ANNALES
POLONICI MATHEMATICI
LIX.3 (1994)

λ-Properties of Orlicz sequence spaces

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Abstract. It is proved that every Orlicz sequence space has the λ-property. Criteria for the uniform λ-property in Orlicz sequence spaces, with Luxemburg norm and Orlicz norm, are given.

1. Notations. Let X be a Banach space, $B(X)$ the closed unit ball, and $S(X)$ the unit sphere. A point e of a convex subset A of X is an *extreme point* of A if $x, y \in A$ and $2e = x + y$ imply $e = x = y$. The set of extreme points of A is denoted by $\operatorname{Ext} A$. For each $x \in B(X)$, we introduce a number

$$\lambda(x) = \sup\{\lambda \in [0,1] : x = \lambda e + (1-\lambda)y,\ y \in B(X),\ e \in \operatorname{Ext} B(X)\}.$$

If $\lambda(x) > 0$, then we call x a λ-*point* of $B(X)$. If $\lambda(x) > 0$ for all $x \in B(X)$, then X is said to have the λ-*property*. Moreover, if

$$\lambda(X) = \inf\{\lambda(x) : x \in B(X)\} > 0$$

then X is said to have the *uniform λ-property*.

It is well known that if X has the λ-property, then $B(X) = \operatorname{co}(\operatorname{Ext} B(X))$ and any element x in $B(X)$ can be expressed as $x = \sum \lambda_i x_i$, where $x_i \in \operatorname{Ext} B(X)$ and $\lambda_i \ge 0$ $(i \in \mathbb{N}), \sum \lambda_i = 1$. Moreover, if X has the uniform λ-property, then the series $x = \sum \lambda_i x_i$ converge uniformly for all x in $B(X)$ (see [1], [2]).

In [4], [5] and [7] the λ-property and uniform λ-property for Orlicz function spaces are discussed. This paper investigates those properties for Orlicz sequence spaces. We first introduce some notations. A function $M : \mathbb{R} \to \mathbb{R}$ is called an *Orlicz function* if it satisfies the following conditions:

(1) M is even, continuous, convex and $M(0) = 0$;
(2) $M(u) > 0$ for all $u \neq 0$, and

1991 *Mathematics Subject Classification*: Primary 46E30.
Key words and phrases: Orlicz function, Orlicz sequence space, extreme point, λ-property.
The authors are supported by the National Science Foundation of China and Heilongjiang Science Foundation.

(3) $\lim_{u\to 0} M(u)/u = 0$ and $\lim_{u\to\infty} M(u)/u = \infty$.

Let $N(v) = \sup\{uv - M(u) : u \in \mathbb{R}\}$. Then N is also an Orlicz function, and it is called the *complementary function* of M.

Let M be an Orlicz function. An interval $[a, b]$ is called a *structural affine interval* of M, or simply SAI of M, if M is affine on $[a, b]$ but it is not affine on either $[a - \varepsilon, b]$ or $[a, b + \varepsilon]$ for any $\varepsilon > 0$. Let $\{[a_i, b_i]\}_i$ be all the SAIs of M. We call

$$S_M = \mathbb{R} \setminus \bigcup_i (a_i, b_i)$$

the set of *strictly convex points* of M. Clearly, if $u, v \in \mathbb{R}$, $\alpha \in (0, 1)$ and $\alpha u + (1 - \alpha)v \in S_M$, then

$$M(\alpha u + (1 - \alpha)v) < \alpha M(u) + (1 - \alpha)M(v)\,.$$

Furthermore, S_M contains infinitely many points near the origin and $0 \in S_M$ since $M(u) > 0$ iff $u \neq 0$ (see [8]).

For any real number sequence $\{u(i)\}$, we introduce its *modular* by

$$\varrho_M(u) = \sum_{i=1}^{\infty} M(u(i))\,.$$

Then the *Orlicz sequence space*

$$l_M = \{u : \varrho_M(\lambda u) < \infty \text{ for some } \lambda > 0\}$$

with *Orlicz norm*

$$\|u\|^\circ = \inf\{k^{-1}[1 + \varrho_M(ku)] : k > 0\}$$

or *Luxemburg norm*

$$\|u\| = \inf\{\varrho_M(u/\alpha) \leq 1 : \alpha > 0\}$$

is a Banach space. We denote $(l_M, \|\ \|^\circ)$ and $(l_M, \|\ \|)$ by l_M° and l_M respectively.

