldots \cup \mathcal{F}^n$. Define

$$\mathcal{F}^d = \{C \in \mathcal{P}^d : \dim C = d, C \cap C' = \emptyset, \forall C' \in \mathcal{F}^{d+1} \cup \ldots \cup \mathcal{F}^n\}.$$

Then, by the construction and Lemma 2.10.1, the family of cells $\mathcal{F} = \bigcup_{0 \leq d \leq n} \mathcal{F}^d$ satisfies (S1)–(S3). ■

2.11. Corollary. *Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{D}$-function. Then there are $a_1 < \ldots < a_k$ such that f is analytic on each interval (a_i, a_{i+1}), $i = 0, \ldots, k$, where $a_0 := -\infty$ and $a_{k+1} := \infty$.*

2.12. Corollary. *Let M be an analytic submanifold of $\mathbb{R}^n$ and $f_i : M \to \mathbb{R}$, $i \in I$, be a family of analytic $\mathcal{D}$-functions. Then there are $i_1, \ldots, i_k \in I$ such that*

$$\bigcap_{i \in I} f_i^{-1}(0) = f_{i_1}^{-1}(0) \cap \ldots \cap f_{i_k}^{-1}(0).$$

Proof. Induction on $d = \dim M$. By Corollary 2.9, it suffices to prove the corollary for connected analytic submanifolds. If $d = 0$ it is clear. Suppose that $d > 0$.

If $f_i \equiv 0$ for every $i \in I$, the corollary is verified. If there is $\mu \in I$ such that $f_\mu \not\equiv 0$, then $\dim f_\mu^{-1}(0) < d$. The corollary follows from Theorem 2.8 and the inductive hypothesis. ■

Acknowledgments. I would like to express my thanks to Prof. Wiesław Pawłucki for his help and encouragement.

References

[1] Z. Denkowska, S. Łojasiewicz and J. Stasica, *Certaines propriétés élémentaires des ensembles sous-analytiques*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), 529–536.

[2] L. van den Dries, *Tame topology and O-minimal structures*, mimeographed notes, 1991.

[3] L. van den Dries and C. Miller, *On the real exponential field with restricted analytic functions*, Israel J. Math. 85 (1994), 19–56.

[4] A. G. Khovanskiĭ, *On a class of systems of transcendental equations*, Dokl. Akad. Nauk SSSR 255 (1980), 804–807 (in Russian).

[5] —, *Fewnomials*, Transl. Math. Monographs 88, Amer. Math. Soc., 1991.

[6] J. Knight, A. Pillay and C. Steinhorn, *Definable sets in ordered structures. II*, Trans. Amer. Math. Soc. 295 (1986), 593–605.

[7] T. L. Loi, *C^k-regular points of sets definable in the structure* $\mathbb{R}_{\exp}$, preprint, 1992.

[8] S. Łojasiewicz, *Ensembles Semi-Analytiques*, mimeographed notes, I.H.E.S., Bures-sur-Yvette, 1965.

[9] J. C. Tougeron, *Sur certaines algèbres de fonctions analytiques*, Séminaire de géométrie algébrique réelle, Paris VII, 1986.
[10] —, *Algèbres analytiques topologiquement noethériennes. Théorie de Khovanskiĭ*, Ann. Inst. Fourier (Grenoble) 41 (4) (1991), 823–840.
[11] A. J. Wilkie, *Some model completeness results for expansions of the ordered field of real numbers by Pfaffian functions*, preprint, 1991.
[12] —, *Model completeness results for expansions of the real field II: The exponential function*, manuscript, 1991.

Permanent address

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DALAT
1 PHU DONG THIEN VUONG
DALAT, VIETNAM

Current address

INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
REYMONTA 4
30-059 KRAKÓW, POLAND

Reçu par la Rédaction le 3.6.1993