ANNALES
POLONICI MATHEMATICI
LIX.3 (1994)

On a class of functional boundary value problems for the equation $x'' = f(t, x, x', x'', \lambda)$

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Abstract. The Leray–Schauder degree theory is used to obtain sufficient conditions for the existence and uniqueness of solutions for the boundary value problem $x'' = f(t, x, x', x'', \lambda)$, $\alpha(x) = 0$, $\beta(\overline{x}) = 0$, $\gamma(\overline{\overline{x}}) = 0$, depending on the parameter λ. Here α, β, γ are linear bounded functionals defined on the Banach space of C^0-functions on $[0,1]$ and $\overline{x}(t) = x(0) - x(t)$, $\overline{\overline{x}}(t) = x(1) - x(t)$.

1. Introduction. Let $\mathbf{X}$ be the Banach space of C^0-functions on $[0,1]$ with the norm $\|x\| = \max\{|x(t)| : 0 \le t \le 1\}$ and $\alpha, \beta, \gamma : \mathbf{X} \to \mathbb{R}$ be linear bounded functionals such that

$$x, y \in \mathbf{X},\ x(t) < y(t) \text{ on } [0,1] \Rightarrow \alpha(x) < \alpha(y), \tag{1}$$

$$x, y \in \mathbf{X},\ x(t) < y(t) \text{ on } (0,1) \Rightarrow \beta(x) < \beta(y), \tag{2}$$

$$x, y \in \mathbf{X},\ x(t) < y(t) \text{ on } [0,1) \Rightarrow \gamma(x) < \gamma(y). \tag{3}$$

For $x \in \mathbf{X}$ define $\overline{x}, \overline{\overline{x}} : [0,1] \to \mathbb{R}$ by

$$\overline{x}(t) = x(0) - x(t), \qquad \overline{\overline{x}}(t) = x(1) - x(t).$$

Consider the boundary value problem (BVP for short)

$$x'' = f(t, x, x', x'', \lambda), \tag{4}$$

$$\alpha(x) = 0, \quad \beta(\overline{x}) = 0, \quad \gamma(\overline{\overline{x}}) = 0, \tag{5}$$

depending on the parameter λ. Here $f \in C^0([0,1] \times \mathbb{R}^4)$.

We say that (x, λ_0) is a *solution* of BVP (4), (5) if $(x, \lambda_0) \in C^2([0,1]) \times \mathbb{R}$ and x is a solution of (4) for $\lambda = \lambda_0$ and satisfies (5).

In this paper, the existence and uniqueness of solutions for BVP (4), (5) is studied. Using the technique of Tineo [11] we shall show that under some

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1991 *Mathematics Subject Classification*: 34B15, 34K10.
Key words and phrases: Leray–Schauder degree theory, functional boundary conditions, boundary value problem depending on the parameter.
Supported by grant no. 201/93/2311 of the Grant Agency of Czech Republic.

assumptions BVP (4), (5) is equivalent to BVP of the type $x'' = g(t, x, x', \lambda)$, (5) and the proof of an existence theorem for the latter is based on the theory of completely continuous mappings and on the homotopy invariance of the Leray–Schauder degree. More precisely, we apply the following theorem.

THEOREM 1 [1, Theorem 1]. *Let* $\mathbf{X}$ *be a Banach space*, $A : \mathbf{X} \to \mathbf{X}$ *be a completely continuous mapping such that* $I - A$ *is one-to-one*, *and let* Ω *be an open bounded set such that* $0 \in (I - A)(\Omega)$. *Then the completely continuous mapping* $T : \overline{\Omega} \to \mathbf{X}$ *has a fixed point in* $\overline{\Omega}$ *if for any* $c \in (0, 1)$, *the equation*

$$x = cTx + (1 - c)Ax$$

has no solution on the boundary $\partial\Omega$ *of* Ω.

We notice that the existence and uniqueness of solutions for the BVP

$$x'' = F(t, x, x', x''), \qquad x \in \mathcal{E},$$

where $F \in C^0([0, 1] \times \mathbb{R}^3)$ and $\mathcal{E}$ is a closed subset of $C^2([0, 1])$ of codimension two such that for all $x \in \mathcal{E}$ there exists $t_0 = t_0(x) \in [0, 1]$ with $|x(t)| \le |x(t_0)|$ $(0 \le t \le 1)$ and $x'(t_0) = 0$ was considered in [11].

In [3] sufficient conditions for the existence of solutions of the differential equation

$$x'' = f(t, x, x', x'') - y(t), \qquad 0 \le t \le T,\ y \in C^0([0, T])$$

subject to either Dirichlet, Neumann, periodic, Sturm–Liouville, or antiperiodic boundary conditions were obtained. The proofs are based on the author's continuation theorem for semilinear A-proper maps and a priori bounds for the solutions of the equation $x'' = f(t, x, x', x'')$.

In [4] the author studied the BVP

$$\begin{aligned} &x''' - 4a^2x' = f(t, x, x', x'', x''', \lambda), \\ &x'(t_1) = x(t_2) = x'(t_2) = x'(t_3) = 0 \end{aligned} \tag{6}$$

$(-\infty < t_1 < t_2 < t_3 < \infty,\ 0 < a < \infty)$ depending on the parameter λ. Sufficient conditions were given for the existence and uniqueness of BVP (6) using the Schauder linearization technique and the Schauder fixed point theorem.

BVPs for second order differential and functional differential equations depending on a parameter were considered in [5]–[8] using the Schauder linearization and quasi-linearization technique and the Schauder fixed point theorem, and in [10] using a surjectivity result in $\mathbb{R}^n$. Applying the Leray–Schauder degree method, sufficient conditions for the existence of solutions of the one-parameter BVP $x'' = f(t, x, x', \lambda)$, $\alpha(x) = A$, $x(0) - x(1) = B$, $x'(0) - x'(1) = C$ were stated in [9].

