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An initial value problem for a third order differential equation

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Abstract. For an initial value problem $u'''(x) = g(u(x))$, $u(0) = u'(0) = u''(0) = 0$, $x > 0$, some theorems on existence and uniqueness of solutions are established.

1. Introduction. We consider the initial value problem

$$\begin{aligned} &u'''(x) = g(u(x)) \quad (x > 0), \\ &u(0) = u'(0) = u''(0) = 0, \end{aligned} \tag{1.1}$$

where

(i) $g : (0, \infty) \to [0, \infty)$ is continuous,
(ii) $x^{1/2} g(x)$ is bounded as $x \to 0$,
(iii) $0 < \int_0^\delta g(s) s^{-1/2}\, ds < \infty$ $(\delta > 0)$.

In the paper δ always denotes some positive constant. We permit it to change its value from paragraph to paragraph.

We are looking for a nonnegative function $u \in C^2[0, \delta)$ satisfying (1.1). It is known (see [1]–[3]) that for g nondecreasing a similar problem for the nth order differential equation

$$\begin{aligned} &u^{(n)}(x) = g(u(x)), \\ &u(0) = u'(0) = \ldots = u^{(n-1)}(0) = 0 \quad (n \text{ an integer}), \end{aligned}$$

has a nontrivial solution $u \not\equiv 0$ if and only if the generalized Osgood condition

$$\int_0^\delta \left(\frac{s}{g(s)} \right)^{1/n} \frac{ds}{s} < \infty \quad (\delta > 0)$$

is satisfied. For further results of this type, see [4], [5].

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Our aim is to give a similar condition for the existence of nontrivial solutions of (1.1) in the case of a class of functions g satisfying (i)–(iii).

To study the problem (1.1) we transform it into its integral form

$$u(x)=\frac{1}{2}\int\limits_0^x (x-s)^2 g(u(s))\,ds \qquad (0<x<\delta)\,.$$

Taking $v(x)=2^{-2/3}[u'(u^{-1}(x))]^2$, where u^{-1} is the inverse function to u, we get the integral equation

$$v(x)=\int\limits_0^x (x-s)g(s)v(s)^{-1/2}\,ds \qquad (0<x<\delta)\,. \tag{1.2}$$

The integral equation (1.2), with an unknown nonnegative function $v\in C[0,\delta)$, will be equivalent to problem (1.1) if we assume

$$\int\limits_0^\delta v(s)^{-1/2}\,ds<\infty \qquad (\delta>0)\,, \tag{1.3}$$

which is motivated by the relation $v(x)^{-1/2}=2^{1/3}(u^{-1})'(x)$.

To analyse (1.2), we introduce the integral operator

$$Tw(x)=\int\limits_0^x (x-s)g(s)w(s)^{-1/2}\,ds \qquad (x>0) \tag{1.4}$$

defined for any continuous function $w:[0,\infty)\to[0,\infty)$ such that the integral on the right-hand side is finite, and set

$$\varphi(x)=\int\limits_0^x (x-s)g(s)s^{-1/2}\,ds \quad \text{and} \quad q(x)=x^{1/3}\varphi(x)^{2/3} \quad (x>0)\,.$$

Some properties of φ and q which we shall use in the sequel are collected in the two following lemmas.

LEMMA 1.1. *Let g satisfy* (i)–(iii). *Then*

(a) *$\varphi(x)/x$ is nondecreasing for $x>0$;*

(b) *there exists a constant $c>0$ such that $x\varphi'^2(x)\le c\varphi(x)$ for $x\in(0,\delta)$;*

(c) *if $x^{1/2}g(x)$ is nondecreasing for $x\in(0,\delta)$, then $x\varphi'^2(x)/\varphi(x)$ is also nondecreasing on $(0,\delta)$.*

Proof. (a) It suffices to observe that φ is convex and $\varphi(0)=0$.

(b) Notice that

$$x\varphi'^2(x)=\int\limits_0^x (2s\varphi''(s)+\varphi'(s))\varphi'(s)\,ds \qquad (x>0)\,. \tag{1.5}$$

Now, our assertion follows by the fact that $2s\varphi''(s)+\varphi'(s)$ is bounded on $(0,\delta)$.