For $u \in l_M^\circ$, let

$$k^* = k^*(u) = \inf\{k > 0 : \varrho_N(p(k|u|)) \geq 1\}\,,$$
$$k^{**} = k^{**}(u) = \sup\{k > 0 : \varrho_N(p(k|u|)) \leq 1\}\,,$$

where p is the right derivative of M. Then

$$K(u) = K_M(u) = [k^*, k^{**}] \neq \emptyset \qquad (u \neq 0)\,.$$

Moreover, $k \in K(u)$ $(u \neq 0)$ iff $\|u\|^\circ = k^{-1}[1 + \varrho_M(ku)]$ (see [8]).

2. λ-property of l_M

LEMMA 1. *Suppose* $\operatorname{Ext} B(X) \neq \emptyset$. *If* $x, y, z \in B(X)$ *and* $x = \alpha y + (1-\alpha)z$ *for some* $\alpha \in (0,1)$, *then* $\lambda(x) \geq \alpha\lambda(y)$. *Consequently*, $\lambda(0) = 1/2$ *and*

$$\lambda(u) \geq \max\{2^{-1}(1 - \|u\|), \lambda(u/\|u\|)\|u\|\}.$$

Proof. For any given $\varepsilon > 0$, choose $e \in \operatorname{Ext} B(X)$ and $u \in B(X)$ such that $y = \lambda e + (1-\lambda)u$ and $\lambda(y) - \varepsilon < \lambda$. Then

$$x = \alpha y + (1-\alpha)z = \alpha\lambda e + (1-\alpha\lambda)\frac{\alpha(1-\lambda)u + (1-\alpha)z}{1-\alpha\lambda}.$$

Since

$$\left\|\frac{\alpha(1-\lambda)u + (1-\alpha)z}{1-\alpha\lambda}\right\| \leq \frac{\alpha(1-\lambda) + (1-\alpha)}{1-\alpha\lambda} = 1,$$

we deduce $\lambda(x) \geq \alpha\lambda(y)$ as $\varepsilon > 0$ is arbitrary.

Pick $e \in \operatorname{Ext} B(X)$ arbitrarily; then $0 = 2^{-1}e + (-2^{-1})e$, hence, $\lambda(0) \geq 2^{-1}\lambda(e) = 2^{-1}$. On the other hand, if $0 = \lambda e + (1-\lambda)y$, where $e \in \operatorname{Ext} B(X)$ and $y \in B(X)$, then $1 \geq \|y\| = \lambda/(1-\lambda)$. Therefore, $\lambda \leq 2^{-1}$. Thus $\lambda(0) = 1/2$.

The last claim follows from

$$u = (1 - \|u\|)0 + \|u\|\frac{u}{\|u\|}. \ \blacksquare$$

LEMMA 2 ([6]). $x = (x(i)) \in \operatorname{Ext} B(l_M)$ *iff* $\varrho_M(x) = 1$ *and* $\operatorname{card}\{i \in \mathbb{N} : x(i) \in \mathbb{R} \setminus S_M\} \leq 1$.

THEOREM 3. *Each* l_M *has the* λ*-property.*

Proof. In view of Lemma 1, we only need to show $\lambda(x) > 0$ for each $x \in S(l_M) \setminus \operatorname{Ext} B(l_M)$. For convenience, we may assume $x(i) \geq 0$ for all $i \in \mathbb{N}$.

First we consider the case $\varrho_M(x) = 1$. This implies that there exist at least two coordinates of x belonging to the interiors of some SAIs of M by Lemma 2. For each $\lambda \in [0,1]$, define

$$y_\lambda(k) = \begin{cases} b_i & \text{if } b_i > x(k) > \lambda a_i + (1-\lambda)b_i \text{ for some } i \geq 1, \\ a_i & \text{if } a_i < x(k) \leq \lambda a_i + (1-\lambda)b_i \text{ for some } i \geq 1, \\ x(k) & \text{otherwise,} \end{cases}$$

and $f(\lambda) = \varrho_M(y_\lambda)$. Then $f(\lambda)$ is a nondecreasing, left-continuous function of λ and $f(0) < \varrho_M(x) = 1 < f(1)$. Moreover, if $y_\lambda(k) = b_i$, then

