An arc-analytic function with nondiscrete singular set

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Abstract. We construct an arc-analytic function (i.e. analytic on every real-analytic arc) in \( \mathbb{R}^2 \) which is analytic outside a nondiscrete subset of \( \mathbb{R}^2 \).

Let \( M \) be a real-analytic manifold. A function \( f : M \to \mathbb{R} \) is called \textit{arc-analytic} iff for every analytic arc \( \gamma : [-\varepsilon, \varepsilon] \to M \) the composition \( f \circ \gamma \) is analytic (see [K1]). For every function \( f \) on \( M \), let \( \text{Sing} f \) denote the set of points of nonanalyticity of \( f \). If \( f \) is an arc-analytic function with subanalytic graph, then \( \text{Sing} f \) is a subanalytic subset of \( M \) (see [T] or [K2]); moreover, \( \dim(\text{Sing} f) \leq \dim M - 2 \) (see [K1]). Actually, in this case a stronger result is true: there exists a locally finite composition \( \pi \) of local blowing-ups of \( M \) such that \( f \circ \pi \) is analytic (see [BM], [P]). Recently examples of arc-analytic functions with nonsubanalytic graphs were given ([K3] and [BMP], where a discontinuous example is given).

Suppose that \( f : \mathbb{R}^2 \to \mathbb{R} \) is an arc-analytic function. Professor Siciak asked whether \( \text{Sing} f \) is always discrete. By the previous remarks this is the case if, for example, \( f \) has subanalytic graph, because \( \text{Sing} f \), being subanalytic of dimension 0, contains only isolated points.

In this note we construct an arc-analytic function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( \text{Sing} f \) is nondiscrete and \( f \) is unbounded at each point of \( \text{Sing} f \). Our construction is based on an idea of [K3].

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Let \( a_\nu \in \mathbb{R}, \nu \in \mathbb{N}, \nu \geq 1 \), be a convergent sequence in \( \mathbb{R} \). Define

\[
\lim_{\nu \to \infty} a_\nu, \quad Z_0 = \{(a_\nu, 0) \in \mathbb{R}^2 : \nu \in \mathbb{N}, \nu \geq 0\}.
\]

We will construct an arc-analytic function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( \text{Sing} f = Z_0 \), and \( f \) is unbounded at each point of \( Z_0 \).

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We will blow up every point of $Z_0$ infinitely many times. To get a formal construction we take the projective limit of the following system.

Set $c_\nu = (a_\nu, 0), P_0 = \{y = 0\} \subset \mathbb{R}^2$.

(i) Let $X_0 = \mathbb{R}^2$, and let $\pi_{1,0} : X_1 \to X_0$ be the blowing-up of $c_0$ in $X_0$. Let $P_1$ be the strict transform of $P_0$. We put

\[ c_0^1 = \pi_{1,0}^{-1}(c_0) \cap P_1 \quad \text{and} \quad c_\nu^1 = \pi_{1,0}^{-1}(c_\nu) \quad \text{for} \quad \nu \geq 1 \]

(we assume that $c_\nu \neq c_\mu$ for $\nu \neq \mu$).

(ii) Suppose we have already constructed $\pi_{n,n-1} : X_n \to X_{n-1}$, and $P_n$ is the strict transform of $P_{n-1}$ by $\pi_{n,n-1}$. Suppose we also have a sequence $c_{\nu}^n \in P_n$ such that $\pi_{n,n-1}(c_{\nu}^n) = c_{\nu}^{n-1} \quad \nu \in \mathbb{N}$. We define $\pi_{n+1,n}$ to be the compositon

\[ X_n \xrightarrow{p_n^{-1}} X_{n-1} \xrightarrow{p_{n-1}^{-2}} \ldots \xrightarrow{p_1^{-1}} X_1 \xrightarrow{p_0^n} X_n = X_0 \]

where each $p_{i}^{n} : X_{i+1} \to X_{i}, \quad i = 0, \ldots, n-1$, is the blowing-up of $(p_{i-1}^n \circ \ldots \circ p_0^1)(c_{\nu}^n)$ in $X_n$. We put $X_{n+1} = X_n$ and

\[ \pi_{n+1,n} = p_{n-1}^{-1} \circ \ldots \circ p_0^n. \]

