An initial value problem for a third order differential equation

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Abstract. For an initial value problem
\[ u''(x) = g(u(x)), \quad u(0) = u'(0) = u''(0) = 0, \]
x > 0, some theorems on existence and uniqueness of solutions are established.

1. Introduction. We consider the initial value problem
\[ u''(x) = g(u(x)) \quad (x > 0), \]
\[ u(0) = u'(0) = u''(0) = 0, \]
where
(i) \( g : (0, \infty) \to [0, \infty) \) is continuous,
(ii) \( x^{1/2}g(x) \) is bounded as \( x \to 0, \)
(iii) \( 0 < \int_0^\delta g(s)s^{-1/2}ds < \infty \) (\( \delta > 0 \)).

In the paper \( \delta \) always denotes some positive constant. We permit it to change its value from paragraph to paragraph.

We are looking for a nonnegative function \( u \in C^2[0, \delta) \) satisfying (1.1). It is known (see [1]–[3]) that for \( g \) nondecreasing a similar problem for the \( n \)th order differential equation
\[ u^{(n)}(x) = g(u(x)), \]
\[ u(0) = u'(0) = \ldots = u^{(n-1)}(0) = 0 \quad (n \text{ an integer}), \]
has a nontrivial solution \( u \not\equiv 0 \) if and only if the generalized Osgood condition
\[ \int_0^\delta \left( \frac{s}{g(s)} \right)^{1/n} \frac{ds}{s} < \infty \quad (\delta > 0) \]
is satisfied. For further results of this type, see [4], [5].

1991 Mathematics Subject Classification: 45D05, 45G10.
Key words and phrases: nonlinear Volterra integral equation, existence and uniqueness of solutions.
Our aim is to give a similar condition for the existence of nontrivial solutions of (1.1) in the case of a class of functions \( g \) satisfying (i)–(iii).

To study the problem (1.1) we transform it into its integral form

\[
    u(x) = \frac{1}{2} \int_0^x (x - s)^2 g(u(s)) \, ds \quad (0 < x < \delta).
\]

Taking \( v(x) = 2^{-2/3}[u'(u^{-1}(x))]^2 \), where \( u^{-1} \) is the inverse function to \( u \), we get the integral equation

\[
    v(x) = \int_0^x (x - s)g(s)v(s)^{-1/2} \, ds \quad (0 < x < \delta).
\]

The integral equation (1.2), with an unknown nonnegative function \( v \in C[0,\delta) \), will be equivalent to problem (1.1) if we assume

\[
    \int_0^\delta v(s)^{-1/2} \, ds < \infty \quad (\delta > 0),
\]

which is motivated by the relation \( v(x)^{-1/2} = 2^{1/3}(u^{-1})'(x) \).

To analyse (1.2), we introduce the integral operator

\[
    Tw(x) = \int_0^x (x - s)g(s)w(s)^{-1/2} \, ds \quad (x > 0)
\]

defined for any continuous function \( w : [0,\infty) \to [0,\infty) \) such that the integral on the right-hand side is finite, and set

\[
    \varphi(x) = \int_0^x (x - s)g(s)s^{-1/2} \, ds \quad \text{and} \quad q(x) = x^{1/3}\varphi(x)^{2/3} \quad (x > 0).
\]

Some properties of \( \varphi \) and \( q \) which we shall use in the sequel are collected in the two following lemmas.

**Lemma 1.1.** Let \( g \) satisfy (i)–(iii). Then

(a) \( \varphi(x)/x \) is nondecreasing for \( x > 0; \)

(b) there exists a constant \( c > 0 \) such that \( x\varphi^2(x) \leq c\varphi(x) \) for \( x \in (0,\delta); \)

(c) if \( x^{1/2}g(x) \) is nondecreasing for \( x \in (0,\delta) \), then \( x\varphi^2(x)/\varphi(x) \) is also nondecreasing on \( (0,\delta) \).

**Proof.** (a) It suffices to observe that \( \varphi \) is convex and \( \varphi(0) = 0. \)

(b) Notice that

\[
    x\varphi^2(x) = \int_0^x (2s\varphi''(s) + \varphi'(s))\varphi'(s) \, ds \quad (x > 0).
\]

Now, our assertion follows by the fact that \( 2s\varphi''(s) + \varphi'(s) \) is bounded on \( (0,\delta). \)
(c) We first note that
\[ \varphi^2(x \varphi^2/\varphi)' = (2x\varphi'' + \varphi')\varphi - x\varphi^2 \varphi'. \]
Further, from (1.5) and by the fact that \(2s\varphi''(s) + \varphi'(s)\) is nondecreasing on \((0, \delta)\) we obtain
\[ x\varphi^2(x) \leq (2x\varphi''(x) + \varphi'(x))\varphi(x) \quad \text{for } x \in (0, \delta). \]
So, \((x\varphi^2/\varphi)' \geq 0\) on \((0, \delta)\), which completes the proof.