$$M(x(k)) > M(\lambda a_i + (1-\lambda)b_i) = \lambda M(a_i) + (1-\lambda)M(b_i) > (1-\lambda)M(b_i)$$

implies

$$f(\lambda) \leq \left(1 + \frac{1}{1+\lambda}\right)\varrho_M(x) < \infty$$

and so, $f(\lambda)$ is continuous at 0 and 1 by its definition and the Levy Theorem. Therefore, if we define $\sigma = \sup\{\lambda : \varrho_M(y_\lambda) \le 1\}$, then $\sigma \in (0,1)$ and $\varrho_M(y_\sigma) \le 1$. Set

$$N_i = \{k \in \mathbb{N} : x(k) = \sigma a_i + (1-\sigma)b_i\}\,.$$

Then there exists $E_i \subset N_i$ $(i \ge 1)$ such that the element $u = (u(k))_k$ defined by

$$u(k) = \begin{cases} b_i & \text{if } b_i > x(k) > \sigma a_i + (1-\sigma)b_i \text{ or } k \in E_i \text{ for some } i \ge 1,\\ a_i & \text{if } a_i < x(k) < \sigma a_i + (1-\sigma)b_i \text{ or } k \in N_i \setminus E_i \text{ for some } i \ge 1,\\ x(k) & \text{otherwise,} \end{cases}$$

satisfies $\varrho_M(u) \le 1$, and for any $k \in N_i \setminus E_i$, if we change the value of $u(k)$ to be b_i, then the modular of u will become greater than one. (By the definition of σ, such $\{E_i\}_i$ do exist.) If $\varrho_M(u) = 1$, then we define $y = u$. If $\varrho_M(u) < 1$, then there exists at least one nonempty set $E_{i'}$. In this case, we pick $k' \in E_{i'}$ arbitrarily and find $\alpha \in (a_{i'}, b_{i'})$ such that $\varrho_M(y) = 1$, where $y = (y(k))_k$ is defined by

$$y(k) = \begin{cases} \alpha, & k = k',\\ u(k), & k \ne k'. \end{cases}$$

Clearly, by Lemma 2, $y \in \operatorname{Ext} B(l_M)$. Set $z = \sigma^{-1}[x - (1-\sigma)y]$ when $\sigma \ge 1/2$. Then $x = (1-\sigma)y + \sigma z$ and $z(k) = y(k)$ when $y(k) = x(k)$. If $y(k) = b_i$, then

$$b_i > x(k) \ge \sigma a_i + (1-\sigma)b_i\,.$$

Therefore

$$\begin{aligned} b_i > x(k) \ge z(k) &= \sigma^{-1}[x(k) - (1-\sigma)y(k)]\\ &\ge \sigma^{-1}[\sigma a_i + (1-\sigma)b_i - (1-\sigma)b_i] = a_i\,. \end{aligned}$$

If $y(k) = a_i$, then by $\sigma \ge 1/2$, we also have

$$\begin{aligned} a_i < z(k) &\le \sigma^{-1}[\sigma a_i + (1-\sigma)b_i - (1-\sigma)a_i]\\ &= a_i + (\sigma^{-1} - 1)(b_i - a_i) \le b_i\,. \end{aligned}$$

Observe that M is affine on each $[a_i, b_i]$. Hence

$$\begin{aligned} 1 = \varrho_M(x) &= \varrho_M((1-\sigma)y + \sigma z)\\ &= (1-\sigma)\varrho_M(y) + \sigma\varrho_M(z) = 1 - \sigma + \sigma\varrho_M(z)\,. \end{aligned}$$

This shows that $\varrho_M(z) = 1$, and thus, $\lambda(x) \ge 1 - \sigma > 0$. Similarly, if $0 < \sigma < 1/2$, then by defining

$$z = \frac{1}{1-\sigma}(x - \sigma y)$$

we can deduce that $\lambda(x) \ge \sigma > 0$.