2. Auxiliary lemmas

LEMMA 1. *Assume* $x \in \mathbf{X}$ *and* $\alpha(x) = 0$ (*resp.* $\beta(\overline{x}) = 0$; *resp.* $\gamma(\overline{\overline{x}}) = 0$). *Then there exists a* $\xi \in [0,1]$ (*resp.* $\tau \in (0,1)$; *resp.* $\kappa \in [0,1)$) *such that* $x(\xi) = 0$ (*resp.* $x(0) = x(\tau)$; *resp.* $x(1) = x(\kappa)$).

Proof. Assume $\alpha(x) = 0$ and, on the contrary, $x(t) \neq 0$ on $[0,1]$. Then $x(t) > 0$ on $[0,1]$ or $x(t) < 0$ on $[0,1]$ and therefore $\alpha(x) > \alpha(0) = 0$ or $\alpha(x) < \alpha(0) = 0$, a contradiction.

Assume $\beta(\overline{x}) = 0$ and $x(0) - x(t) \neq 0$ on $(0,1)$. Then $x(0) - x(t) > 0$ on $(0,1)$ or $x(0) - x(t) < 0$ on $(0,1)$, hence $\beta(\overline{x}) > \beta(0) = 0$ or $\beta(\overline{x}) < \beta(0) = 0$, a contradiction.

Analogously we can prove that $\gamma(\overline{\overline{x}}) = 0$ implies $x(1) - x(\kappa) = 0$ for a $\kappa \in [0,1)$. ■

COROLLARY 1. *Assume* $x \in \mathbf{X}$ *and* $\beta(\overline{x}) = \gamma(\overline{\overline{x}}) = 0$. *Then* $x(0) = x(\tau)$ *and* $x(1) = x(\varepsilon)$ *for some* $\tau, \varepsilon \in (0,1)$.

Proof. By Lemma 1 we have $x(0) = x(\tau)$ for a $\tau \in (0,1)$ and $x(1) = x(\kappa)$ for a $\kappa \in [0,1)$. If $\kappa = 0$, then $(x(\kappa) =)\, x(0) = x(1) = x(\tau)$ and therefore setting

$$\varepsilon = \begin{cases} \kappa & \text{if } \kappa \in (0,1), \\ \tau & \text{if } \kappa = 0, \end{cases}$$

we see that $\varepsilon \in (0,1)$ and $x(1) = x(\varepsilon)$. ■

Remark 1. By the well-known general form of linear bounded functionals on $\mathbf{X}$ it is clear that the functional α (resp. β; resp. γ) defined on $\mathbf{X}$ by

$$\alpha(x) = \int_0^1 x(s)\, dv_1(s)$$

$$\Big(\text{resp. } \beta(x) = \int_0^1 x(s)\, dv_2(s);\ \text{resp. } \gamma(x) = \int_0^1 x(s)\, dv_3(s)\Big)$$

is linear bounded and satisfies (1) (resp. (2); resp. (3)), where $v_1 \in \mathcal{A}_1 := \{x : x \text{ is nondecreasing on } [0,1],\ x(1) > x(0)\}$ (resp. $v_2 \in \mathcal{A}_2 := \{x : x$ is nondecreasing on $[0,1]$, $x(1) > x(0)$ and x is continuous at $t = 0$ and $t = 1\}$; resp. $v_3 \in \mathcal{A}_3 := \{x : x$ is nondecreasing on $[0,1]$, $x(1) > x(0)$ and x is continuous at $t = 1\}$).

EXAMPLE 1. The boundary conditions ($\xi \in (0,1)$)

(i) $$x(0) = x(\xi) = x(1) = 0,$$

(ii) $$\int_0^1 x(s)\, ds = 0, \qquad x(0) = x(\xi) = x(1),$$

(iii) $$\int_0^1 x(s)\,ds = 0, \qquad x(0) = x(1) = 0$$

are examples of boundary conditions (5). We get (i)–(iii) putting $\alpha(x) = \gamma(x) = x(0)$, $\beta(x) = x(\xi)$ and $\alpha(x) = \int_0^1 x(s)\,ds$, $\beta(x) = x(\xi)$, $\gamma(x) = x(0)$ and $\alpha(x) = \gamma(x) = x(0)$, $\beta(x) = \int_0^1 x(s)\,ds$, respectively.

LEMMA 2. *There exists an $\varepsilon_0 > 0$ such that*

(7) $$\beta(\sin(\varepsilon t))\gamma(\cos(\varepsilon t) - \cos(\varepsilon)) - \beta(1 - \cos(\varepsilon t))\gamma(\sin(\varepsilon) - \sin(\varepsilon t)) \neq 0$$

and

(8) $$\beta(1 - e^{\varepsilon t})\gamma(e^{-\varepsilon} - e^{-\varepsilon t}) - \beta(1 - e^{-\varepsilon t})\gamma(e^{\varepsilon} - e^{\varepsilon t}) \neq 0$$

for all $0 < \varepsilon \leq \varepsilon_0$.

Proof. Since $0 < t^2 < t$ on $(0,1)$ and $0 < 1 - t \leq 1 - t^2$ on $[0,1)$ we have $0 < \beta(t^2) < \beta(t)$, $0 < \gamma(1-t) \leq \gamma(1-t^2)$, and consequently

(9) $$\beta(t^2)\gamma(1-t) - \beta(t)\gamma(1-t^2) < 0.$$

Since

$$\lim_{\varepsilon\to 0}\left[\frac{\sin(\varepsilon t)}{\varepsilon} - t\right] = 0, \qquad \lim_{\varepsilon\to 0}\left[\frac{\cos(\varepsilon t) - \cos(\varepsilon)}{\varepsilon^2} + \frac{t^2-1}{2}\right] = 0,$$
$$\lim_{\varepsilon\to 0}\left[\frac{1 - \cos(\varepsilon t)}{\varepsilon^2} - \frac{t^2}{2}\right] = 0, \qquad \lim_{\varepsilon\to 0}\left[\frac{\sin(\varepsilon) - \sin(\varepsilon t)}{\varepsilon} - 1 + t\right] = 0$$

uniformly on $[0,1]$, we have (cf. (9))

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}[\beta(\sin(\varepsilon t))\gamma(\cos(\varepsilon t) - \cos(\varepsilon)) - \beta(1-\cos(\varepsilon t))\gamma(\sin(\varepsilon) - \sin(\varepsilon t))]$$
$$= \frac{1}{2}[\beta(t)\gamma(1-t^2) - \beta(t^2)\gamma(1-t)] > 0,$$

and therefore there exists an $\widetilde{\varepsilon} > 0$ such that (7) holds for all $0 < \varepsilon \leq \widetilde{\varepsilon}$.