**Lemma 1.2.** Let \(g\) satisfy (i)–(iii). Then
(a) \(Tq(x)\) is well defined for \(x > 0\);
(b) if \(x^{1/2}g(x)\) is nondecreasing on \((0, \delta)\), then \(Tq(x) \leq \frac{9}{2}q(x)\) for \(x \in (0, \delta)\).

**Proof.** (a) We first observe that
\[
\int_0^x g(s)q(s)^{-1/2} \, ds = \int_0^x \varphi''(s) \left( \frac{\varphi(s)}{s} \right)^{-1/3} \, ds \quad (0 < x) .
\]
By the estimate given in Lemma 1.1(b) we can integrate by parts to obtain
\[
\begin{align}
\int_0^x \varphi''(s) \left( \frac{\varphi(s)}{s} \right)^{-1/3} \, ds &= \varphi'(x) \left( \frac{\varphi(x)}{x} \right)^{-1/3} \\
&\quad + \frac{1}{3} \int_0^x \varphi'(s) \left( \frac{\varphi(s)}{s} \right)^{-1/2} \left( \frac{\varphi(s)}{s} \right)' \left( \frac{\varphi(s)}{s} \right)^{-5/6} \, ds .
\end{align}
\]
In order to estimate the right-hand side of (1.6) we use Lemma 1.1(b) once more, which gives
\[
\int_0^x \varphi''(s) \left( \frac{\varphi(s)}{s} \right)^{-1/3} \, ds < \infty \quad \text{for } x \in (0, \delta) .
\]
Hence our assertion follows immediately.

(b) We use Lemma 1.1(c) in order to estimate the integral on the right-hand side of (1.6). Then we obtain
\[
\int_0^x \varphi''(s) \left( \frac{\varphi(s)}{s} \right)^{-1/3} \, ds \leq 3\varphi'(x) \left( \frac{\varphi(x)}{x} \right)^{-1/3} \quad \text{for } x \in (0, \delta) .
\]
Now, we derive
\[
Tq(x) = \int_0^x \left( \int_0^x g(\xi)q(\xi)^{-1/2} \, d\xi \right) \, ds \\
= 3 \int_0^x \varphi'(s) \left( \frac{\varphi(s)}{s} \right)^{-1/3} \, ds \leq \frac{9}{2}q(x) \quad \text{for } x \in (0, \delta) ,
\]
which completes the proof.
In view of Lemma 1.2(a), we can put
\[ Q(x) = Tq(x) \quad \text{for } x \in (0, \delta). \]

In the following theorem necessary and sufficient conditions for the existence of nontrivial solutions \( u \neq 0 \) of (1.1) are established.

**Theorem 1.1.** Let (i)–(iii) be satisfied. If
\[ \int_0^\delta q(s)^{-1/2} \, ds < \infty \quad (\delta > 0), \]
then (1.1) has a nontrivial solution. Conversely, if (1.1) has a nontrivial solution, then
\[ \int_0^\delta Q(s)^{-1/2} \, ds < \infty \quad (\delta > 0). \]

Define
\[ K_0 = \{ g : \text{s}^{1/2} g(s) \text{ is nondecreasing for } 0 < s < \delta \ (\delta > 0) \}. \]

Let \( h^*(x) = \sup s^{1/2} g(s) \text{ for } 0 < s \leq x \) and \( g^*(x) = x^{-1/2} h^*(x) \ (x > 0) \) where \( g \) is any function satisfying (i)–(iii). For \( g^* \) we define \( T^*, \varphi^*, q^* \) and \( Q^* \) similarly to those corresponding to \( g \). Define
\[ J(x) = \int_0^x g(s)s^{-1/2} \, ds, \quad J^*(x) = \int_0^x g^*(s)s^{-1/2} \, ds, \]
\[ K = \{ g : \sup(J^*(x)/J(x)) < \infty \text{ for } 0 < x \leq \delta \}. \]

Of course, \( K_0 \subseteq K \). Moreover, both \( K_0 \) and \( K \) contain nondecreasing functions \( g \) satisfying (i). We shall also present an example of \( g \in K \) which takes the value 0 at some points \( x_n, n = 1, 2, \ldots \), such that \( x_n \to 0 \) as \( n \to \infty \).