(c) We first note that

$$\varphi^2(x\varphi'^2/\varphi)' = \{(2x\varphi'' + \varphi')\varphi - x\varphi'^2\}\varphi' .$$

Further, from (1.5) and by the fact that $2s\varphi''(s) + \varphi'(s)$ is nondecreasing on $(0, \delta)$ we obtain

$$x\varphi'^2(x) \le (2x\varphi''(x) + \varphi'(x))\varphi(x) \quad \text{for } x \in (0, \delta) .$$

So, $(x\varphi'^2/\varphi)' \ge 0$ on $(0, \delta)$, which completes the proof.

LEMMA 1.2. *Let g satisfy* (i)–(iii). *Then*

(a) *$Tq(x)$ is well defined for $x > 0$;*

(b) *if $x^{1/2}g(x)$ is nondecreasing on $(0, \delta)$, then $Tq(x) \le \frac{9}{2}q(x)$ for $x \in (0, \delta)$.*

Proof. (a) We first observe that

$$\int_0^x g(s)q(s)^{-1/2}\, ds = \int_0^x \varphi''(s)\left(\frac{\varphi(s)}{s}\right)^{-1/3} ds \quad (0 < x) .$$

By the estimate given in Lemma 1.1(b) we can integrate by parts to obtain

$$\int_0^x \varphi''(s)\left(\frac{\varphi(s)}{s}\right)^{-1/3} ds = \varphi'(x)\left(\frac{\varphi(x)}{x}\right)^{-1/3} + \frac{1}{3}\int_0^x \varphi'(s)\left(\frac{\varphi(s)}{s}\right)^{-1/2}\left(\frac{\varphi(s)}{s}\right)'\left(\frac{\varphi(s)}{s}\right)^{-5/6} ds . \tag{1.6}$$

In order to estimate the right-hand side of (1.6) we use Lemma 1.1(b) once more, which gives

$$\int_0^x \varphi''(s)\left(\frac{\varphi(s)}{s}\right)^{-1/3} ds < \infty \quad \text{for } x \in (0, \delta) .$$

Hence our assertion follows immediately.

(b) We use Lemma 1.1(c) in order to estimate the integral on the right-hand side of (1.6). Then we obtain

$$\int_0^x \varphi''(s)\left(\frac{\varphi(s)}{s}\right)^{-1/3} ds \le 3\varphi'(x)\left(\frac{\varphi(s)}{s}\right)^{-1/3} \quad \text{for } x \in (0, \delta) .$$

Now, we derive

$$\begin{aligned} Tq(x) &= \int_0^x \Big(\int_0^s g(\xi)q(\xi)^{-1/2}\, d\xi\Big)\, ds \\ &= 3\int_0^x \varphi'(s)\left(\frac{\varphi(s)}{s}\right)^{-1/3} ds \le \frac{9}{2}q(x) \quad \text{for } x \in (0, \delta) , \end{aligned}$$

which completes the proof.

In view of Lemma 1.2(a), we can put

$$Q(x) = Tq(x) \quad \text{for } x \in (0, \delta)\,.$$

In the following theorem necessary and sufficient conditions for the existence of nontrivial solutions $u \not\equiv 0$ of (1.1) are established.

THEOREM 1.1. *Let* (i)–(iii) *be satisfied. If*

$$\int_0^\delta q(s)^{-1/2}\, ds < \infty \quad (\delta > 0)\,, \tag{1.7}$$

then (1.1) *has a nontrivial solution. Conversely*, *if* (1.1) *has a nontrivial solution*, *then*

$$\int_0^\delta Q(s)^{-1/2}\, ds < \infty \quad (\delta > 0)\,. \tag{1.8}$$

Define

$$K_0 = \{g : s^{1/2} g(s) \text{ is nondecreasing for } 0 < s < \delta\ (\delta > 0)\}\,.$$

Let

$$h^*(x) = \sup s^{1/2} g(s) \quad \text{for } 0 < s \le x \quad \text{and} \quad g^*(x) = x^{-1/2} h^*(x) \quad (x > 0)$$

where g is any function satisfying (i)–(iii). For g^* we define T^*, φ^*, q^* and Q^* similarly to those corresponding to g. Define

$$J(x) = \int_0^x g(s) s^{-1/2}\, ds\,, \quad J^*(x) = \int_0^x g^*(s) s^{-1/2}\, ds\,,$$
$$K = \{g : \sup(J^*(x)/J(x)) < \infty \text{ for } 0 < x \le \delta\}\,.$$

Of course, $K_0 \subseteq K$. Moreover, both K_0 and K contain nondecreasing functions g satisfying (i). We shall also present an example of $g \in K$ which takes the value 0 at some points x_n, $n = 1, 2, \ldots$, such that $x_n \to 0$ as $n \to \infty$.