Assume (8) does not hold on a right neighbourhood of 0. Then there is a decreasing sequence $\{\varepsilon_n\}$ of positive numbers such that $\lim_{n\to\infty}\varepsilon_n = 0$ and

$$\beta(1 - e^{\varepsilon_n t})\gamma(e^{-\varepsilon_n} - e^{-\varepsilon_n t}) - \beta(1 - e^{-\varepsilon_n t})\gamma(e^{\varepsilon_n} - e^{\varepsilon_n t}) = 0 \qquad \text{for all } n \in \mathbb{N}.$$

Since

$$\begin{aligned} 0 &= \beta(1 - e^{\varepsilon_n t})(\gamma(e^{-\varepsilon_n} - e^{-\varepsilon_n t}) + \gamma(e^{\varepsilon_n} - e^{\varepsilon_n t})) \\ &\quad - (\beta(1 - e^{\varepsilon_n t}) + \beta(1 - e^{-\varepsilon_n t}))\gamma(e^{\varepsilon_n} - e^{\varepsilon_n t}) \\ &= 2\beta(1 - e^{\varepsilon_n t})\gamma(\cosh(\varepsilon_n) - \cosh(\varepsilon_n t)) \\ &\quad - 2\beta(1 - \cosh(\varepsilon_n t))\gamma(e^{\varepsilon_n} - e^{\varepsilon_n t}) \end{aligned}$$

and

$$\lim_{n\to\infty}\left[\frac{1-e^{\varepsilon_n t}}{\varepsilon_n}+t\right]=0,\qquad \lim_{n\to\infty}\left[\frac{\cosh(\varepsilon_n)-\cosh(\varepsilon_n t)}{\varepsilon_n^2}-\frac{1-t^2}{2}\right]=0,$$

$$\lim_{n\to\infty}\left[\frac{1-\cosh(\varepsilon_n t)}{\varepsilon_n^2}+\frac{t^2}{2}\right]=0,\qquad \lim_{n\to\infty}\left[\frac{e^{\varepsilon_n}-e^{\varepsilon_n t}}{\varepsilon_n}-1+t\right]=0$$

uniformly on $[0,1]$, we obtain

$$\begin{aligned}0&=\lim_{n\to\infty}\frac{1}{\varepsilon_n^3}[\beta(1-e^{\varepsilon_n t})\gamma(\cosh(\varepsilon_n)-\cosh(\varepsilon_n t))\\&\quad-\beta(1-\cosh(\varepsilon_n t))\gamma(e^{\varepsilon_n}-e^{\varepsilon_n t})]\\&=-\frac{1}{2}[\beta(t)\gamma(1-t^2)-\beta(t^2)\gamma(1-t)],\end{aligned}$$

which contradicts (9). This proves that there exists a constant $\varepsilon^*>0$ such that (8) holds for all $0<\varepsilon\le\varepsilon^*$. The assertion of Lemma 2 is true for $\varepsilon_0=\min\{\widetilde{\varepsilon},\varepsilon^*\}$. ∎

LEMMA 3. *Assume* $h\in C^0([0,1]\times\mathbb{R}^3)$ *and there are constants* $\lambda_1<0$, $\lambda_2>0$, $M>0$, $N>0$, $T>0$ *and a nondecreasing function* $w:[0,\infty)\to(0,\infty)$ *such that*

(10′) $\quad h(t,x,0,\lambda_2)>0 \quad$ *for all* $(t,x)\in[0,1]\times[0,M]$,

(10″) $\quad h(t,x,0,\lambda_1)<0 \quad$ *for all* $(t,x)\in[0,1]\times[-N,0]$,

(11) $\quad h(t,-N,0,\lambda)<0<h(t,M,0,\lambda) \quad$ *for all* $(t,\lambda)\in[0,1]\times[\lambda_1,\lambda_2]$,

(12) $\quad |h(t,x,y,\lambda)|\le w(|y|)$

for all $(t,x,y,\lambda)\in[0,1]\times[-N,M]\times\mathbb{R}\times[\lambda_1,\lambda_2]$

and

(13) $$\int_0^T\frac{s\,ds}{w(s)}>2\max\{M,N\}.$$

Assume $x(t)$ *is a solution of the differential equation*

$$x''=h(t,x,x',\lambda)$$

for $\lambda=\lambda_0$ *satisfying* (5) *and* $\lambda_1\le\lambda_0\le\lambda_2$, $-N\le x(t)\le M$ *for* $t\in[0,1]$. *Then*

$$\lambda_1<\lambda_0<\lambda_2,\quad -N<x(t)<M,\quad |x'(t)|<T,\quad |x''(t)|<w(T)+1 \quad \textit{for all } t\in[0,1].$$

Proof. Let $\lambda_0=\lambda_2$. By Lemma 1, there exists a $\xi\in[0,1]$ such that $x(\xi)=0$, and therefore $0\le\max\{x(t):0\le t\le1\}=x(\varepsilon)$ for an $\varepsilon\in[0,1]$. Without loss of generality we may assume $\varepsilon\in(0,1)$ (cf. Corollary 1). Then $x(\varepsilon)\ge0$, $x'(\varepsilon)=0$, $x''(\varepsilon)\le0$, which contradicts $x''(\varepsilon)=f(\varepsilon,x(\varepsilon),0,\lambda_2)>0$

(cf. (10′)). Analogously we can prove that $\lambda_0 = \lambda_1$ is impossible; hence $\lambda_1 < \lambda_0 < \lambda_2$.