If $\varrho_M(x) < 1$, then for any $\alpha \in (0,1)$, since $\varrho_M(x/(1-\alpha)) = \infty$, we can select $n' \in \mathbb{N}$ and $0 < \alpha' < \alpha$ such that

$$\sum_{k=1}^{n'} M\left(\frac{x(k)}{1-\alpha'}\right) + \sum_{k>n'} M(x(k)) = 1\,.$$

Define $v = x$ and $u = (u(k))_k$ by

$$u(k) = \begin{cases} x(k)/(1-\alpha'), & k \le n',\\ x(k), & k > n'.\end{cases}$$

Then $x = (1-\alpha')u + \alpha' v$, $\varrho_M(u) = 1$, $\varrho_M(v) < 1$. Thus, $\lambda(u) > 0$, by the first part of the proof. Finally, Lemma 1 shows that $\lambda(x) \ge (1-\alpha')\lambda(u) > 0$. ■

Theorem 4. *l_M has the uniform λ-property iff M is strictly convex near the origin.*

Proof. $\Leftarrow$: Let M be strictly convex on $[0,d]$. Define $\beta = 1/M(d) + 2$. Referring to the proof of Theorem 3, we only need to show $\lambda(x) \ge 1/\beta$ for all $x = (x(i))_i \in S(l_M) \setminus \operatorname{Ext} B(l_M)$ with $\varrho_M(x) = 1$ and $x(i) \ge 0$ ($i \in \mathbb{N}$). For any $\lambda \in (0,1)$, we define y_λ and $\sigma \in (0,1)$ as in the proof of Theorem 3. First we assume $\sigma \ge 1/2$. If $\sigma \le 1 - 1/\beta$, then by the proof of Theorem 3, $\lambda(x) \ge 1-\sigma \ge 1/\beta$. Now, we consider the case $\sigma \ge 1-1/\beta$. Let $I = \{i \in \mathbb{N} : x(i) \in \mathbb{R} \setminus S_M\}$. Without loss of generality, we may assume $I = \{1,\ldots,m\}$ (clearly, $m < \beta$) and $x(i) \in (a_i, b_i)$ ($i \le m$), where $\{[a_i,b_i]\}_{i\le m}$ are SAIs of M. Set

$$J = \{i \le m : \lambda_i \le 1/\beta,\ x(i) = (1-\lambda_i)a_i + \lambda_i b_i\}\,.$$

Then $J \ne \emptyset$ since $\sigma > 1 - 1/\beta$. For convenience, we assume $J = \{1,\ldots,r\}$ and

$$\lambda_r[M(b_r) - M(a_r)] = \max_{i\le r}\{\lambda_i[M(b_i) - M(a_i)]\}\,.$$

For any $\delta \in [0,1]$, if we define $u_\delta = (u_\delta(i))_i$ by

$$u_\delta(i) = \begin{cases} (1-\delta)a_r + \delta b_r, & i = r,\\ a_i, & i < r,\\ b_i, & r < i \le m,\\ x(i), & i > m,\end{cases}$$

then since $r\lambda_r < 1$, and

$$\varrho_M(u_0) = \varrho_M(y_{1-1/\beta}) < \varrho_M(y_\sigma) \le 1$$

and

$$\begin{aligned}\varrho_M(u_\delta)-1&=\varrho_M(u_\delta)-\varrho_M(x)\\&=\sum_{i=1}^{r}M(a_i)+\delta[M(b_r)-M(a_r)]\\&\quad-\Big\{\sum_{i=1}^{r}[(1-\lambda_i)M(a_i)+\lambda_iM(b_i)]+\sum_{i=r+1}^{m}M(x(i))\Big\}\\&\ge\delta[M(b_r)-M(a_r)]-\sum_{i=1}^{r}\lambda_i[M(b_i)-M(a_i)]\\&\ge(\delta-r\lambda_r)[M(b_r)-M(a_r)]\end{aligned}$$

we can find $\delta'\in[0,r\lambda_r]$ such that $\varrho_M(u_{\delta'})=1$. Let

$$y=u_{\delta'}\quad\text{and}\quad z=\frac{1}{1-1/\beta}(x-y/\beta)\,.$$

Then

$$z(r)=\frac{\beta x(r)-y(r)}{\beta-1}=a_i+\frac{1}{\beta+1}(\beta\lambda_r-\delta)(b_r-a_r)>a_r\,.$$

It remains to show $\lambda(x)\ge 1/\beta$; this is similar to the proof of Theorem 3. Symmetrically, if $\sigma<1/2$, we also derive $\lambda(x)\ge 1/\beta$.