If we are interested only in functions \( g \in K \), then conditions (1.7) and (1.8) turn out to be equivalent. This is stated in the following theorem.

**Theorem 1.2.** If \( g \in K \), then the condition
\[ \int_0^\delta \frac{ds}{(s^{1/2} \varphi(s))^{1/3}} < \infty \quad (\delta > 0) \]
is necessary and sufficient for the existence of a unique solution \( u \) of (1.1) such that \( u(x) > 0 \) for \( x > 0 \).

2. **Proofs of theorems.** To prove Theorems 1.1 and 1.2 we need three lemmas. In the first one we give an a priori estimate of solutions of (1.2).
Lemma 2.1. For any continuous solution \( v \) of (1.2) we have

\[ q(x) \leq v(x) \leq Q(x) \quad (x > 0). \]

Proof. Using (1.2) we easily check that \( v(x)/x, x > 0 \), is nondecreasing. Therefore

\[ (2.1) \quad q(x) = \int_0^x (x - s) g(s) v(s) - 1/2 \, ds \geq \left( \frac{v(x)}{x} \right)^{-1/2} \varphi(x) \quad (x > 0). \]

Hence we get the left-hand inequality. Now the right-hand inequality follows from the monotonicity properties of \( T \).

The problem of the existence of solutions of (1.2) is considered in the following lemmas.

Lemma 2.2. There exists at least one continuous solution of (1.2).

Proof. We regularize (1.2) as follows:

\[ (2.2) \quad v_\varepsilon(x) = \varepsilon x + \int_0^x (x - s) g(s) v_\varepsilon(s) - 1/2 \, ds \quad (\varepsilon, x > 0), \]

where \( v_\varepsilon \) is sought in \( C[0, \delta) \). Define

\[ T_\varepsilon w(x) = \varepsilon x + \int_0^x (x - s) g(s) w(s) - 1/2 \, ds \]

for any continuous \( w : (0, \delta) \rightarrow [0, \infty) \) such that the integral on the right-hand side is finite and set

\[ w_0(x) = \varepsilon x \quad \text{for } 0 < x < \delta \quad \text{and} \quad w_n = T_\varepsilon^n w_0, \quad n = 1, 2, \ldots \]

The following useful properties of \( T_\varepsilon \) can easily be obtained:

\[ (2.3) \quad w_0 \leq T_\varepsilon v \quad \text{for any } v \geq 0; \]
\[ (2.4) \quad \text{if } 0 \leq v_1 \leq v_2, \quad \text{then } T_\varepsilon v_1 \geq T_\varepsilon v_2; \]
\[ (2.5) \quad T_\varepsilon(cv) \geq c^{-1/2} T_\varepsilon v \quad \text{for any } c \geq 1 \text{ and } v \geq 0. \]

From (2.3) and (2.4) it follows that

\[ (2.6) \quad w_0 \leq w_2 \leq w_4 \leq \ldots \leq w_5 \leq w_3 \leq w_1. \]

Since

\[ T_\varepsilon w_0(x) = \varepsilon x + \varepsilon^{-1/2} \int_0^x (x - s) g(s) s^{-1/2} \, ds, \]

from (iii) it follows that there exists \( \delta_\varepsilon > 0 \) such that

\[ w_0(x) \leq w_1(x) \leq \frac{3}{2} w_0(x) \quad \text{for } 0 < x < \delta_\varepsilon. \]
Applying (2.5) and using an inductive argument we can see that

\[ 1 \leq \frac{w_{2k+1}(x)}{w_{2k}(x)} \leq \left( \frac{3}{2} \right)^{\frac{1}{2^{k}}} \quad \text{for } 0 < x < \delta_{\varepsilon} \text{ and } k = 0, 1, 2, \ldots \]

From (2.6) and (2.7) we conclude that the sequence \( \{w_{n}\} \) is convergent. Since the function \( w(x) = \lim_{n \to \infty} w_{n}(x) \) with \( 0 < x < \delta_{\varepsilon} \), satisfies (2.2) on \( (0, \delta_{\varepsilon}) \), from its construction it follows that \( v_{\varepsilon}(x) = w(x) \) is a unique continuous solution of (2.2) on \( (0, \delta_{\varepsilon}) \). Now, noting that \( v_{\varepsilon}(x)/x \) is nondecreasing, we get the a priori estimates

\[ q(x) \leq v_{\varepsilon}(x) \leq x + Q(x), \quad v_{\varepsilon}'(x) \leq \varepsilon + Q'(x) \quad \text{for } x > 0, \]

from which we conclude that \( v_{\varepsilon} \) can be extended from \( [0, \delta_{\varepsilon}) \) to a unique continuous solution of (2.2) defined on a whole interval independent of \( \varepsilon \).