Assume, on the contrary, $x(\varrho) = -N$ (resp. $x(\varrho) = M$) for a $\varrho \in [0,1]$. By Corollary 1 we may assume $\varrho \in (0,1)$. Then $x(\varrho) = -N$, $x'(\varrho) = 0$, $x''(\varrho) \geq 0$ (resp. $x(\varrho) = M$, $x'(\varrho) = 0$, $x''(\varrho) \leq 0$), which contradicts (cf. (11)) $x''(\varrho) = h(\varrho, -N, 0, \lambda_0) < 0$ (resp. $x''(\varrho) = h(\varrho, M, 0, \lambda_0) > 0$).

Finally, since $\beta(\overline{x}) = 0$, there exists a $\tau \in (0,1)$ such that $x(0) = x(\tau)$ (cf. Lemma 1) and therefore $x'(\nu) = 0$ for a $\nu \in (0,\tau)$. Using (12) and (13) and a standard procedure (see e.g. [2]) we can prove $|x'(t)| < T$ for $t \in [0,1]$ and, moreover, $|x''(t)| = |h(t, x(t), x'(t), \lambda_0)| \leq w(|x'(t)|) < w(T) + 1$ for $t \in [0,1]$. ∎

LEMMA 4. *Assume that* $r \in C^0([0,1] \times \mathbb{R}^3)$ *and that there are constants* $\lambda_1 < 0$, $\lambda_2 > 0$, $M > 0$, $N > 0$, $T > 0$ *and a nondecreasing function* $w : [0,\infty) \to (0,\infty)$ *such that*

$$r(t,x,0,\lambda_2) \geq 0 \quad \textit{for all } (t,x) \in [0,1] \times [0,M],$$

$$r(t,x,0,\lambda_1) \leq 0 \quad \textit{for all } (t,x) \in [0,1] \times [-N,0],$$

$$r(t,-N,0,\lambda) \leq 0 \leq r(t,M,0,\lambda) \quad \textit{for all } (t,\lambda) \in [0,1] \times [\lambda_1,\lambda_2],$$

$$|r(t,x,y,\lambda)| \leq w(|y|) \quad \textit{for all } (t,x,y,\lambda) \in [0,1] \times [-N,M] \times \mathbb{R} \times [\lambda_1,\lambda_2]$$

and

$$\int_0^T \frac{s\,ds}{w(s)} > 2\max\{M,N\}.$$

Then the BVP

$$x'' = r(t,x,x',\lambda), \quad (5) \tag{14}$$

has at least one solution.

P r o o f. Denote by $\mathbf{Y}$ the Banach space of C^1-functions on $[0,1]$ with the norm $\|x\|_1 = \|x\| + \|x'\|$, $\mathbf{Z}$ the Banach space of C^2-functions on $[0,1]$ with the norm $\|x\|_2 = \|x\|_1 + \|x''\|$, $\mathbf{Y}_0 = \{x : x \in \mathbf{Y},\ x \text{ satisfies } (5)\}$, $\mathbf{Z}_0 = \mathbf{Y}_0 \cap \mathbf{Z}$. Let $\mathbf{X} \times \mathbb{R}$, $\mathbf{Y}_0 \times \mathbb{R}$ and $\mathbf{Z}_0 \times \mathbb{R}$ be the Banach spaces with the norms $\|(x,\lambda)\| = \|x\| + |\lambda|$, $\|(x,\lambda)\|_1 = \|x\|_1 + |\lambda|$ and $\|(x,\lambda)\|_2 = \|x\|_2 + |\lambda|$, respectively. Set $\mathbf{S} = \{(x,\alpha(x)) : x \in \mathbf{X}\}$ $(\subset \mathbf{X} \times \mathbb{R})$. Clearly $\mathbf{S}$ is a Banach space.

Let

$$0 < \varepsilon = \min\left\{\varepsilon_0, \sqrt{\frac{w(0)}{2\max\{M,N\}}}\right\}, \qquad 0 < k = \frac{\varepsilon^2 \min\{M,N\}}{2\max\{\lambda_2, -\lambda_1\}}$$

be constants, where ε_0 is defined in Lemma 2.

Define the operators $L, F, K : \mathbf{Z}_0 \times \mathbb{R} \to \mathbf{S}$ by

$$(L(x,\lambda))(t) = (x''(t) + \varepsilon^2 x(t) + k\lambda, \alpha(x''(t) + k\lambda)),$$

$$(F(x,\lambda))(t) = (r(t,x(t),x'(t),\lambda), \alpha(r(t,x(t),x'(t),\lambda))),$$
$$(K(x,\lambda))(t) = (\varepsilon^2 x(t) + k\lambda, \alpha(k\lambda)).$$

Consider the operator equation

$$(15_c) \qquad L(x,\lambda) = c(F(x,\lambda) + K(x,\lambda)) + 2(1-c)K(x,\lambda), \qquad c \in [0,1].$$

We see that BVP (14) has a solution (x,λ_0) if and only if (x,λ_0) is a solution of (15_1). We use Theorem 1 to prove the existence of a solution of (15_1).