$\Rightarrow$: If M is not strictly convex near origin, then for any $n\in\mathbb{N}$, M has a SAI $[a,b]$ such that $nM(b)\le 1$. Define $x(i)=(1-1/n)a+b/n$ for $i\le n$ and find $x(j)\in S_M$ $(j>n)$ such that $\sum_{i=1}^{\infty}M(x(i))=1$. Then $x=(x(i))_i\in S(l_M)$. Now, for any $\lambda\in(0,1)$, $e\in\operatorname{Ext}B(l_M)$ and $u\in B(l_M)$ satisfying $x=\lambda e+(1-\lambda)u$, we have $e(i)=x(i)$ for all $i>n$ and $e(i)=a$ or b for all $i\le n$ except at most one $i'\le n$ according to Lemma 2. Since $e(i')\in[a,b]$ and

$$\sum_{i\le n}M(e(i))=\sum_{i\le n}M(x(i))=(n-1)M(a)+M(b)$$

we deduce that $e(j)=b$ for some $j\le n$ and $e(i)=a$ for all $i\le n$ other than j. Observe $u(j)\in[a,b]$; we find

$$(1-1/n)a+b/n=x(j)=\lambda e(j)+(1-\lambda)u(j)\ge\lambda b+(1-\lambda)a\,,$$

i.e., $\lambda\le 1/n$. This shows $\lambda(x)\le 1/n$, and so, $\lambda(l_M)=0$ since $n\in\mathbb{N}$ is arbitrary. ∎

3. λ-property of l_M°

LEMMA 5 ([3]). *$x=(x(i))_i\in\operatorname{Ext}B(l_M^\circ)$ iff $\operatorname{card}\{i\in\mathbb{N}:x(i)\ne 0\}=1$ or $kx(i)\in S_M$ for all $k\in K(x)$ and all $i\in\mathbb{N}$.*

THEOREM 6. *Each Orlicz space l_M° has the λ-property.*

Proof. We shall prove $\lambda(x) > 0$ for all $x \in S(l_M^\circ) \setminus \operatorname{Ext} B(l_M^\circ)$. Let $\{[a_i, b_i]\}_i$ be the set of all SAIs of M.

First, we select a point $k \in K(x)$ in the following way: if $K(x) = \{k\}$, then we have no alternative; if $K(x)$ contains more than one point, then for each $h \in \operatorname{int} K(x)$ and each $j \in \mathbb{N}$, $hx(j) = 0$ or $hx(i) \in (a_i, b_i)$ for some $i \in \mathbb{N}$. Hence, by Lemma 5, we can choose $k \in K(x)$ such that neither

$$\{j \in \mathbb{N} : a_i \le kx(j) \le (a_i + b_i)/2 \text{ for some } i \in \mathbb{N}\}$$

nor

$$\{j \in \mathbb{N} : b_i \ge kx(j) \ge (a_i + b_i)/2 \text{ for some } i \in \mathbb{N}\}$$

is empty. Therefore, for each $i \ge 1$, we can divide the set

$$\{j \in \mathbb{N} : a_i \le kx(j) \le b_i\}$$

into two sets E_i and F_i such that neither $\bigcup_i E_i$ nor $\bigcup_i F_i$ is empty and

$$j \in E_i \ \Rightarrow\ kx(j) \le (a_i + b_i)/2; \qquad j \in F_i \ \Rightarrow kx(j) \ge (a_i + b_i)/2\,.$$

Next, we define a sequence $\{y(j)\}_j$ by considering two cases. If $K(x) = \{k\}$, then let

$$y(j) = \begin{cases} a_i & \text{if } a_i < kx(j) < (a_i + b_i)/2 \text{ for some } i \ge 1, \\ b_i & \text{if } b_i > kx(j) \ge (a_i + b_i)/2 \text{ for some } i \ge 1, \\ kx(j) & \text{otherwise.} \end{cases}$$

If $K(x)$ contains more than one point, then we set

$$y(j) = \begin{cases} a_i & \text{if } j \in E_i,\ i \ge 1, \\ b_i & \text{if } j \in F_i,\ i \ge 1, \\ kx(j) & \text{otherwise.} \end{cases}$$