Now, estimate (2.8) allows us to apply the Arzelà–Ascoli theorem. Thus we see that there exists a convergent sequence \( \{v_{\varepsilon_{n}}\} \) of solutions of (2.2), where \( \varepsilon_{n} \to 0 \), as \( n \to \infty \). The limit \( v(x) = \lim_{n \to \infty} v_{\varepsilon_{n}}(x), \quad x > 0 \), is a solution of (1.2), which completes the proof.

A uniqueness result for equation (1.2) is given in the following lemma.

**Lemma 2.3.** If there exists a constant \( m \) such that

\[ Q(x) \leq mq(x) \quad (0 < x \leq \delta), \]

then the equation (1.2) has exactly one continuous solution \( v \).

**Proof.** Let \( v_1, v_2 \) be two continuous solutions of (1.2). By Lemma 2.1 we have

\[ q(x) \leq v_1(x), \quad v_2(x) \leq mq(x) \quad (0 < x < \delta). \]

Therefore there exist constants \( c_1, c_2 > 0 \) such that

\[ c_1 v_1(x) \leq v_2(x) \leq c_2 v_1(x) \quad (0 \leq x \leq \delta). \]

Note also that for any \( c > 0 \) and for any solution \( v \) of (1.2) we have

\[ T^n(cv) = e^{(-1/2)^n v}. \]

Therefore applying \( T^n \) to each term of (2.10), by the monotonicity properties of \( T \) we easily get \( v_1 = v_2 \), which completes the proof.

Now we are ready to prove our theorems.

**Proof of Theorem 1.1.** It is easily seen that Theorem 1.1 is a simple corollary of Lemmas 2.1 and 2.2.

**Proof of Theorem 1.2.** In view of Theorem 1.1 and Lemma 2.3, to prove Theorem 1.2 it suffices to show that there exists a constant \( m \) such that (2.9) is satisfied.
It follows from the definition of $K$ that there exists a constant $c > 0$ such that
\begin{equation}
(2.11) \quad cq^*(x) \leq q(x) \leq q^*(x) \quad \text{for } x \in (0, \delta).
\end{equation}
Hence we get
\begin{equation}
(2.12) \quad Q(x) = Tq(x) \leq c^{-1/2}Tq^*(x) \\
\leq c^{-1/2}T^*q^*(x) = c^{-1/2}Q^*(x) \quad \text{for } x \in (0, \delta).
\end{equation}
Now it suffices to note that $g^* \in K_0$. Therefore, by Lemma 1.2(b), $Q^*(x) \leq \frac{9}{q^*(x)}$ for $x \in (0, \delta)$, which combined with (2.11) and (2.12) gives the required result.

3. Examples. In this section we give two examples of application of the previous results.

Example 3.1. Let $g(x) = x^{-1/2}(-\ln x)^{\gamma}$, $\gamma > 0$, $0 < x < \delta$, for some $\delta > 0$. If $0 < \gamma \leq 1$, then we can easily check that $g$ does not satisfy condition (iii) which, in view of inequality (2.1), is one of the necessary conditions for the existence of nontrivial solutions of (1.1).

If $\gamma > 1$, then $g$ belongs to $K_0$ and we can easily check that the condition given in Theorem 1.2 is satisfied. Therefore in the case of $\gamma > 1$ the problem (1.1) has a unique solution $u$ such that $u(x) > 0$ for $0 < x < \delta$.

Before giving a second example we consider another subclass of $K$. Namely, let a bounded function $g$ satisfy condition (i). We set
\begin{equation}
\overline{g}(x) = \sup_{0 < s \leq x} g(s) \quad \text{for } 0 < x < \delta,
\end{equation}
and we define
\begin{equation}
G(x) = \int_0^x g(s) \, ds, \quad \overline{G}(x) = \int_0^x \overline{g}(s) \, ds \quad (0 < x < \delta).
\end{equation}

Denote by $K_1$ the class of bounded functions $g$ satisfying (i) and such that
\begin{equation}
\sup_{x \in (0, \delta)} \left( \overline{G}(x)/G(x) \right) < \infty.
\end{equation}

Remark 3.1. $K_1 \subseteq K$.