We shall show that $L : \mathbf{Z}_0 \times \mathbb{R} \to \mathbf{S}$ is one-to-one and onto. Let $(u,\alpha(u)) \in \mathbf{S}$ and consider the equation

$$L(x,\lambda) = (u,\alpha(u)),$$

that is, the equations (see the definition of L)

$$(16) \qquad \begin{aligned} x'' + \varepsilon^2 x + k\lambda &= u(t), \\ \alpha(x'' + k\lambda) &= \alpha(u), \end{aligned}$$

where $x \in \mathbf{Z}_0$ and $\lambda \in \mathbb{R}$. The function $x(t) = c_1 \sin(\varepsilon t) + c_2 \cos(\varepsilon t) - (k/\varepsilon^2)\lambda + v(t)$ is the general solution of (16), where

$$v(t) = (1/\varepsilon) \int_0^t u(s) \sin(\varepsilon(t-s))\, ds$$

and c_1, c_2 are integration constants. The function $x(t)$ satisfies the boundary conditions $\beta(\overline{x}) = \gamma(\overline{\overline{x}}) = 0$ if and only if c_1, c_2 are solutions of the linear system

$$(17) \qquad \begin{aligned} c_1\beta(-\sin(\varepsilon t)) + c_2\beta(1-\cos(\varepsilon t)) &= -\beta(\overline{v}), \\ c_1\gamma(\sin(\varepsilon) - \sin(\varepsilon t)) + c_2\gamma(\cos(\varepsilon) - \cos(\varepsilon t)) &= -\gamma(\overline{\overline{v}}). \end{aligned}$$

This system has a unique solution, say $c_1 = a$, $c_2 = b$, since its determinant

$$-\beta(\sin(\varepsilon t))\gamma(\cos(\varepsilon) - \cos(\varepsilon t)) - \beta(1-\cos(\varepsilon t))\gamma(\sin(\varepsilon) - \sin(\varepsilon t))$$

is different from zero by Lemma 2. Hence

$$x^*(t) = a\sin(\varepsilon t) + b\cos(\varepsilon t) - (k/\varepsilon^2)\lambda + v(t), \qquad \lambda \in \mathbb{R},$$

are all solutions of (16) satisfying $\beta(\overline{x}^*) = \gamma(\overline{\overline{x}}^*) = 0$. The function $q : \mathbb{R} \to \mathbb{R}$, $q(\lambda) = \alpha(x^{*\prime\prime}(t) + k\lambda)$, is continuous increasing, $\lim_{\lambda\to-\infty} q(\lambda) = -\infty$, $\lim_{\lambda\to\infty} q(\lambda) = \infty$, and therefore there exists a unique solution of the equation $\alpha(x^{*\prime\prime}(t) + k\lambda) = \alpha(u)$, say $\lambda = \lambda_0$. Then $x(t) = a\sin(\varepsilon t) + b\cos(\varepsilon t) - (k/\varepsilon^2)\lambda_0 + v(t)$ is the unique solution of (16) satisfying (5). Hence $L^{-1} : \mathbf{S} \to \mathbf{Z}_0 \times \mathbb{R}$ exists, L^{-1} is a linear bounded operator by the Banach theorem and (15_c) can be written in the equivalent form

$$(18_c) \qquad (x,\lambda) = c(L^{-1}Fj(x,\lambda) + L^{-1}Kj(x,\lambda)) + 2(1-c)L^{-1}Kj(x,\lambda), \qquad c \in [0,1],$$

where $j : \mathbf{Z}_0 \times \mathbb{R} \to \mathbf{Y}_0 \times \mathbb{R}$ is the natural embedding, which is completely continuous by the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem. Define

$$\Omega = \{(x, \lambda) : (x, \lambda) \in \mathbf{Z}_0 \times \mathbb{R},\ -N < x(t) < M \text{ on } [0,1], \\ \|x'\| < T,\ \|x''\| < w(T) + 1,\ \lambda_1 < \lambda < \lambda_2\}.$$

Then Ω is a bounded open subset of $\mathbf{Z}_0 \times \mathbb{R}$. Moreover, $L^{-1}Fj + L^{-1}Kj$ is a compact operator on $\overline{\Omega}$ and $2L^{-1}Kj$ is completely continuous on $\mathbf{Z}_0 \times \mathbb{R}$. In order to prove that (15_1) has a solution, that is, $L^{-1}Fj + L^{-1}Kj$ has a fixed point, we have to show (cf. Theorem 1) that $(x, \lambda) - 2L^{-1}Kj(x, \lambda) = (0, 0)$ implies $(x, \lambda) = (0, 0)$ and for any $c \in (0, 1)$ equation (18_c) has no solution on the boundary $\partial\Omega$ of Ω.

Consider the equation $(x, \lambda) - 2L^{-1}Kj(x, \lambda) = (0, 0)$ which is equivalent to

$$L(x, \lambda) = 2K(x, \lambda). \tag{19}$$

A pair $(x, \lambda) \in \mathbf{Z}_0 \times \mathbb{R}$ is a solution of (19) if and only if x and λ are solutions of the system

$$\begin{aligned} x'' &= \varepsilon^2 x + k\lambda, \\ \alpha(x'') &= \alpha(k\lambda), \end{aligned} \tag{20}$$

and, moreover, $\beta(\overline{x}) = \gamma(\overline{\overline{x}}) = 0$. Since $x(t) = c_1 e^{\varepsilon t} + c_2 e^{-\varepsilon t} - (k/\varepsilon^2)\lambda$ is the general solution of (20), where c_1, c_2 are integration constants, we see that (x, λ) is a solution of (19) if and only if c_1, c_2, λ are solutions of the linear system

$$\begin{aligned} &c_1\alpha(e^{\varepsilon t}) + c_2\alpha(e^{-\varepsilon t}) - \lambda\alpha(k/\varepsilon^2) = 0, \\ &c_1\beta(1 - e^{\varepsilon t}) + c_2\beta(1 - e^{-\varepsilon t}) = 0, \\ &c_1\gamma(e^{\varepsilon} - e^{\varepsilon t}) + c_2\gamma(e^{-\varepsilon} - e^{-\varepsilon t}) = 0. \end{aligned}$$

This linear system has only the trivial solution $(c_1, c_2, \lambda) = (0, 0, 0)$ since its determinant

$$-\alpha(k/\varepsilon^2)(\beta(1 - e^{\varepsilon t})\gamma(e^{-\varepsilon} - e^{-\varepsilon t}) - \beta(1 - e^{-\varepsilon t})\gamma(e^{\varepsilon} - e^{\varepsilon t}))$$

is different from zero by Lemma 2; hence $(x, \lambda) = (0, 0)$ is the unique solution of (19).

