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An arc-analytic function with nondiscrete singular set

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Abstract. We construct an arc-analytic function (i.e. analytic on every real-analytic arc) in $\mathbb{R}^2$ which is analytic outside a nondiscrete subset of $\mathbb{R}^2$.

Let M be a real-analytic manifold. A function $f : M \to \mathbb{R}$ is called *arc-analytic* iff for every analytic arc $\gamma :]-\varepsilon, \varepsilon[\to M$ the composition $f \circ \gamma$ is analytic (see [K1]). For every function f on M, let $\operatorname{Sing} f$ denote the set of points of nonanalyticity of f. If f is an arc-analytic function with subanalytic graph, then $\operatorname{Sing} f$ is a subanalytic subset of M (see [T] or [K2]); moreover, $\dim(\operatorname{Sing} f) \le \dim M - 2$ (see [K1]). Actually, in this case a stronger result is true: there exists a locally finite composition π of local blowing-ups of M such that $f \circ \pi$ is analytic (see [BM], [P]). Recently examples of arc-analytic functions with nonsubanalytic graphs were given ([K3] and [BMP], where a discontinuous example is given).

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is an arc-analytic function. Professor Siciak asked whether $\operatorname{Sing} f$ is always discrete. By the previous remarks this is the case if, for example, f has subanalytic graph, because $\operatorname{Sing} f$, being subanalytic of dimension 0, contains only isolated points.

In this note we construct an arc-analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{Sing} f$ is nondiscrete and f is unbounded at each point of $\operatorname{Sing} f$. Our construction is based on an idea of [K3].

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Let $a_\nu \in \mathbb{R}$, $\nu \in \mathbb{N}$, $\nu \ge 1$, be a convergent sequence in $\mathbb{R}$. Define

$$a_0 = \lim_{\nu \to \infty} a_\nu, \qquad Z_0 = \{(a_\nu, 0) \in \mathbb{R}^2 : \nu \in \mathbb{N},\ \nu \ge 0\}.$$

We will construct an arc-analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{Sing} f = Z_0$, and f is unbounded at each point of Z_0.

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We will blow up every point of Z_0 infinitely many times. To get a formal construction we take the projective limit of the following system.

Set $c_\nu = (a_\nu, 0)$, $P_0 = \{y = 0\} \subset \mathbb{R}^2$.

(i) Let $X_0 = \mathbb{R}^2$, and let $\pi_{1,0} : X_1 \to X_0$ be the blowing-up of c_0 in X_0. Let P_1 be the strict transform of P_0. We put

$$c_0^1 = \pi_{1,0}^{-1}(c_0) \cap P_1 \quad \text{and} \quad c_\nu^1 = \pi_{1,0}^{-1}(c_\nu) \quad \text{for } \nu \geq 1$$

(we assume that $c_\nu \neq c_\mu$ for $\nu \neq \mu$).

(ii) Suppose we have already constructed $\pi_{n,n-1} : X_n \to X_{n-1}$, and P_n is the strict transform of P_{n-1} by $\pi_{n,n-1}$. Suppose we also have a sequence $c_\nu^n \in P_n$ such that $\pi_{n,n-1}(c_\nu^n) = c_\nu^{n-1}$, $\nu \in \mathbb{N}$. We define $\pi_{n+1,n}$ to be the compositon

$$X_n^n \xrightarrow{p_n^{n-1}} X_n^{n-1} \xrightarrow{p_n^{n-2}} \ldots \xrightarrow{p_n^1} X_n^1 \xrightarrow{p_n^0} X_n^0 = X_n$$

where each $p_n^i : X_n^{i+1} \to X_n^i$, $i = 0, \ldots, n-1$, is the blowing-up of $(p_n^{i-1} \circ \ldots \circ p_n^0)^{-1}(c_i^n)$ in X_n. We put $X_{n+1} = X_n^n$ and

$$\pi_{n+1,n} = p_n^{n-1} \circ \ldots \circ p_n^0 .$$

Finally, let P_{n+1} be the strict transform of P_n by $\pi_{n+1,n}$ and let

$$c_\nu^{n+1} = (\pi_{n+1,n})^{-1}(c_\nu^n) \cap P_{n+1}, \qquad \nu \in \mathbb{N} .$$