Finally, let $P_{n+1}$ be the strict transform of $P_n$ by $\pi_{n+1,n}$ and let

\[ c_{\nu}^{n+1} = (\pi_{n+1,n})^{-1}(c_{\nu}^n) \cap P_{n+1}, \quad \nu \in \mathbb{N}. \]

We define $Z_n = \{c_{\nu}^n \in X_n : \nu \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ we put $\pi_{n,n} = \text{id}_{X_n}$, and for $m \leq n$ we put

\[ \pi_{n,m} = \pi_{n,n-1} \circ \ldots \circ \pi_{m+1,m}. \]

Hence we have constructed a projective system $\pi_{n,m} : X_n \to X_m, \quad m \leq n$, with a subsystem $Z_n \to Z_m$. Then there exist topological spaces $X = \lim X_n, Z = \lim Z_n$ and continuous mappings $\text{pr}_n : X \to X_n$ such that $\text{pr}_m = \pi_{n,m} \circ \text{pr}_n$ for $m \leq n$.

Set $L = X \setminus Z$. Clearly $L$ is a Hausdorff $\sigma$-compact topological space. We will define a structure of a real-analytic manifold on $L$.

Let $x = (x_n)_{n \in \mathbb{N}} \in L$. Then there exists an open neighbourhood $U$ of $x$ in $L$ and $n_0 \in \mathbb{N}$ such that if $y = (y_n) \in U$ then $y_k = \text{pr}_k(y) \notin Z_k$ for $k \geq n_0$. Hence $\text{pr}_k|_U : U \to \text{pr}_k(U)$ is a homeomorphism. Notice that $\text{pr}_k(U)$ is an open subset of the real-analytic manifold $X_k$. Clearly the family of all such mappings defines a structure of a real-analytic manifold on $L$. Moreover, each $\text{pr}_k$ is analytic on $L$. Notice also that

\[ \text{pr}_n : L \setminus \text{pr}_1^{-1}(Z_n) \to X_n \setminus Z_n \]

is an analytic diffeomorphism. In the sequel we need $\text{pr}_0^{-1} : \mathbb{R}^2 \setminus Z_0 \to L$, which we denote by $q$. The mapping $q$ has the following property:
Let \( \gamma : ]-\varepsilon, \varepsilon[ \to \mathbb{R}^2 \) be an analytic arc such that \( \gamma(t) \not\in P_0 = \{ y = 0 \} \) for \( t \neq 0 \). Then the mapping

\[ q \circ \gamma : ]-\varepsilon, 0] \cup [0, \varepsilon[ \to \mathcal{L} \]

extends to an analytic mapping from \( ]-\varepsilon, \varepsilon[ \) to \( \mathcal{L} \).

Proof. If \( \gamma(0) \not\in Z_0 \) then the assertion is trivial. Suppose that \( \gamma(0) = c_{v_0} \) for some \( v_0 \in \mathbb{N} \). The order of contact of \( \gamma([-\varepsilon, \varepsilon[) \) and \( P_0 = \{ y = 0 \} \) at \( \gamma(0) \) is finite. If we blow up the point \( \gamma(0) \), then either the strict transforms of those curves are disjoint or the order of their contact decreases by 1. Hence for some \( n \in \mathbb{N} \),

\[ \lim_{t \to 0} (\pi_n, 0)^{-1} \circ \gamma(t) = \tilde{\gamma}_n(0) \not\in P_n. \]

Clearly \( \tilde{\gamma}_n = (\pi_n, 0)^{-1} \circ \gamma \) has an analytic extension through 0. Since

\[ q \circ \gamma = \text{pr}_n^{-1} \circ (\pi_n, 0)^{-1} \circ \gamma \]

and \( \text{pr}_n^{-1} \) is analytic outside \( Z_n \) (recall that \( Z_n \subset P_n \)), it follows that \( q \circ \gamma \) extends to a function analytic at 0. This ends the proof of the lemma.

Recall that our analytic manifold \( L \) has a countable basis of topology, hence by the Grauert Embedding Theorem ([G]) there exists a proper analytic embedding \( \varphi : L \to \mathbb{R}^N \) for some \( N \in \mathbb{N} \).