If we are interested only in functions $g \in K$, then conditions (1.7) and (1.8) turn out to be equivalent. This is stated in the following theorem.

THEOREM 1.2. *If* $g \in K$, *then the condition*

$$\int_0^\delta \frac{ds}{\{s^{1/2}\varphi(s)\}^{1/3}} < \infty \quad (\delta > 0)$$

is necessary and sufficient for the existence of a unique solution u *of* (1.1) *such that* $u(x) > 0$ *for* $x > 0$.

2. Proofs of theorems. To prove Theorems 1.1 and 1.2 we need three lemmas. In the first one we give an a priori estimate of solutions of (1.2).

LEMMA 2.1. *For any continuous solution v of* (1.2) *we have*

$$q(x) \le v(x) \le Q(x) \quad (x > 0).$$

Proof. Using (1.2) we easily check that $v(x)/x$, $x > 0$, is nondecreasing. Therefore

$$v(x) = \int_0^x (x-s)g(s)v(s)^{-1/2}\, ds \ge \left(\frac{v(x)}{x}\right)^{-1/2} \varphi(x) \quad (x > 0). \tag{2.1}$$

Hence we get the left-hand inequality. Now the right-hand inequality follows from the monotonicity properties of T.

The problem of the existence of solutions of (1.2) is considered in the following lemmas.

LEMMA 2.2. *There exists at least one continuous solution of* (1.2).

Proof. We regularize (1.2) as follows:

$$v_\varepsilon(x) = \varepsilon x + \int_0^x (x-s)g(s)v_\varepsilon(s)^{-1/2}\, ds \quad (\varepsilon, x > 0), \tag{2.2}$$

where v_ε is sought in $C[0,\delta)$. Define

$$T_\varepsilon w(x) = \varepsilon x + \int_0^x (x-s)g(s)w(s)^{-1/2}\, ds$$

for any continuous $w : (0,\delta) \to [0,\infty)$ such that the integral on the right-hand side is finite and set

$$w_0(x) = \varepsilon x \quad \text{for } 0 < x < \delta \quad \text{and} \quad w_n = T_\varepsilon^n w_0, \quad n = 1, 2, \ldots$$

The following useful properties of T_ε can easily be obtained:

$$w_0 \le T_\varepsilon v \quad \text{for any } v \ge 0; \tag{2.3}$$

$$\text{if } 0 \le v_1 \le v_2, \text{ then } T_\varepsilon v_1 \ge T_\varepsilon v_2; \tag{2.4}$$

$$T_\varepsilon(cv) \ge c^{-1/2} T_\varepsilon v \quad \text{for any } c \ge 1 \text{ and } v \ge 0. \tag{2.5}$$

From (2.3) and (2.4) it follows that

$$w_0 \le w_2 \le w_4 \le \ldots \le w_5 \le w_3 \le w_1. \tag{2.6}$$

Since

$$T_\varepsilon w_0(x) = \varepsilon x + \varepsilon^{-1/2} \int_0^x (x-s)g(s)s^{-1/2}\, ds,$$

from (iii) it follows that there exists $\delta_\varepsilon > 0$ such that

$$w_0(x) \le w_1(x) \le \tfrac{3}{2} w_0(x) \quad \text{for } 0 < x < \delta_\varepsilon.$$

Applying (2.5) and using an inductive argument we can see that

$$1 \le \frac{w_{2k+1}(x)}{w_{2k}(x)} \le \left(\frac{3}{2}\right)^{(1/2)^{2k}} \qquad \text{for } 0 < x < \delta_\varepsilon \text{ and } k = 0, 1, 2, \ldots \tag{2.7}$$

From (2.6) and (2.7) we conclude that the sequence $\{w_n\}$ is convergent. Since the function $w(x) = \lim_{n\to\infty} w_n(x)$, $0 < x < \delta_\varepsilon$, satisfies (2.2) on $(0, \delta_\varepsilon)$, from its construction it follows that $v_\varepsilon(x) = w(x)$ is a unique continuous solution of (2.2) on a $(0, \delta_\varepsilon)$. Now, noting that $v_\varepsilon(x)/x$ is nondecreasing, we get the a priori estimates

$$q(x) \le v_\varepsilon(x) \le \varepsilon x + Q(x)\,, \quad v'_\varepsilon(x) \le \varepsilon + Q'(x) \quad \text{for } x > 0\,, \tag{2.8}$$

from which we conclude that v_ε can be extended from $[0, \delta_\varepsilon)$ to a unique continuous solution of (2.2) defined on a whole interval independent of ε.