We define $Z_n = \{c_\nu^n \in X_n : \nu \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ we put $\pi_{n,n} = \mathrm{id}_{X_n}$, and for $m \leq n$ we put

$$\pi_{n,m} = \pi_{n,n-1} \circ \ldots \circ \pi_{m+1,m} .$$

Hence we have constructed a projective system $\pi_{n,m} : X_n \to X_m$, $m \leq n$, with a subsystem $Z_n \to Z_m$. Then there exist topological spaces $X = \varprojlim X_n$, $Z = \varprojlim Z_n$ and continuous mappings $\mathrm{pr}_n : X \to X_n$ such that $\mathrm{pr}_m = \pi_{n,m} \circ \mathrm{pr}_n$ for $m \leq n$.

Set $L = X \setminus Z$. Clearly L is a Hausdorff σ-compact topological space. We will define a structure of a real-analytic manifold on L.

Let $x = (x_n)_{n \in \mathbb{N}} \in L$. Then there exists an open neighbourhood U of x in L and $n_0 \in \mathbb{N}$ such that if $y = (y_n) \in U$ then $y_k = \mathrm{pr}_k(y) \notin Z_k$ for $k \geq n_0$. Hence $\mathrm{pr}_k|_U : U \to \mathrm{pr}_k(U)$ is a homeomorphism. Notice that $\mathrm{pr}_k(U)$ is an open subset of the real-analytic manifold X_k. Clearly the family of all such mappings defines a structure of a real-analytic manifold on L. Moreover, each pr_k is analytic on L. Notice also that

$$\mathrm{pr}_n : L \setminus \mathrm{pr}_n^{-1}(Z_n) \to X_n \setminus Z_n$$

is an analytic diffeomorphism. In the sequel we need $\mathrm{pr}_0^{-1} : \mathbb{R}^2 \setminus Z_0 \to L$, which we denote by q. The mapping q has the following property:

LEMMA. *Let $\gamma :]-\varepsilon,\varepsilon[\to \mathbb{R}^2$ be an analytic arc such that $\gamma(t) \notin P_0 = \{y=0\}$ for $t \neq 0$. Then the mapping*

$$q \circ \gamma :]-\varepsilon, 0[\cup]0, \varepsilon[\to L$$

extends to an analytic mapping from $]-\varepsilon,\varepsilon[$ to L.

Proof. If $\gamma(0) \notin Z_0$ then the assertion is trivial. Suppose that $\gamma(0) = c_{\nu_0}$ for some $\nu_0 \in \mathbb{N}$. The order of contact of $\gamma(]-\varepsilon,\varepsilon[)$ and $P_0 = \{y=0\}$ at $\gamma(0)$ is finite. If we blow up the point $\gamma(0)$, then either the strict transforms of those curves are disjoint or the order of their contact decreases by 1. Hence for some $n \in \mathbb{N}$,

$$\lim_{t \to 0} (\pi_{n,0})^{-1} \circ \gamma(t) = \widetilde{\gamma}_n(0) \notin P_n\,.$$

Clearly $\widetilde{\gamma}_n = (\pi_{n,0})^{-1} \circ \gamma$ has an analytic extension through 0. Since

$$q \circ \gamma = \mathrm{pr}_n^{-1} \circ (\pi_{n,0})^{-1} \circ \gamma$$

and pr_n^{-1} is analytic outside Z_n (recall that $Z_n \subset P_n$), it follows that $q \circ \gamma$ extends to a function analytic at 0. This ends the proof of the lemma.

Recall that our analytic manifold L has a countable basis of topology, hence by the Grauert Embedding Theorem ([G]) there exists a proper analytic embedding $\varphi : L \to \mathbb{R}^N$ for some $N \in \mathbb{N}$.