Obviously, $y(j) \in S_M$ for all $j \in \mathbb{N}$. Now, we prove $y/\|y\|^\circ \in \operatorname{Ext} B(l_M^\circ)$. To show this, it suffices to verify $K(y/\|y\|^\circ) = \{\|y\|^\circ\}$, i.e., $K(y) = \{1\}$ according to Lemma 5. Indeed, by the definition of E_i, F_i and the fact that p is a constant on each $[a_i, b_i)$, when $K(x) = \{k\}$ we have, for any ε in $(0, 1)$, the following implications: if $y(j) = a_i$, then $(1 + \varepsilon/2)|kx(j)| < b_i$ implies $p(y(j)) = p((1 + \varepsilon/2)|kx(j)|)$, and $(1 + \varepsilon/2)|kx(j)| \ge b_i$ implies

$$2|kx(j)| < a_i + b_i \le a_i + (1 + \varepsilon/2)|kx(j)|\,.$$

Thus, $y(j) = a_i \ge (1 - \varepsilon/2)|kx(j)|$. Hence, we always have

$$\varrho_N(p((1 + \varepsilon)|y|)) \ge \varrho_N(p((1 + \varepsilon)(1 - \varepsilon/2)|kx|)) > 1\,.$$

Similarly, we also have

$$\varrho_N(p((1 - \varepsilon)|y|)) \le \varrho_N(p((1 - \varepsilon)(1 + \varepsilon/2)|kx|)) < 1\,.$$

When $K(x)$ contains more than one point, we have

$$\varrho_N(p((1 + \varepsilon)|y|)) > \varrho_N(p(|kx|)) = 1\,,$$
$$\varrho_N(p((1 - \varepsilon)|y|)) < \varrho_N(p(|kx|)) = 1\,.$$

Hence, $K(y) = \{1\}$.

Finally, we set $z = 2kx - y$. Then $y(j) = kx(j)$ implies $z(j) = kx(j)$; $y(j) = a_i$ implies $a_i \le kx(j) \le z(j) \le b_i$; and $y(j) = b_i$ implies $b_i \ge kx(j) \ge z(j) \ge a_i$. Moreover, by the same method, we can verify $1 \in K(z)$. Hence

$$\begin{aligned} k = \|kx\|^\circ &= 1 + \varrho_M(kx) = 1 + \varrho_M\left(\frac{y+z}{2}\right) \\ &= \tfrac{1}{2}[1 + \varrho_M(y)] + \tfrac{1}{2}[1 + \varrho_M(z)] \\ &= \tfrac{1}{2}\|y\|^\circ + \tfrac{1}{2}\|z\|^\circ \end{aligned}$$

and so

$$\begin{aligned} x &= \frac{1}{2k}y + \frac{1}{2k}z = \frac{\|y\|^\circ}{2k} \cdot \frac{y}{\|y\|^\circ} + \frac{\|z\|^\circ}{2k} \cdot \frac{z}{\|z\|^\circ} \\ &= \frac{\|y\|^\circ}{2k} \cdot \frac{y}{\|y\|^\circ} + \frac{2k - \|y\|^\circ}{2k} \cdot \frac{z}{\|z\|^\circ}, \end{aligned}$$

which implies $\lambda(x) \ge \|y\|^\circ/(2k) > 0$. ∎

Theorem 7. *l_M° has the uniform λ-property iff*

$$\sup\{b_i/a_i : 0 < b_i \le 1\} < \infty$$

where $\{[a_i, b_i]\}_i$ is the set of all SAIs of M.

Proof. $\Leftarrow$: For each $x \in S(l_M^\circ) \setminus \operatorname{Ext} B(l_M^\circ)$, define y as in Theorem 6. We have already proved that $\lambda(x) \ge \|y\|^\circ/(2k)$. Let

$$c_M = 1 + \sup\{b_i/a_i : 0 < b_i \le 1\}.$$

Then $|y(j)| \ge (k/c_M)|x(j)|$, $j \in \mathbb{N}$. Hence

$$\lambda(x) \ge \frac{1}{2k}\|y\|^\circ \ge \frac{k}{2kc_M}\|x\|^\circ \ge \frac{1}{2c_M}.$$

Combining this with Lemma 1, we find that $\lambda(l_M^\circ) \ge 1/(4c_M)$.