Now, estimate (2.8) allows us to apply the Arzelà–Ascoli theorem. Thus we see that there exists a convergent sequence $\{v_{\varepsilon_n}\}$ of solutions of (2.2), where $\varepsilon_n \to 0$, as $n \to \infty$. The limit $v(x) = \lim_{n\to\infty} v_{\varepsilon_n}(x)$, $x > 0$, is a solution of (1.2), which completes the proof.

A uniqueness result for equation (1.2) is given in the following lemma.

LEMMA 2.3. *If there exists a constant* m *such that*

$$Q(x) \le mq(x) \quad (0 < x \le \delta)\,, \tag{2.9}$$

then the equation (1.2) *has exactly one continuous solution* v.

Proof. Let v_1, v_2 be two continuous solutions of (1.2). By Lemma 2.1 we have

$$q(x) \le v_1(x), \quad v_2(x) \le mq(x) \quad (0 < x < \delta)\,.$$

Therefore there exist constants $c_1, c_2 > 0$ such that

$$c_1 v_1(x) \le v_2(x) \le c_2 v_1(x) \quad (0 \le x \le \delta)\,. \tag{2.10}$$

Note also that for any $c > 0$ and for any solution v of (1.2) we have

$$T^n(cv) = c^{(-1/2)^n} v\,.$$

Therefore applying T^n to each term of (2.10), by the monotonicity properties of T we easily get $v_1 = v_2$, which completes the proof.

Now we are ready to prove our theorems.

Proof of Theorem 1.1. It is easily seen that Theorem 1.1 is a simple corollary of Lemmas 2.1 and 2.2.

Proof of Theorem 1.2. In view of Theorem 1.1 and Lemma 2.3, to prove Theorem 1.2 it suffices to show that there exists a constant m such that (2.9) is satisfied.

It follows from the definition of K that there exists a constant $c > 0$ such that

$$cq^*(x) \le q(x) \le q^*(x) \quad \text{for } x \in (0,\delta)\,. \tag{2.11}$$

Hence we get

$$\begin{aligned} Q(x) = Tq(x) &\le c^{-1/2}Tq^*(x) \\ &\le c^{-1/2}T^*q^*(x) = c^{-1/2}Q^*(x) \quad \text{for } x \in (0,\delta)\,. \end{aligned} \tag{2.12}$$

Now it suffices to note that $g^* \in K_0$. Therefore, by Lemma 1.2(b), $Q^*(x) \le \frac{9}{2}q^*(x)$ for $x \in (0,\delta)$, which combined with (2.11) and (2.12) gives the required result.

3. Examples. In this section we give two examples of application of the previous results.

EXAMPLE 3.1. Let $g(x) = x^{-1/2}(-\ln x)^{-\gamma}$, $\gamma > 0$, $0 < x < \delta$, for some $\delta > 0$. If $0 < \gamma \le 1$, then we can easily check that g does not satisfy condition (iii) which, in view of inequality (2.1), is one of the necessary conditions for the existence of nontrivial solutions of (1.1).

If $\gamma > 1$, then g belongs to K_0 and we can easily check that the condition given in Theorem 1.2 is satisfied. Therefore in the case of $\gamma > 1$ the problem (1.1) has a unique solution u such that $u(x) > 0$ for $0 < x < \delta$.

Before giving a second example we consider another subclass of K. Namely, let a bounded function g satisfy condition (i). We set

$$\overline{g}(x) = \sup g(s) \quad \text{for } 0 < s \le x \ (0 < x < \delta)\,,$$

and we define

$$G(x) = \int_0^x g(s)\,ds\,, \quad \overline{G}(x) = \int_0^x \overline{g}(s)\,ds \quad (0 < x < \delta)\,.$$

Denote by K_1 the class of bounded functions g satisfying (i) and such that

$$\sup(\overline{G}(x)/G(x)) < \infty \quad \text{for } x \in (0,\delta)\,.$$

Remark 3.1. $K_1 \subseteq K$.