Take now a countable subset A of $P_0 \setminus Z_0$ which is discrete in $\mathbb{R}^2 \setminus Z_0$ and $\overline{A} \setminus A = Z_0$. Notice that for every $n \in \mathbb{N}$ the set $\pi_{n,0}^{-1}(A)$ is also discrete in $X_n \setminus Z_n$; moreover, pr_n maps homeomorphically $L \setminus \mathrm{pr}_n^{-1}(Z_n)$ onto $X_n \setminus Z_n$. Hence $q^{-1}(A)$ is also discrete in $L = \varprojlim X_n \setminus \varprojlim Z_n$. Thus the set $\widetilde{A} = \varphi(q^{-1}(A))$ is discrete in $\mathbb{R}^N$, since φ is proper.

We claim that there exists a discrete subset $\widetilde{B}$ of $\varphi(L)$ such that if we set $B = \mathrm{pr}_0 \circ \varphi^{-1}(\widetilde{B})$ then

$$B \cap P_0 = \emptyset \quad \text{and} \quad \overline{B} \setminus B = Z_0\,.$$

To get such a $\widetilde{B}$ let us arrange the elements of $\widetilde{A}$ in a sequence $\widetilde{a}_k,\ k \in \mathbb{N}$. Notice that $\varphi(\mathrm{pr}_0^{-1}(P_0))$ is nowhere dense in $\varphi(L)$. Hence there exists a sequence

$$\widetilde{b}_k \in \varphi(L) \setminus \varphi(\mathrm{pr}_0^{-1}(P_0))$$

such that $\|\widetilde{a}_k - \widetilde{b}_k\| < 1/k$. We put $B = \{b_0, b_1, \ldots\}$, where $b_k = \mathrm{pr}_0 \circ \varphi^{-1}(\widetilde{b}_k)$. Let us write, in coordinates in $\mathbb{R}^2$, $b_k = (x_k, y_k)$. Notice that $y_k \neq 0$ for all $k \in \mathbb{N}$.

Now take an analytic function $\widetilde{h} : \mathbb{R}^N \to \mathbb{R}$ such that $\widetilde{h}(\widetilde{b}_k) = y_k^{-2}$. Such an $\widetilde{h}$ exists since $\varphi(L)$ is closed in $\mathbb{R}^{\mathbb{N}}$, hence $\widetilde{B}$ is discrete in $\mathbb{R}^N$. Now put $h = \widetilde{h} \circ \varphi \circ q$ and observe that h is analytic in $\mathbb{R}^2 \setminus Z_0$. Finally, put

$$f(x,y) = \begin{cases} y h(x,y) & \text{if } (x,y) \notin Z_0\,, \\ 0 & \text{if } (x,y) \in Z_0\,. \end{cases}$$

To see that f is arc-analytic take an analytic arc $\gamma :]-\varepsilon, \varepsilon[\to \mathbb{R}^2$. If $\gamma(]-\varepsilon, \varepsilon[) \subset P_0 = \{y = 0\}$ then $f \circ \gamma \equiv 0$ is analytic. Otherwise the set $\gamma^{-1}(Z_0)$ is discrete in $]-\varepsilon, \varepsilon[$ and by the lemma $q \circ \gamma$ extends to an analytic mapping from $]-\varepsilon, \varepsilon[$ to L. Hence also $h \circ \gamma$ extends to an analytic mapping on $]-\varepsilon, \varepsilon[$. Thus $f \circ \gamma$ is analytic on $]-\varepsilon, \varepsilon[$.

Clearly f is analytic in $\mathbb{R}^2 \setminus Z_0$. Observe that $f(b_k) = f(x_k, y_k) = y_k^{-1}$ and $\lim_{k\to\infty} y_k = 0$. Since $\overline{B} \setminus B = Z_0$, for every $(x_0, y_0) \in Z_0$ we have

$$\limsup_{(x,y)\to(x_0,y_0)} |f(x, y)| = +\infty\,.$$

This proves that $\operatorname{Sing} f = Z_0$.

Remark. This example raises two questions about arc-analytic functions.

1) Can one find an arc-analytic function on a manifold M such that $\operatorname{Sing} f$ is dense in the analytic Zariski topology (i.e. every analytic function vanishing on $\operatorname{Sing} f$ must vanish on M)?

2) Given an arc-analytic function $f : M \to \mathbb{R}$, can one find a countable composition π of blowing-ups such that $f \circ \pi$ is analytic? Here countable composition might be understood as a projective limit as in our example.

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