$\Rightarrow$: Suppose that M has SAIs $\{[a_n, b_n]\}_n$ satisfying $b_n > n^3 a_n > 0$ and $n(n+2)N(p(b_n)) < 1$ ($n \in \mathbb{N}$). Then there exist $m_n \in \mathbb{N}$ such that

$$(nm_n + 1)N(p(a_n)) + N(p(b_n)) \le 1,$$

and

$$(1) \qquad (n(m_n + 1) + 1)N(p(a_n)) + N(p(b_n)) > 1$$

($n \in \mathbb{N}$). Let

$$c_n = \sup\{c \ge 0 : N(p(c)) + (nm_n + 1)N(p(a_n)) + N(p(b_n)) \le 1\}.$$

Then by (1) and $n(n+2)N(p(b_n)) \le 1$,

$$N(p(c_n)) \le nN(p(a_n)) < n^{-1} \to 0 \quad (n \to \infty).$$

For each $n \in \mathbb{N}$, $r \le n$, if we define

$$G_n(r) = \{i \in \mathbb{N} : (r-1)m_n + 4 \le i \le rm_n + 3\}$$

and

$$x_n = c_n e_1 + a_n e_2 + b_n e_3 + \sum_{r=1}^{n} \sum_{i\in G_n(r)} \left[\left(1-\frac{1}{r\ln n}\right)a_n + \frac{1}{r\ln n} b_n\right] e_i$$

where $\{e_i\}$ is the natural basis of l_1, then by the definition of K, c_n and SAIs of M, it is obvious that $K(x_n) = \{1\}$, i.e., $K(x_n/\|x_n\|^\circ) = \|x_n\|^\circ$. We shall complete the proof by showing that $\lambda(x_n/\|x_n\|^\circ) \to 0$ as $n\to\infty$. Let $\lambda_n \in (0,1)$, $y_n \in \operatorname{Ext} B(l_M^\circ)$ and $u_n \in B(l_M^\circ)$ satisfying $x_n/\|x_n\|^\circ = \lambda_n y_n + (1-\lambda_n)u_n$. We have to show $\lambda_n \to 0$.

First, we take $k_n \in K(y_n)$ and $h_n \in K(u_n)$. Then by the convexity of M and Theorem 1.26 of [2], we have

$$\begin{aligned}
1 &= \lambda_n\|y_n\|^\circ + (1-\lambda_n)\|u_n\|^\circ \\
&= \frac{\lambda_n}{k_n}[1+\varrho_M(k_n y_n)] + \frac{1-\lambda_n}{h_n}[1+\varrho_M(h_n u_n)] \\
&= \frac{(1-\lambda_n)k_n + \lambda_n h_n}{\lambda_n h_n}\Big(1 + \frac{\lambda_n h_n}{(1-\lambda_n)k_n+\lambda_n h_n}\varrho_M(k_n y_n) \\
&\qquad + \frac{(1-\lambda_n)k_n}{(1-\lambda_n)k_n+\lambda_n h_n}\varrho_M(h_n u_n)\Big) \\
&\ge \frac{(1-\lambda_n)k_n + \lambda_n h_n}{k_n h_n} \\
&\qquad \times \left[1 + \varrho_M\left(\frac{k_n h_n}{(1-\lambda_n)k_n+\lambda_n h_n}(\lambda_n y_n + (1-\lambda_n)u_n)\right)\right] \\
&\ge \left\|\frac{x_n}{\|x_n\|^\circ}\right\|^\circ = 1\,.
\end{aligned}$$

This implies

$$(2)\qquad \|x_n\|^\circ = \frac{k_n h_n}{(1-\lambda_n)k_n + \lambda_n h_n}\,,\quad \text{i.e.,}\quad \frac{1}{\|x_n\|^\circ} = \frac{1-\lambda_n}{h_n} + \frac{\lambda_n}{k_n}$$

and that $x_n(i)$, $k_n y_n(i)$, $h_n u_n(i)$ are in the same SAI of M for each $i\in\mathbb{N}$. Hence, by Lemma 5 and $y_n \in \operatorname{Ext} B(l_M^\circ)$, we derive $k_n y_n(1) \to 0$ and $k_n y_n(i) = a_n$ or b_n for all $i > 1$.