Finally, we shall prove that for any $c \in (0, 1)$ equation (18_c) has no solution on $\partial\Omega$. To this purpose we study the differential equation

$$x'' = cr(t, x, x', \lambda) + (1 - c)(\varepsilon^2 x + k\lambda), \qquad c \in (0, 1). \tag{21_c}$$

Assume (x_c, λ_c) is a solution of BVP (21_c), (5). We have to show that $(x_c, \lambda_c) \notin \partial\Omega$. Set $p_c(t, x, y, \lambda) = cr(t, x, y, \lambda) + (1 - c)(\varepsilon^2 x + k\lambda)$ for (t, x, y, λ)

$\in [0,1]\times\mathbb{R}^3$ and $c\in(0,1)$. Then $p_c\in C^0([0,1]\times\mathbb{R}^3)$ and (as $c\in(0,1)$)

$$p_c(t,x,0,\lambda_2)=cr(t,x,0,\lambda_2)+(1-c)(\varepsilon^2x+k\lambda_2)>0 \quad \text{for } (t,x)\in[0,1]\times[0,M],$$

$$p_c(t,x,0,\lambda_1)=cr(t,x,0,\lambda_1)+(1-c)(\varepsilon^2x+k\lambda_1)<0 \quad \text{for } (t,x)\in[0,1]\times[-N,0],$$

$$\begin{aligned}p_c(t,-N,0,\lambda)&\le cr(t,-N,0,\lambda)+(1-c)\bigg(-\varepsilon^2N+\varepsilon^2\lambda_2\frac{\min\{M,N\}}{2\max\{\lambda_2,-\lambda_1\}}\bigg)\\ &\le -(1-c)(\varepsilon^2N/2)<0 \quad \text{for } (t,\lambda)\in[0,1]\times[\lambda_1,\lambda_2],\end{aligned}$$

$$\begin{aligned}p_c(t,M,0,\lambda)&\ge cr(t,M,0,\lambda)+(1-c)\bigg(\varepsilon^2M+\varepsilon^2\lambda_1\frac{\min\{M,N\}}{2\max\{\lambda_2,-\lambda_1\}}\bigg)\\ &\ge(1-c)(\varepsilon^2M/2)>0 \quad \text{for } (t,\lambda)\in[0,1]\times[\lambda_1,\lambda_2],\end{aligned}$$

$$\begin{aligned}|p_c(t,x,y,\lambda)|&\le c|r(t,x,y,\lambda)|+(1-c)\bigg(\frac{w(0)}{2\max\{M,N\}}\bigg\{\max\{M,N\}\\ &\quad+\frac{\min\{M,N\}}{2\max\{\lambda_2,-\lambda_1\}}\max\{\lambda_2,-\lambda_1\}\bigg\}\bigg)\\ &\le cw(|y|)+(1-c)w(0)\le w(|y|)\\ &\qquad \text{for } (t,x,y,\lambda)\in[0,1]\times[-N,M]\times\mathbb{R}\times[\lambda_1,\lambda_2].\end{aligned}$$

We see that for any $c\in(0,1)$, the function p_c satisfies the same assumptions as h in Lemma 3 and since $|x_c''(t)|\le w(|x_c'(t)|)\le w(T)$ on $[0,1]$, it follows that $(x_c,\lambda_c)\notin\partial\Omega$ for any solution (x_c,λ_c) of BVP (21_c), (5). The proof is finished. ■

LEMMA 5. *Assume there are constants $\lambda_1<0$, $\lambda_2>0$, $M>0$, $N>0$, $0<a<1$ and a nondecreasing function $w(\cdot;\mathcal{D}_0):[0,\infty)\to(0,\infty)$ for any bounded subset $\mathcal{D}_0$ of $\mathbb{R}^2$ such that*

(H_1) $f(t,x,0,0,\lambda_2)\ge 0$ *for all* $(t,x)\in[0,1]\times[0,M]$,
$f(t,x,0,0,\lambda_1)\le 0$ *for all* $(t,x)\in[0,1]\times[-N,0]$;

(H_2) $f(t,-N,0,0,\lambda)\le 0\le f(t,M,0,0,\lambda)$ *for all* $(t,\lambda)\in[0,1]\times[\lambda_1,\lambda_2]$;

(H_3) $|f(t,x,y,z,\lambda)|\le w(|y|;\mathcal{D}_0)+a|z|$ *for all* $(t,x,\lambda)\in[0,1]\times\mathcal{D}_0$, $(y,z)\in\mathbb{R}^2$;

(H_4) *the function* $p:\mathbb{R}\to\mathbb{R}$, $p(z)=z-f(t,x,y,z,\lambda)$, *is increasing on* $\mathbb{R}$ *for each fixed* $(t,x,y,\lambda)\in[0,1]\times\mathbb{R}^3$.

Then there exists a unique continuous function $g:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ such that

(22) $$g(t,x,y,\lambda)=f(t,x,y,g(t,x,y,\lambda),\lambda) \quad \textit{for all } (t,x,y,\lambda)\in[0,1]\times\mathbb{R}^3,$$

(23′) $g(t,x,0,\lambda_2)\ge 0$ *for all* $(t,x)\in[0,1]\times[0,M]$,

(23″) $g(t,x,0,\lambda_1)\le 0$ *for all* $(t,x)\in[0,1]\times[-N,0]$,

(24) $g(t,-N,0,\lambda)\le 0\le g(t,M,0,\lambda)$ *for all* $(t,\lambda)\in[0,1]\times[\lambda_1,\lambda_2]$,

(25) $|g(t,x,y,\lambda)|\le(1/(1-a))w(|y|;\mathcal{D}_0)$ *for all* $(t,x,\lambda)\in[0,1]\times\mathcal{D}_0, y\in\mathbb{R}$.