Proof. Let $g \in K_1$. Since $g$ is bounded, an integration by parts shows that
\begin{equation}
\int_0^x g(s)s^{-1/2} \, ds = x^{-1/2}G(x) + \frac{1}{2} \int_0^x G(s)s^{-3/2} \, ds.
\end{equation}
Using the same formula for $\overline{g}$ we obtain
\begin{equation}
\int_0^x \overline{g}(s)s^{-1/2} \, ds \leq c \int_0^x g(s)s^{-1/2} \, ds \quad \text{for } x \in (0, \delta),
\end{equation}
where \( c > 0 \) is some constant. Hence and by the fact that \( g^* \leq g \) on \((0, \delta)\) it follows that \( g \in K \), which ends the proof.

In the case of \( g \in K_1 \) the following theorem gives a necessary and sufficient condition for the existence of nontrivial solutions of (1.1).

**Theorem 3.1.** Let \( g \in K_1 \). Then (1.1) has a unique solution \( u \) such that \( u(x) > 0 \) for \( x > 0 \) if and only if

\[
\int_0^\delta \left( \frac{s}{g(s)} \right)^{1/3} \frac{ds}{s} < \infty \quad (\delta > 0).
\]

**Proof.** Since \( g \) is nondecreasing, we get

\[
2 - \frac{1}{g(x/2)} x^{1/2} \leq x \int_0^g g(s)s^{-1/2} ds \leq 2g(x)x^{1/2} \quad (0 < x < \delta).
\]

Therefore the required result follows from Theorem 1.2 by applying (3.1).

**Example 3.2.** Let \( x_k = (2k+1)^{1/2}, \ y_k = (2k\pi)^{-1}, \ a_k = (2+5/6)^{-1}\pi^{-1}, \ b_k = (2+1/6)^{-1}\pi^{-1} \) for any integer \( k \geq k_0 \), where \( k_0 \) is a fixed integer number such that \( y_{k_0} \in (0, \delta) \). We define

\[
g(x) = \begin{cases} 
\sin 1/x & x \leq y_k, \\
0 & y_{k+1} < x < x_k \ (k > k_0)
\end{cases}
\]

and

\[
g_1(x) = \begin{cases} 
1/2 & a_k \leq x \leq b_k, \\
0 & b_{k+1} < x < a_k \ (k > k_0).
\end{cases}
\]

Thus we have \( g_1(x) \leq g(x) \leq \tilde{g}(x) = 1 \) for \( x \in (0, \delta) \). Set

\[
G_1(x) = \int_0^x g_1(s) ds, \quad \tilde{G}(x) = \int_0^x \tilde{g}(s) ds = x \quad \text{for } x \in (0, \delta),
\]

\[
r_k = \sum_{l=k}^\infty (b_l - a_l) \quad \text{for } k > k_0.
\]

For \( x \in (a_{k+1}, a_k) \ (k > k_0) \) we get

\[
\frac{x}{G_1(x)} - \frac{x}{\tilde{G}(x)} \leq \frac{a_k}{r_{k+1}^2} \leq 2 \frac{a_k}{r_{k+1}^2} = 2 \frac{a_{k+1}}{r_{k+1}^2} \frac{a_k}{r_{k+1}}.
\]

Of course, \( a_k/a_{k+1} < c \) for some constant \( c \) and \( k > k_0 \). Since

\[
r_k = \pi^{-1} \sum_{l=k}^\infty (2l + 5/6)^{-1}(2l + 1/6)^{-1},
\]

we can compare \( r_k \) with \( \int_0^\infty s^{-2} ds \). As a result, \( a_k/r_k < c \) for some constant \( c \) and \( k > k_0 \). Now, by (3.3) we conclude that \( g \in K_1 \). Applying Theorem 3.1 we see that the problem (1.1) with \( g \) defined in (3.2) has a nontrivial solution.
Acknowledgements. This paper was written while the author was visiting Freie Universität Berlin in November 1992 as a guest to the research group “Regularization”. The author wishes to thank Prof. R. Gorenflo for helpful discussions on convolution type integral equations.

References


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Reçu par la Rédaction le 25.1.1993
Révisé le 3.6.1993