Take now a countable subset \( A \) of \( P_0 \setminus Z_0 \) which is discrete in \( \mathbb{R}^2 \setminus Z_0 \) and \( \bar{A} \setminus A = Z_0 \). Notice that for every \( n \in \mathbb{N} \) the set \( \pi_n, 0(A) \) is also discrete in \( X_n \setminus Z_n \); moreover, \( \text{pr}_n \) maps homeomorphically \( L \setminus \text{pr}_n^{-1}(Z_n) \) onto \( X_n \setminus Z_n \). Hence \( q^{-1}(A) \) is also discrete in \( L = \lim X_n \setminus \lim Z_n \). Thus the set \( \hat{A} = \varphi(q^{-1}(A)) \) is discrete in \( \mathbb{R}^N \), since \( \varphi \) is proper.

We claim that there exists a discrete subset \( \tilde{B} \) of \( \varphi(L) \) such that if we set \( B = \text{pr}_0 \circ \varphi^{-1}(\tilde{B}) \) then

\[ B \cap P_0 = \emptyset \quad \text{and} \quad \overline{B} \setminus B = Z_0. \]

To get such a \( \tilde{B} \) let us arrange the elements of \( \hat{A} \) in a sequence \( \tilde{a}_k, k \in \mathbb{N} \). Notice that \( \varphi(\text{pr}_n^{-1}(P_0)) \) is nowhere dense in \( \varphi(L) \). Hence there exists a sequence

\[ \tilde{b}_k \in \varphi(L) \setminus \varphi(\text{pr}_0^{-1}(P_0)) \]

such that \( \| \tilde{a}_k - \tilde{b}_k \| < 1/k \). We put \( B = \{ b_0, b_1, \ldots \} \), where \( b_k = \text{pr}_0 \circ \varphi^{-1}(\tilde{b}_k) \).

Let us write, in coordinates in \( \mathbb{R}^2 \), \( b_k = (x_k, y_k) \). Notice that \( y_k \neq 0 \) for all \( k \in \mathbb{N} \).

Now take an analytic function \( \tilde{h} : \mathbb{R}^N \to \mathbb{R} \) such that \( \tilde{h}(\tilde{b}_k) = y_k^{-2} \). Such an \( \tilde{h} \) exists since \( \varphi(L) \) is closed in \( \mathbb{R}^N \), hence \( \tilde{B} \) is discrete in \( \mathbb{R}^N \). Now put \( h = \tilde{h} \circ \varphi \circ q \) and observe that \( h \) is analytic in \( \mathbb{R}^2 \setminus Z_0 \). Finally, put

\[ f(x, y) = \begin{cases} yh(x, y) & \text{if } (x, y) \not\in Z_0, \\ 0 & \text{if } (x, y) \in Z_0. \end{cases} \]
To see that \( f \) is arc-analytic take an analytic arc \( \gamma : ]-\varepsilon,\varepsilon[ \rightarrow \mathbb{R}^2 \). If \( \gamma([]-\varepsilon,\varepsilon[) \subset P_0 = \{y = 0\} \) then \( f \circ \gamma \equiv 0 \) is analytic. Otherwise the set \( \gamma^{-1}(Z_0) \) is discrete in \( ]-\varepsilon,\varepsilon[ \) and by the lemma \( q \circ \gamma \) extends to an analytic mapping from \( ]-\varepsilon,\varepsilon[ \) to \( L \). Hence also \( h \circ \gamma \) extends to an analytic mapping on \( ]-\varepsilon,\varepsilon[ \). Thus \( f \circ \gamma \) is analytic on \( ]-\varepsilon,\varepsilon[ \).

Clearly \( f \) is analytic in \( \mathbb{R}^2 \setminus Z_0 \). Observe that \( f(b_k) = f(x_k, y_k) = y_k^{-1} \) and \( \lim_{k \to \infty} y_k = 0 \). Since \( \overline{B} \setminus B = Z_0 \), for every \( (x_0, y_0) \in Z_0 \) we have
\[
\limsup_{(x,y) \to (x_0,y_0)} |f(x,y)| = +\infty.
\]
This proves that \( \text{Sing} f = Z_0 \).

**Remark.** This example raises two questions about arc-analytic functions.

1) Can one find an arc-analytic function on a manifold \( M \) such that \( \text{Sing} f \) is dense in the analytic Zariski topology (i.e. every analytic function vanishing on \( \text{Sing} f \) must vanish on \( M \))? 

2) Given an arc-analytic function \( f : M \to \mathbb{R} \), can one find a countable composition \( \pi \) of blowing-ups such that \( f \circ \pi \) is analytic? Here countable composition might be understood as a projective limit as in our example.

### References