If, *moreover*,

(H_5) $f(t,\cdot,y,z,\lambda)$ *is increasing on* $\mathbb{R}$ *for each* $(t,y,z,\lambda)\in[0,1]\times\mathbb{R}^3$,

(H_6) $f(t,x,y,z,\cdot)$ *is increasing on* $\mathbb{R}$ *for each* $(t,x,y,z)\in[0,1]\times\mathbb{R}^3$,

then

(26) $g(t,\cdot,y,\lambda)$ *is increasing on* $\mathbb{R}$ *for each* $(t,y,\lambda)\in[0,1]\times\mathbb{R}^2$,

(27) $g(t,x,y,\cdot)$ *is increasing on* $\mathbb{R}$ *for each* $(t,x,y)\in[0,1]\times\mathbb{R}^2$.

Proof. Fix $(t,x,y,\lambda)\in[0,1]\times\mathbb{R}^3$. The function $p:\mathbb{R}\to\mathbb{R}$, $p(u)=u-f(t,x,y,u,\lambda)$, is continuous increasing on $\mathbb{R}$ (by (H_4)), $\lim_{u\to-\infty}p(u)=-\infty$, $\lim_{u\to\infty}p(u)=\infty$ (by (H_3)), hence there exists a unique $z\in\mathbb{R}$ such that $p(z)=0$. If we put $z=g(t,x,y,\lambda)$ we obtain the function $g:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ satisfying (22).

Assume g is discontinuous at a point $(t_0,x_0,y_0,\lambda_0)\in[0,1]\times\mathbb{R}^3$. Then there are a sequence $\{(t_n,x_n,y_n,\mu_n)\}\subset[0,1]\times\mathbb{R}^3$ and an $\varepsilon>0$ such that

$$(t_n,x_n,y_n,\mu_n)\to(t_0,x_0,y_0,\mu_0)\qquad\text{as } n\to\infty$$

and

$$|g(t_n,x_n,y_n,\mu_n)-g(t_0,x_0,y_0,\mu_0)|\ge\varepsilon\qquad\text{for all } n\in\mathbb{N}. \tag{28}$$

Since

$$\begin{aligned}|g(t_n,x_n,y_n,\mu_n)|&=|f(t_n,x_n,y_n,g(t_n,x_n,y_n,\mu_n),\mu_n)|\\&\le w(|y_n|;\mathcal{D}_0)+a|g(t_n,x_n,y_n,\mu_n)|\\&\le w(B;\mathcal{D}_0)+a|g(t_n,x_n,y_n,\mu_n)|\end{aligned}$$

(by (H_3)), where $B=\sup\{|y_n|:n\in\mathbb{N}\}$ $(<\infty)$ and $\mathcal{D}_0\subset\mathbb{R}^2$ is a bounded set with $\{(x_n,\lambda_n)\}\subset\mathcal{D}_0$, we have

$$|g(t_n,x_n,y_n,\mu_n)|\le\frac{w(B;\mathcal{D}_0)}{1-a},\qquad n\in\mathbb{N},$$

and consequently $\{g(t_n,x_n,y_n,\mu_n)\}$ is bounded. Without loss of generality we may assume that $\{g(t_n,x_n,y_n,\mu_n)\}$ is convergent, say $\lim_{n\to\infty}g(t_n,x_n,y_n,\mu_n)=d$. Then

$$d=\lim_{n\to\infty}f(t_n,x_n,y_n,g(t_n,x_n,y_n,\mu_n),\mu_n)=f(t_0,x_0,y_0,d,\mu_0),$$

and therefore $d=g(t_0,x_0,y_0,\mu_0)$, which contradicts (28).

Assume $f(t,x,0,0,\lambda) \ge 0$ (resp. ≤ 0) for a point $(t,x,\lambda) \in [0,1] \times \mathbb{R}^2$. The function $r : \mathbb{R} \to \mathbb{R}$, $r(z) = z - f(t,x,0,z,\lambda)$, is continuous increasing, $\lim_{z\to-\infty} r(z) = -\infty$, $\lim_{z\to\infty} r(z) = \infty$ (by (H_3) and (H_4)) and $r(0) = -f(t,x,0,0,\lambda) \le 0$ (resp. ≥ 0), hence there is a unique $z_0 \in \mathbb{R}$, $z_0 \ge 0$ (resp. ≤ 0), such that $r(z_0) = 0$, that is, $z_0 = f(t,x,0,z_0,\lambda)$. Then we have $(z_0 =) g(t,x,0,\lambda) \ge 0$ (resp. ≤ 0), which proves (23) and (24).

Now, using assumption (H_3) we obtain

$$|g(t,x,y,\lambda)| = |f(t,x,y,g(t,x,y,\lambda),\lambda)| \le w(|y|;\mathcal{D}_0) + a|g(t,x,y,\lambda)|$$

and thus inequality (25) is true.