Proof. Let $g \in K_1$. Since g is bounded, an integration by parts shows that

$$\int_0^x g(s)s^{-1/2}\,ds = x^{-1/2}G(x) + \frac{1}{2}\int_0^x G(s)s^{-3/2}\,ds\,.$$

Using the same formula for $\overline{g}$ we obtain

$$\int_0^x \overline{g}(s)s^{-1/2}\,ds \le c\int_0^x g(s)s^{-1/2}\,ds \quad \text{for } x \in (0,\delta)\,, \tag{3.1}$$

where $c > 0$ is some constant. Hence and by the fact that $g^* \le \overline{g}$ on $(0,\delta)$ it follows that $g \in K$, which ends the proof.

In the case of $g \in K_1$ the following theorem gives a necessary and sufficient condition for the existence of nontrivial solutions of (1.1).

THEOREM 3.1. *Let $g \in K_1$. Then* (1.1) *has a unique solution u such that $u(x) > 0$ for $x > 0$ if and only if*

$$\int_0^\delta \left(\frac{s}{\overline{g}(s)}\right)^{1/3} \frac{ds}{s} < \infty \qquad (\delta > 0)\,.$$

Proof. Since $\overline{g}$ is nondecreasing, we get

$$2^{-1}\overline{g}(x/2)x^{1/2} \le \int_0^x \overline{g}(s)s^{-1/2}\,ds \le 2\overline{g}(x)x^{1/2} \qquad (0 < x < \delta)\,.$$

Therefore the required result follows from Theorem 1.2 by applying (3.1).

EXAMPLE 3.2. Let $x_k = (2k+1)^{-1}\pi^{-1}$, $y_k = (2k\pi)^{-1}$, $a_k = (2k+5/6)^{-1}\pi^{-1}$, $b_k = (2k+1/6)^{-1}\pi^{-1}$ for any integer $k \ge k_0$, where k_0 is a fixed integer number such that $y_{k_0} \in (0,\delta)$. We define

$$g(x) = \begin{cases} \sin 1/x & \text{for } x_k \le x \le y_k, \\ 0 & \text{for } y_{k+1} < x < x_k \ (k > k_0) \end{cases} \tag{3.2}$$

and

$$g_1(x) = \begin{cases} 1/2 & \text{for } a_k \le x \le b_k, \\ 0 & \text{for } b_{k+1} < x < a_k \ (k > k_0). \end{cases}$$

Thus we have $g_1(x) \le g(x) \le \overline{g}(x) = 1$ for $x \in (0,\delta)$. Set

$$G_1(x) = \int_0^x g_1(s)\,ds\,, \quad \overline{G}(x) = \int_0^x \overline{g}(s)\,ds = x \quad \text{for } x \in (0,\delta)\,,$$

$$r_k = \sum_{l=k}^{\infty}(b_l - a_l) \qquad \text{for } k > k_0\,.$$

For $x \in (a_{k+1}, a_k)$ $(k > k_0)$ we get

$$\frac{\overline{G}(x)}{G(x)} \le \frac{x}{G_1(x)} \le 2\frac{x}{r_{k+1}} \le 2\frac{a_k}{r_{k+1}} = 2\frac{a_{k+1}}{r_{k+1}}\frac{a_k}{a_{k+1}}\,. \tag{3.3}$$

Of course, $a_k/a_{k+1} < c$ for some constant c and $k > k_0$. Since

$$r_k = \pi^{-1}\sum_{l=k}^{\infty}(2l+5/6)^{-1}(2l+1/6)^{-1}\,,$$

we can compare r_k with $\int_k^\infty s^{-2}\,ds$. As a result, $a_k/r_k < c$ for some constant c and $k > k_0$. Now, by (3.3) we conclude that $g \in K_1$. Applying Theorem 3.1 we see that the problem (1.1) with g defined in (3.2) has a nontrivial solution.

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References

[1] P. J. Bushell and W. Okrasiński, *Uniqueness of solutions for a class of nonlinear Volterra integral equations with convolution kernel*, Math. Proc. Cambridge Philos. Soc. 106 (1989), 547–552.

[2] G. Gripenberg, *Unique solutions of some Volterra integral equations*, Math. Scand. 48 (1981), 59–67.

[3] W. Mydlarczyk, *The existence of nontrivial solutions of Volterra equations*, ibid. 68 (1991), 83–88.

[4] W. Okrasiński, *Nontrivial solutions to nonlinear Volterra integral equations*, SIAM J. Math. Anal. 22 (1991), 1007–1015.

[5] —, *Nontrivial solutions for a class of nonlinear Volterra equations with convolution kernel*, J. Integral Equations Appl. 3 (1991), 399–409.

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