Second, since

$$M(b_n) = \int_0^{b_n} p(t)\,dt > \int_{a_n}^{b_n} p(t)\,dt \ge (b_n - a_n)p(a_n)\,,$$

$$N(p(a_n)) = a_n p(a_n) - M(a_n) < a_n p(a_n)\,,$$

we find

$$n m_n M(b_n) \ge n m_n (b_n/a_n - 1) N(p(a_n)) \ge (1-1/n)(n^3-1)\,.$$

Let

$$H_n = \{i \in \mathbb{N} : k_n y_n(i) = b_n\}\,, \qquad r(n) = \max\{r \le n : G_n(r) \cap H_n \ne \emptyset\}\,.$$

Then for any $i \in H_n \cap G_n(r(n))$,

$$\begin{aligned}(3)\qquad \left(1 - \frac{1}{r(n)\ln n}\right)a_n + \frac{1}{r(n)\ln n}b_n &= x_n(i)\\ &= \frac{\lambda_n\|x_n\|^\circ}{k_n}k_n y_n(i) + \frac{(1-\lambda_n)\|x_n\|^\circ}{h_n}h_n u_n(i)\\ &> \frac{\lambda_n\|x_n\|^\circ}{k_n}b_n(i)\,.\end{aligned}$$

Combining this with $\sum_{i=1}^n 1/i > \ln n$ and

$$M(b_n) \ge M(n^3 a_n) > n^3 M(a_n)$$

we have

$$\begin{aligned}&\lim_n \frac{k_n}{\|x_n\|^\circ}\\ &\quad = \lim_n \frac{1+\varrho_M(k_n y_n)}{1+\varrho_M(x_n)}\\ &\quad \le \lim_n \frac{1 + M(k_n y_n(1)) + n m_n M(a_n) + \sum_{i\le r(n)} M(b_n)\,\mathrm{card}(H_n \cap G_n(i))}{1 + \sum_{r\le n}\frac{1}{r\ln n}m_n M(b_n)}\\ &\quad \le \lim_n \frac{1 + n m_n M(a_n) + \frac{r(n)}{n} n m_n M(b_n)}{n m_n M(b_n)/n}\\ &\quad \le \lim_n \left(\frac{n}{(n^3-1)(1-1/n)} + \frac{1}{n^2} + r(n)\right).\end{aligned}$$

Hence, if $r(n) = 0$, then (2) implies

$$\lim_n \lambda_n \le \lim_n \frac{k_n}{\|x_n\|^\circ} = 0$$

and if $r(n) \ne 0$, then (3) also implies

$$\begin{aligned}\lim_n \lambda_n &\le \lim_n \frac{k_n}{\|x_n\|^\circ}\left\{\left(1 - \frac{1}{r(n)\ln n}\right)\frac{a_n}{b_n} + \frac{1}{r(n)\ln n}\right\}\\ &< \lim_n \left(n\frac{a_n}{b_n} + \frac{1}{\ln n}\right) \le \lim_n \left(\frac{1}{n^2} + \frac{1}{\ln n}\right) = 0\,. \ \blacksquare\end{aligned}$$

References

[1] R. M. Aron and R. H. Lohman, *A geometric function determined by extreme points of the unit ball of a normed space*, Pacific J. Math. 127 (1987), 209–231.
[2] R. M. Aron, R. H. Lohman and A. Suárez, *Rotundity, the C.S.R.P., and the λ-property in Banach spaces*, Proc. Amer. Math. Soc. 111 (1991), 151–155.
[3] S. Chen and Y. Shen, *Extreme points and rotundity of Orlicz spaces*, J. Harbin Normal Univ. 2 (1985), 1–5.
[4] S. Chen, H. Sun and C. Wu, *λ-property of Orlicz spaces*, Bull. Polish Acad. Sci. Math. 39 (1991), 63–69.
[5] A. Suárez, *λ-property in Orlicz spaces*, ibid. 37 (1989), 421–431.
[6] Z. Wang, *Extreme points of sequence Orlicz spaces*, J. Daqing Petroleum College 1 (1983), 112–121.
[7] C. Wu and H. Sun, *On the λ-property of Orlicz space L_M*, Comment. Math. Univ. Carolin. 31 (1990), 731–741.
[8] C. Wu, T. Wang, S. Chen and Y. Wang, *Geometry of Orlicz Spaces*, Harbin Institute of Technology Press, Harbin, 1986.

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Reçu par la Rédaction le 6.5.1993