Assume that, moreover, (H_5) and (H_6) are satisfied and $x_1 < x_2$. Then $f(t,x_1,y,z,\lambda) < f(t,x_2,y,z,\lambda)$ and for $q_i : \mathbb{R} \to \mathbb{R}$, $q_i(z) = z - f(t,x_i,y,z,\lambda)$ $(i = 1,2)$, we have $q_1(z) > q_2(z)$ on $\mathbb{R}$. Since q_1, q_2 are continuous increasing and $\lim_{z\to-\infty} q_i(z) = -\infty$, $\lim_{z\to\infty} q_i(z) = \infty$, we have $q_1(z_1) = 0 = q_2(z_2)$ for unique $z_1, z_2 \in \mathbb{R}$, $z_1 < z_2$, and consequently $(z_1 =)\ g(t,x_1,y,\lambda) < g(t,x_2,y,\lambda)\,(= z_2)$. This proves (26). The proof of (27) is similar. ∎

4. Existence theorem

THEOREM 2. *Assume there exist constants* $\lambda_1 < 0$, $\lambda_2 > 0$, $M > 0$, $N > 0$, $T > 0$, $0 < a < 1$ *and a nondecreasing function* $w(\cdot;\mathcal{D}_0) : [0,\infty) \to (0,\infty)$ *for any bounded subset* $\mathcal{D}_0$ *of* $\mathbb{R}^2$ *such that assumptions* (H_1)–(H_4) *are satisfied and*

$$\int_0^T \frac{s\,ds}{w(s;\mathcal{D}^*)} > 2\max\{-N, M\} \qquad \textit{with } \mathcal{D}^* = [-N,M] \times [\lambda_1,\lambda_2].$$

Then BVP (4), (5) *has at least one solution.*

Proof. By Lemma 5, there exists a continuous function $g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfying (22)–(25). Hence BVP (4), (5) has a solution (x,λ_0) if and only if (x,λ_0) is a solution of BVP

$$x'' = g(t,x,x',\lambda), \quad (5). \tag{29}$$

The function g satisfies the same assumptions as r in Lemma 4, and therefore BVP (29) has a solution by Lemma 4. ∎

Remark 2. Let the assumptions of Theorem 2 be satisfied. It follows from Lemma 4 and its proof that there exists a solution (x,λ_0) of BVP (4), (5) such that $-N \le x(t) \le M$, $-T \le x'(t) \le T$ for $t \in [0,1]$ and $\lambda_0 \in [\lambda_1,\lambda_2]$.

EXAMPLE 2. Let $p,q \in C^0([0,1])$, $n \in \mathbb{N}$, $b \in \mathbb{R}$, $|b| < 1$. Consider the differential equation

$$x'' = x^{2n+1} - x + q(t)x'^2 + b\sin(x'') + \lambda. \tag{30}$$

We can apply Theorem 2 to BVP (30), (5) (for example $-\lambda_1 = \lambda_2 = \|p\|+1$, $M = N = \max\{2, (2\|p\|+1)^{1/(2n)}\}$).

5. Uniqueness theorem

THEOREM 3. *Assume there exist constants* $\lambda_1 < 0$, $\lambda_2 > 0$, $M > 0$, $N > 0$, $T > 0$, $0 < a < 1$ *and a nondecreasing function* $w(\cdot;\mathcal{D}_0) : [0,\infty) \to (0,\infty)$ *for any bounded subset* $\mathcal{D}_0$ *of* $\mathbb{R}^2$ *such that assumptions* (H_1)–(H_6) *are satisfied and*

$$\int_0^T \frac{s\,ds}{w(s;\mathcal{D}^*)} > 2\max\{-N, M\} \qquad \text{with } \mathcal{D}^* = [-N, M] \times [\lambda_1, \lambda_2].$$

Then BVP (4), (5) *has a unique solution.*

Proof. By Theorem 2, there exists a solution (x_1, μ_1) of BVP (4), (5). This solution as well as all solutions of BVP (4), (5) are solutions of BVP (29), where g is defined in Lemma 5. The function $g(t,\cdot,y,\lambda)$ (resp. $g(t,x,y,\cdot)$) is increasing on $\mathbb{R}$ for each fixed $(t,y,\lambda) \in [0,1] \times \mathbb{R}^2$ (resp. $(t,x,y) \in [0,1] \times \mathbb{R}^2$) (cf. Lemma 5). Assume (x_2, μ_2) is another solution of BVP (29). We have to show that $(x_1,\mu_1) = (x_2,\mu_2)$. Set $u = x_2 - x_1$ and assume for example $\mu_2 \ge \mu_1$. Then $\alpha(u) = \beta(\overline{u}) = \gamma(\overline{\overline{u}}) = 0$. Since $u(\xi) = 0$ for a $\xi \in [0,1]$ by Lemma 1, $0 \le \max\{u(t) : 0 \le t \le 1\} = u(\tau)$ for a $\tau \in [0,1]$ (cf. Corollary 1); thus $u(\tau) \ge 0$, $u'(\tau) = 0$, $u''(\tau) \le 0$. On the other hand,

$$u''(\tau) = g(\tau, x_2(\tau), x_2'(\tau), \mu_2) - g(\tau, x_1(\tau), x_2'(\tau), \mu_1) \ge 0,$$

which implies $u(t) \le u(\tau) = 0$ on $[0,1]$ and $\mu_1 = \mu_2$. Let $(0 \ge) \min\{u(t) : 0 \le t \le 1\} = u(\varepsilon)$ for an $\varepsilon \in (0,1)$ (see Corollary 1). Then $u(\varepsilon) \le 0$, $u'(\varepsilon) = 0$, $u''(\varepsilon) \ge 0$ and since $u''(\varepsilon) = g(\varepsilon, x_2(\varepsilon), x_2'(\varepsilon), \mu_1) - g(\varepsilon, x_1(\varepsilon), x_2'(\varepsilon), \mu_1) \le 0$ we obtain $u(\varepsilon) = 0$; hence $u = 0$. This completes the proof. ∎

EXAMPLE 3. Let $p, q \in C^0([0,1])$, $n \in \mathbb{N}$, $b \in \mathbb{R}$, $|b| < 1$. Consider the differential equation

$$x'' = x^{2n+1} + q(t)x'^2 + b\sin(x'') + p(t) + \lambda. \tag{31}$$

We can apply Theorem 3 to BVP (31), (5) (for example $-\lambda_1 = \lambda_2 = \|p\|$, $N = M = (2\|p\|)^{1/(2n+1)}$).

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Reçu par la Rédaction le 15.4.1993
Révisé le 28.10.1993